HW 8 Hints and Solutions
22.1.2 \( \sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \). Weierstrass M-test with \( M_k = \frac{R^{2k+1}}{(2k+1)!} \).
Note \( |(-1)^k \frac{x^{2k+1}}{(2k+1)!}| \leq M_k \) on \([-R, R]\) and \( \sum_{k=0}^{\infty} M_k \) converges. In fact even \( \sum_{k=0}^{\infty} \frac{R^k}{k!} = e^R \) converges.
22.2.2 (a) \( f_n(x) = \frac{x}{x+n} \rightarrow f(x) = 0 \) on \([0, R]\) Indeed \( |f_n(x) - f(x)| = \frac{x}{x+n} \leq R/n \) so the convergence is uniform.
(b) \( |\cos(x/n) - 1| = |\cos(x/n) - \cos(0/n)| = |-(1/n)\sin(c/n)(x-0)| \) where \( c \) is between 0 and \( x \). At any rate \( |\sin(c/n)| \leq 1 \) and \( |\cos(x/n) - 1| \leq R/n \) if \( |x| < R \). Thus \( \cos(x/n) \rightarrow 1 \) uniformly for \( |x| < R \).
(c) Weierstrass M-test with \( M_n = 1/n^2 \).
(d) same as (c) with \( M_n = 1/n^2 \).
22.3.1 Since the terms of the series are continuous and since the series converges uniformly by Weierstrass M-test with \( M_n = 1/(n-1)^2 \), the series is continuous.
22.4.2 \( e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \) converges on \((\infty, \infty)\) thus \( e^{-t^2/2} = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{2^k k!} \) converges on \((\infty, \infty)\). For any fixed \( R > 0 \), the convergence of this series is uniform on \([-R, R] \). Use the Weierstrass M test with \( M_k = \frac{R^{2k}}{2^k k!} \).
(\( \sum_{k=0}^{\infty} M_k = e^{M^2/2} \) converges)! Thus for any \( x \) with \( |x| \leq R \) we can integrate the series term by term, so that \( e^f(x) = \int_0^x e^{-t^2/2} dt = \int_0^x \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{2^k k!} dt = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{2^k k!} \). Since \( R \) was arbitrary we have \( erf(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2^k k!(2k+1)} \) on \((\infty, \infty)\).
22.4.3 \( f(x) = \sum_{n=1}^{\infty} u_n(x) \) where \( u_n(x) = \frac{\cos(nx)}{(n-1)!} \). Let \( M_n = 1/(n-1)! \).
We have \( |u_n(x)| \leq M_n \) on the interval \([0, \pi/2]\) (or any compact interval for that matter), and \( \sum_{n=1}^{\infty} M_n = e \) converges. Thus by the Weierstrass M-test, the series for \( f(x) \) converges uniformly. This means we can perform the integration \( \int_0^{\pi/2} f(x) dx \) term-by-term, which gives \( \int_0^{\pi/2} f(x) dx = \sum_{n=1}^{\infty} \int_0^{\pi/2} u_n(x) dx = \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n!} \).
22.5.1 We are assuming the series for \( f(x) \) is convergent. What remains to apply Theorem 22.5A is to show is that the series for \( f'(x) \) is convergent. But the Weierstrass M-test with \( M_n = na_n \) will give this since we are assuming \( \sum M_n \) converges.
22.6.3 Plugging in \( y(x) = \sum_{n=0}^{\infty} a_n x^n \) into \( y' - y = e^x \) we get \( \sum_{n=0}^{\infty} ((n+1)a_{n+1} - a_n)x^n = \sum_{n=0}^{\infty}(1/n!)x^n \). Thus \( (n+1)a_{n+1} - a_n = 1/n! \) for \( n \geq 0 \). If \( a_n = b_n/n! \) then this becomes \( b_{n+1} - b_n = 1 \) for \( n \geq 0 \). Since \( y(0) = a_0 = 0 \), \( b_0 = 0 \) and this solves to \( b_n = n \). Thus \( a_n = b_n/n! = 1/(n-1)! \) for \( n \geq 1 \),
and \( a_0 = 0 \). Thus \( y(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!} = xe^x \).

22.6.2 Let \( f(x) \) be the series, then \( f'(x) = xg(x) \) where \( g(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \).

But then \( g'(x) = \sum_{n=1}^{\infty} x^{n-1} = 1/(1 + x) \). Thus \( g(x) = \log(1 + x) + C \), but \( C = 0 \). So \( f'(x) = x \log(1 + x) \). Thus after some pain, \( f(x) = x/2 - x^2/4 - (1/2) \log(1 + x) + (1/2)x^2 \log(1 + x) + K \). Since \( f(0) = 0 \) we have \( K = 0 \).