Problem 1

Taking partial derivatives:
\[
\begin{align*}
\frac{\partial z}{\partial x} &= -\frac{y}{x^2} + \frac{x}{x^2 + y^2} \Rightarrow \frac{\partial^2 z}{\partial x^2} = \frac{2xy}{(x^2 + y^2)^2}. \\
\frac{\partial z}{\partial y} &= \frac{1}{x} - \frac{x}{x^2 + y^2} \Rightarrow \frac{\partial^2 z}{\partial y^2} = -\frac{2xy}{(x^2 + y^2)^2}. \\
\therefore \quad z_{xx} + z_{yy} &= \frac{2xy}{(x^2 + y^2)^2} - \frac{2xy}{(x^2 + y^2)^2} = 0. \quad \text{QED}
\end{align*}
\]

Problem 2

If a surface is described by \( z = f(x, y) \) then its normal is \( \mathbf{N} = (f_x, f_y, -1) \).

\( \Rightarrow \) a normal to the first surface is \( \mathbf{N}_1 = (2x, 2y, -1) \).

If a surface is given by \( g(x, y, z) = 0 \) then its normal is \( \mathbf{N} = (g_x, g_y, g_z) \).

\( \Rightarrow \) a normal to the second surface is \( \mathbf{N}_2 = (4x, 4y, -2z) \).

At \((1, 1, 2): \quad \mathbf{N}_1 = (2, 2, -1) \) and \( \mathbf{N}_2 = (4, 4, -4) \), rescaling we take \( \mathbf{N}_2 = (1, 1, -1) \).

The tangent to the intersection curve is orthogonal to both normals.

\[ \mathbf{N}_1 \times \mathbf{N}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 1 & 1 & -1 \end{vmatrix} = -1 + \mathbf{j} = \text{tangent vector}. \]

Problem 3

This is a problem where the general case is easier because we avoid some of the computational clutter of the special case. We will see that the point \((0, 0, 0)\) is on every tangent plane.

If \( z = x f(x/y) \) then \( \frac{\partial z}{\partial x} = f(x/y) + \frac{x}{y} f'(x/y) \) and \( \frac{\partial z}{\partial y} = -\frac{x^2}{y^2} f'(x/y) \).

\( \Rightarrow \) the graph has normal \( \mathbf{N} = (f(x/y) + \frac{x}{y} f'(x/y), -\frac{x^2}{y^2} f'(x/y), -1) \).

So, for the point \((x_0, y_0, z_0)\) on the graph (so \( z_0 = x_0 f(x_0/y_0) \) -keep this in mind) the tangent plane has equation

\[
\left( f(x_0/y_0) + \frac{x_0}{y_0} f'(x_0/y_0) \right) \cdot (x - x_0) - \left( \frac{x_0^2}{y_0^2} f'(x_0/y_0) \right) \cdot (y - y_0) - (z - z_0) = 0.
\]

Simplifying this becomes

\[
a x - b y - z = \left( f(x_0/y_0) + \frac{x_0}{y_0} f'(x_0/y_0) \right) \cdot x_0 - \left( \frac{x_0^2}{y_0^2} f'(x_0/y_0) \right) \cdot y_0 - z_0,
\]

where \(a\) and \(b\) are the coefficients of \(x\) and \(y\) (and we are not interested in their values).

The right hand side simplifies further so that

\[
a x - b y - z = f(x_0/y_0) x_0 - z_0 = 0.
\]

(The last equality follows from the fact you were asked to keep in mind.) Planes of this form all contain the origin. \( \text{QED} \)