1. Let $e = \{v, w\}$ be an edge of the unique cycle $C$ of $G$. Remove $e$ from $G$ to get a graph $G'$. $G'$ has no cycles. (If it did, then $G'$ and hence $G$ would contain a cycle $C'$ that does not contain $e$ and is hence different from $C$, a contradiction.) $G'$ is also connected. (Look at any two vertices $x$ and $y$. Since $G$ is connected there is an $x$-$y$ walk $W$ in $G$. Since $e$ is on the cycle $C$, $G$ contains two $v$-$w$ paths, one that uses $e$ and one that does not. Thus $W$ doesn’t need to use the edge $e$ and hence is in $G'$.) Thus $G'$ is a tree and has $n - 1$ edges. So $G$ has $n$ edges.

2. Because $\chi(W_n) = 1 + \chi(C_n)$, $\chi(W_n) = 3$ if $n$ is even and 4 if $n$ is odd. A proper coloring of $C_n$ can be extended to a proper coloring of $W_n$ by giving vertex $a$ a distinct color from the ones already used, so $\chi(W_n) \leq 1 + \chi(C_n)$. On the other hand, a proper coloring of $W_n$ is also a proper coloring of $C_n$ and so must use $\chi(C_n)$ colors on $\{1, \ldots, n\}$ and one more distinct color for vertex $a$, since it is connected to all the others. Thus $\chi(W_n) \geq 1 + \chi(C_n)$.

3. Let $A = \{(i, j, k) : i + j + k \text{ odd}\}$, $B = \{(i, j, k) : i + j + k \text{ even}\}$. If $e = \{(i, j, k), (i', j', k')\}$ is an edge $(i, j, k)$ and $(i', j', k')$ differ in exactly one coordinate and differ in that coordinate by exactly 1. Thus if $e$ is an edge we have $i + j + k = i' + j' + k' + 1$ or $i' + j' + k' - 1$. So $e$ has one end point in $A$ and the other in $B$. Thus $G$ is bipartite and has no odd cycles.

4. Since column $i$ has no repeats it must use $n - k$ distinct numbers from $\{1, 2, \ldots, n\}$ and hence $k$ numbers must be missing. Thus $a_i$ is connected to $k$ vertices $b_j$. On the other hand, since each row is a permutation, color $j$ is used in each row, and since the columns have no repeats, it must be used in a different column each time. Since color $j$ is used in $n - k$ columns and is missing from $k$ columns, $b_j$ is connected to $k$ vertices $a_i$. By the result in class, $G$ has a perfect matching (see page 11 of the class notes on matchings). But a perfect matching in $G$ is an assignment to each $a_i$, a distinct $b_j$ such that $\{a_i, b_j\}$ is an edge. Thus each column $i$ gets a distinct number $j$ that was missing from column $i$. Thus we get a row that is a permutation of $\{1, \ldots, n\}$ such that each column still has no repeats. So $M'$ is extended by one
more row to a partial latin square with $k - 1$ rows missing. Repeat this process until no rows are missing.