HW6: due 2 Apr 2003

1. Solve the following recurrence relations (find a formula for $a_n$):

(a) $a_0 = 1, a_1 = 1, a_n = 3a_{n-1} - 9a_{n-2}; n \geq 2$

$$a_n = (3/4)(1 - 1/3\sqrt{3})(1 + \sqrt{3})^n + (3/4)(1 + 1/3\sqrt{3})(1 - \sqrt{3})^n.$$  

Plug $a_n = r^n$ into $a_n = 3a_{n-1} - 9a_{n-2}$. Get $r^2 = 3r - 9$. Get $r = (3/2)(1 \pm \sqrt{3})$. Thus $a_n = A((3/2)(1 + \sqrt{3})^n + B((3/2)(1 - \sqrt{3})^n)$. Plugging in $n = 0$ and $n = 1$ get, $A + B = a_0 = 1$ and $(3/2)(A + B) + (3/2)\sqrt{3}(A - B) = a_1 = 1$. Solving for $A, B$, get $A = (1/2)(1 - 1/3\sqrt{3})$ and $B = (1/2)(1 + 1/3\sqrt{3})$. Thus $a_n = (1/2)(1 - 1/3\sqrt{3})(3/2)(1 + \sqrt{3})^n + (1/2)(1 + 1/3\sqrt{3})(3/2)(1 - \sqrt{3})^n$.

(b) $a_0 = 1, a_1 = 1, a_n = -4a_{n-2}; n \geq 2$

$$a_n = (1/2 - i/4)(2i)^n + (1/2 + i/4)(-2i)^n.$$  

Solve $r^2 + 4 = 0$. Get $r = \pm 2i$. Get $a_n = A(2i)^n + B(-2i)^n$. 

Plugging in $n = 0, 1$ get $1 = a_0 = A + B$ and $1 = a_1 = 2i(A - B)$ or $A - B = -i/2$. Thus $A = 1/2 - i/4, B = 1/2 + i/4$, and so $a_n = (1/2 - i/4)(2i)^n + (1/2 + i/4)(-2i)^n$.

2. Find a formula for $a_n$ by using generating functions.

$a_0 = 1, a_1 = 1, a_n = 3a_{n-1} - 2a_{n-2}; n \geq 2$

$a_n = 1$ for all $n$.

Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$. We have $a(x) = a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n = 1 + x + \sum_{n=2}^{\infty} (3a_{n-1} - 2a_{n-2}) = 1 + x + 3x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} - 2x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} = 1 + x + 3x(a(x) - 1) + 2x^2 a(x)$. Solving for $a(x)$ we get $a(x) = (1 - 2x)/(1 - 3x + 2x^2)$. By partial fractions, $(1 - 2x)/(1 - 3x + 2x^2) = (1 - 2x)/((1 - x)(1 - 2x)) = 1/(1 - x)$ so $a(x) = 1/(1 - x) = \sum_{n=0}^{\infty} x^n$. Thus $a_n = 1$.

3. Let $a_n$ be the number of strings of length $n$ that are made of 0’s and 1’s that don’t contain the substring string 10. Find a formula for $a_n$. Example: $a_1 = 2, a_2 = 3$. (Find a recurrence relation for $a_n$ and solve it.)

$a_n = n + 1$ (for $a_1 = 2, a_n = a_{n-1} + 1$ for $n \geq 2$). Consider how a string of $n$ characters ends. If it ends in a 1, the remaining string of $n - 1$
characters form a string without a 10 and thus there are \( a_{n-1} \) ways of filling them in. On the other hand if the string ends in a 0 it can contain no 1’s at all (otherwise it would contain a 10) so there is only one such string. Thus \( a_n = a_{n-1} + 1 \).

Thus we have the recurrence relation \( a_1 = 2, a_n = a_{n-1} + 1 \) for \( n \geq 2 \). We claim \( a_n = n + 1 \). We prove this by induction on \( n \). Clearly \( a_n = n + 1 \) for \( n = 1 \) since \( a_1 = 2 \). Suppose we have the claim for \( n - 1 \), or \( a_{n-1} = (n - 1) + 1 \) or \( a_{n-1} = n \). Then \( a_n = a_{n-1} + 1 = n + 1 \).

(One can make a simpler argument without getting a recurrence relation. A string is valid if and only if it consists of a string of 0’s followed by a (possibly empty) string of 1’s. There are only \( n + 1 \) possibilities. More formally, one possibility is that the string is all 0’s. If it is not, there are \( n \) possibilities for the position \( j \) of the first 1 and once you know what \( j \) is, you know the rest of the string. Since the string contains no 10, after a 1 occurs, the string must consist entirely of 1’s. Thus the string consists of \( j - 1 \) 0’s followed by \( n - j \) 1’s.)