To solve a linear 2-term recurrence relation:
\[ a_0 = u, a_1 = v, a_n = aa_{n-1} + ba_{n-2}; n \geq 2, \] where \( u, v, a, b \) are constants \((b \neq 0)\):

1. let \( a_n = x^n \) and plug into the recurrence relation
2. get \( x^n = ax^{n-1} + bx^{n-2} \) or \( x^2 = ax + b \).
3. solve this quadratic for the the roots \( x = x_1, x_2 \) (if \( x_1 \) and \( x_2 \) are imaginary that’s ok)
4. if \( x_1 \neq x_2 \) the general formula for \( a_n \) is \( a_n = A(x_1)^n + B(x_2)^n \) otherwise it is \( a_n = A(x_1)^n + Bn(x_1)^n \), where \( A \) and \( B \) are constants.
5. solve \( a_0 = u, a_1 = v \) for \( A \) and \( B \).

Example: \( a_0 = 1, a_1 = 3, a_n = 2a_{n-1} - a_{n-2}; n \geq 2 \)
Solve \( x^2 - 2x + 1 = 0 \). Get roots \( x_1 = x_2 = 1 \). So the general solution is \( a_n = A(1)^n + Bn(1)^n = A + Bn \). Solve \( a_0 = 1 = A + B(0), a_1 = 3 = A + B(1) \) for \( A, B \). Get \( A = 1, B = 2 \), and so \( a_n = 1 + 2n \) is the solution for \( a_n \).

We now solve the same recurrence relation but this time we use generating functions (see the lecture notes on generating functions on Blackboard).

Let \( a(x) = \sum_{n=0}^{\infty} a_n x^n \). \((a(x)\) is the ordinary generating function for \( a_n\).\)

We have \( a(x) = a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n = 1 + 3x + \sum_{n=2}^{\infty} a_n x^n = 1 + 3x + \sum_{n=2}^{\infty} (2a_{n-1} - a_{n-2}) x^n \).

Now by partial fractions, there exist constants \( A \) and \( B \) such that \( a(x) = \frac{(1 + x)}{(1 - x)^2} = A/\(1 - x\) + B/(1 - x)^2 \). Solving we get \( A = -1, B = 2 \) so \( a(x) = -1/(1 - x) + 2/(1 - x)^2 \).

We have \( 1/(1 - x) = \sum_{n=0}^{\infty} x^n \) and \( 1/(1 - x)^2 = \sum_{n=0}^{\infty} (n + 1) x^n \). (We did both of these in class, see also the lecture notes on generating functions on Blackboard). So \( a(x) = \sum_{n=0}^{\infty} (-1 + 2(n + 1)) x^n = \sum_{n=0}^{\infty} (1 + 2n) x^n \).
Since \( a(x) = \sum_{n=0}^{\infty} a_n x^n \) we have \( a_n = 1 + 2n \).