18.02A Practice Exam 2 (50 minutes)

Problem 1.

a) Let \( w = \frac{x^3}{y^2} \). Find \( \text{grad } w \), at the point (1, 2).

b) Use it to find the approximate value of \( w \) when \( x = 1.04 \) and \( y = 1.96 \).

c) Find a vector at (1, 2) in whose direction \( w \) is instantaneously not changing.

Problem 2. The map shows the region around Mt. Monadnock in southern New Hampshire. The level curves for the height function \( h(x, y) \) are marked (in feet); adjacent level curves represent a difference of 100 feet in height. A scale is given (1 unit = 1000 ft.). Two points \( P \) and \( Q \) are marked.

a) Estimate to .1 the directional derivative \( \frac{dh}{ds} \) at \( P \) in the direction of \( -i + j \).

b) Mark on the map a point \( R \) where \( h = 2800 \) and \( \frac{\partial h}{\partial y} = 0 \).

c) Draw on the map the gradient vector at the point \( Q \).

Problem 3. Let \( w = f(x, y) \) and let \( x(t) = x(t) i + y(t) j \) be the position vector for a point moving in the plane. Then along the path of motion, \( w \) is a function of \( t \), that is, \( w = f(x(t), y(t)) \).

a) Assuming everything is differentiable, show that \( \frac{dw}{dt} = (\text{grad } w) \cdot \frac{dx}{dt} \), for every \( t \). (You can use anything but this fact itself.)

b) Suppose the motion traces out a level curve of the function \( f(x, y) \). Show that at every point of the level curve, \( \text{grad } w \) is perpendicular to the curve.

Problem 4.

a) Find the tangent plane to the surface \( x^2 y + y^2 + 2xz^2 = 4 \) at the point \((1, 1, 1)\) on the surface, giving its equation in the form \( ax + by + cz = d \).

b) At the point \((1, 1, 1)\), in which direction (\( a \) a unit vector) would you start moving in order to decrease the value of \( x^2 y + y^2 + 2xz^2 \) most rapidly?

Problem 5. Let \( w \) be a function of the form \( w = f(xy) \), i.e., \( w = f(u) \), where \( u = xy \).

Assuming \( f \) is differentiable, we have \( x\left(\frac{\partial w}{\partial x}\right) - y\left(\frac{\partial w}{\partial y}\right) = 0 \).

a) Verify this equation for the function \( w = \sin(xy) \).

b) Show the equation is true in general.

Problem 6. Let \( w = x^2 + y^2 + z^2 \), where \( x = f(y, z) \). Express \( \left(\frac{\partial w}{\partial z}\right)_x \) in terms of the formal partial derivatives of \( f(y, z) \), by using either a) or b) below:

a) the chain rule (hint: find \( \frac{\partial}{\partial z} \) for both sides of the second equation)

b) differentials.

Problem 7. A rectangular box is placed in the first octant as shown, with one vertex at the origin and the three adjacent faces in the coordinate planes. The vertex \( P : (x, y, z) \) furthest from the origin lies on the plane \( 2x + 3y + z = 6 \). Find the largest volume such a box can have, as follows:

a) Show the problem leads one to maximize \( f(x, y) = 6xy - 2x^2y - 3xy^2 \).

b) In the first quadrant \( f(x, y) \) has a critical point \( P_0 \) for which \( x_0 = 1 \), i.e., of the form \( P_0 : (1, y_0) \). Find \( y_0 \) and show the resulting \( P_0 \) is a critical point.

c) Show \( P_0 \) is a local maximum point of \( f(x, y) \) by using the second derivative test.

d) Finish the problem.

e) If Lagrange multipliers are used to solve the problem, write down one of the equations involving the multiplier \( \lambda \), and determine from it the value of \( \lambda \) corresponding to \( P_0 \).