HW 6 Hints and Solutions

15.2.1 Let $a$ be any point in the interval $I$. Let $b$ be any other. Then by using the Mean Value Theorem, $f(b) - f(a) = f'(c)$ for some $c$ between $a$ and $b$. Since $|f'(c)| \leq M$ this means $|f(b) - f(a)| \leq M|b - a|$ for all $b$ in the interval $I$. Since $I$ is finite, $|b - a| \leq K$ for some $K$, so $|f(b) - f(a)| \leq MK$ for all $b$ in $I$. This shows that $f$ is bounded on $I$. If $I = \mathbb{R}$ then $f(x) = x$ is a counterexample. ($f'(x)$ is bounded, but $f(x)$ is not.)

(b) Let $I = (0, 1]$. Let $f(x) = 2\sqrt{x}$ then $f(x)$ is bounded, but $f'(x) = 1/\sqrt{x}$ is not.

15.4.2 Use L'Hospital's Rule repeatedly. $\lim_{x \to 0} \frac{x \tan(x)}{x^3}$ $\leq$ $\lim_{x \to 0} \frac{1 - \sec^2(x)}{1 - \cos(x)}$ $= \lim_{x \to 0} \frac{2\sec^2(x) \tan(x)}{\sin(x)}$ $= \lim_{x \to 0} \frac{4\sec^2(x) \tan^2(x) + 2\sec^4(x)}{\cos(x)} = 2$.

16.1.1 (a) $e^x = 1 + x + e^x x^2/2$ where $0 < c < x$. Thus $1 + x + x^2/2 < e^x < 1 + x + e^x x^2/2$ for $0 < x < 1$, as for these values, $1 = e^0 < e^c < e^1 < e$.

(b) $\ln(1 + x) = 0 + 1x + (-1/(1 + c^2)) x^2$ where $0 < c < x$. Thus $x - x^2 < \ln(1 + x) < x - (1/4)x^2$ for $0 < x \leq 1$, as for these values, $-1 < -1/(1 + 0)^2 < (-1/(1 + c^2)) < -1/(1 + 1)^2 = -1/4$.

16.2.4 (a) The midpoint of the chord from $(a, f(a))$ to $(b, f(b))$ occurs at $((a + b)/2, (f(a) + f(b))/2)$. Since the function is geometrically concave, $f((a + b)/2) \geq (f(a) + f(b))/2$.

(b) If $f(x) = \ln(x)$ then $f''(x) = -x^2$ and $f(x)$ is concave/geomterically concave. Thus $\ln((a+b)/2) \geq (\ln(a) + \ln(b))/2 = \ln((ab)^{1/2})$. Thus $(a+b)/2 \geq \sqrt{ab}$.

17.3.3 The error term for the $n$th order Taylor polynomial to $f(x) = \sin(x)$ at 0 will be of the form $f^{(n+1)}(c)x^{n+1}/(n+1)!$ where $c$ is between 0 and $x$. Since $f^{(n+1)}(c) = \pm \sin(c)$ or $\pm \cos(c)$ we upper bound that factor by 1. Thus the largest error term can get on the interval $|x| < 1/2$ is $\frac{1}{2^{n+1}(n+1)!}$. To ensure a certain amount of digits of accuracy requires in general differing estimates of error. For instance, if the approximation is 0, the error must be less than .001, to have three-decimal place accuracy. However if the approximation is .0009 the error should be less than .0001 to get three digit accuracy. Let's just say in this problem that we want the error to be smaller than .0001. The first time $\frac{1}{2^{n+1}(n+1)!} < .0001$ is when $n = 5$. So in order to get an error of no more than .0001 on $|x| < 1/2$ use $x - x^3/6 + x^5/120$.

17.3.4 Similar to previous.