FUNDAMENTAL GROUPS OF COMPLEMENTS OF PLANE CURVES AND SYMPLECTIC INVARIANTS

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Abstract. Introducing the notion of stabilized fundamental group for the complement of a branch curve in \( \mathbb{CP}^2 \), we define effectively computable invariants of symplectic 4-manifolds that generalize those previously introduced by Moishezon and Teicher for complex projective surfaces. Moreover, we study the structure of these invariants and formulate conjectures supported by calculations on new examples.

1. Introduction

Using approximately holomorphic techniques first introduced in [5], it was shown in [2] (see also [1]) that compact symplectic 4-manifolds with integral symplectic class can be realized as branched covers of \( \mathbb{CP}^2 \) and can be investigated using the braid group techniques developed by Moishezon and subsequently by Moishezon and Teicher for the study of complex surfaces (see e.g. [13]):

**Theorem 1.1** ([2]). Let \((X, \omega)\) be a compact symplectic 4-manifold, and let \( L \) be a line bundle with \( c_1(L) = \frac{1}{2\pi}[\omega] \). Then there exist branched covering maps \( f_k : X \to \mathbb{CP}^2 \) defined by approximately holomorphic sections of \( L^\otimes k \) for all large enough values of \( k \); the corresponding branch curves \( D_k \subset \mathbb{CP}^2 \) admit only nodes (both orientations) and complex cusps as singularities, and give rise to well-defined braid monodromy invariants. Moreover, up to admissible creations and cancellations of pairs of nodes in the branch curve, for large \( k \) the topology of \( f_k \) is a symplectic invariant.

This makes it possible to associate to \((X, \omega)\) a sequence of invariants (indexed by \( k \gg 0 \)) consisting of two objects: the braid monodromy characterizing the branch curve \( D_k \), and the geometric monodromy representation \( \theta_k : \pi_1(\mathbb{CP}^2 - D_k) \to S_n \) \((n = \deg f_k)\) characterizing the \( n \)-fold covering of \( \mathbb{CP}^2 - D_k \) induced by \( f_k \) [2]. These invariants are extremely powerful (from them one can recover \((X, \omega)\) up to symplectomorphism) but too complicated to handle in practical cases.

In the study of complex surfaces, Moishezon and Teicher have shown that the fundamental group \( \pi_1(\mathbb{CP}^2 - D) \) (or, restricting to an affine subset, \( \pi_1(\mathbb{C}^2 - D) \)) can be computed explicitly in some simple examples; generally speaking, this group has been expected to provide a valuable invariant for distinguishing diffeomorphism types of complex surfaces of general type. However, in the symplectic case, it is affected by creations and cancellations of pairs of nodes and cannot be used immediately as an invariant.

We will introduce in \( \S 2 \) a certain quotient \( G_k \) (resp. \( \bar{G}_k \)) of \( \pi_1(\mathbb{C}^2 - D_k) \) (resp. \( \pi_1(\mathbb{CP}^2 - D_k) \)), the stabilized fundamental group, which remains invariant under creations and cancellations of pairs of nodes. As an immediate corollary of the construction and of Theorem 1.1, we obtain the following
Theorem 1.2. For large enough $k$, the stabilized groups $G_k = G_k(X, \omega)$ (resp. $\tilde{G}_k(X, \omega)$) and their reduced subgroups $G^0_k = \tilde{G}^0_k(X, \omega)$ are symplectic invariants of the manifold $(X, \omega)$.

These invariants can be computed explicitly in various examples, some due to Moishezon, Teicher and Robb, others new; these examples will be presented in §4, and a brief overview of the techniques involved in the computations is given in §6 and §7. The new examples include double covers of $\mathbb{CP}^1 \times \mathbb{CP}^1$ branched along arbitrary complex curves (Theorem 4.6 and §7); similar methods should apply to other double covers as well, thus providing results for both types of so-called Horikawa surfaces. The calculations described in §7, which rely on various innovative tools in addition to a suitable reformulation of the methods developed by Moishezon and Teicher, go well beyond the scope of results accessible using only the previously known techniques, and may present interest of their own for applications in algebraic geometry.

The available data suggest several conjectures about the structure of the stabilized fundamental groups.

First of all, it appears that in most examples the stabilization operation does not actually affect the fundamental group. The only known exceptions are given by “small” linear systems with insufficient ampleness properties, where the stabilization is a quotient by a non-trivial subgroup (see §4). Therefore we have the following

Conjecture 1.3. Assume that $(X, \omega)$ is a complex surface, and let $D_k$ be the branch curve of a generic projection to $\mathbb{CP}^2$ of the projective embedding of $X$ given by the linear system $|kL|$. Then, provided that $k$ is large enough, the stabilization operation is trivial, i.e. $G_k(X, \omega) \cong \pi_1(C^2 - D_k)$ and $\tilde{G}_k(X, \omega) \cong \pi_1(\mathbb{CP}^2 - D_k)$.

An important class of fundamental groups for which the conjecture holds will be described in §3.

Moreover, the structure of the stabilized fundamental groups seems to be remarkably simple, at least when the manifold $X$ is simply connected; in all known examples they are extensions of a symmetric group by a solvable group, while there exist plane curves with much more complicated complements [4, 6]. In fact these groups seem to be largely determined by intersection pairing data in $H_2(X, \mathbb{Z})$. More precisely, the following result will be proved in §5:

Definition 1.4. Let $\Lambda_k$ be the image of the map $\lambda_k : H_2(X, \mathbb{Z}) \to \mathbb{Z}^2$ defined by $\lambda_k(\alpha) = (\alpha \cdot L_k, \alpha \cdot R_k)$, where $L_k = k c_1(L)$ and $R_k = c_1(K_X) + 3L_k$ are the classes in $H^2(X, \mathbb{Z})$ Poincaré dual to a hyperplane section and to the ramification curve respectively.

Theorem 1.5. If the symplectic manifold $X$ is simply connected, then there exists a natural surjective homomorphism $\phi_k : \text{Ab} \, G_0^0(X, \omega) \to (\mathbb{Z}^2/\Lambda_k) \otimes \mathcal{R}_{n_k} \cong (\mathbb{Z}^2/\Lambda_k)^{n_k - 1}$, where $n_k = \deg f_k = L_k \cdot L_k$, and $\mathcal{R}_{n_k}$ is the reduced regular representation of $S_{n_k}$ (isomorphic to $\mathbb{Z}^{n_k - 1}$).

The map $\phi_k$ is $(G_k, S_{n_k})$-equivariant, in the sense that $\phi_k(g^{-1} \gamma g) = \theta_k(g) \cdot \phi_k(\gamma)$ for any elements $g \in G_k(X, \omega)$ and $\gamma \in \text{Ab} \, G_0^0(X, \omega)$ (cf. also Lemma 5.2).

In the examples discussed in §4, the group $G^0_k$ is always close to being abelian, and $\phi_k$ is always an isomorphism. It seems likely that the injectivity of $\phi_k$ can be proved using techniques similar to those described in §6–7. Therefore, it makes sense to formulate the following
Conjecture 1.6. If the symplectic manifold $X$ is simply connected and $k$ is large enough, then $\text{Ab} G^0_k(X, \omega) \simeq (\mathbb{Z}^2/\Lambda_k) \otimes \mathcal{R}_{n_k}$, and the commutator subgroup $[G^0_k, G^0_k]$ is a quotient of $(\mathbb{Z}_2)^2$.

Conjectures 1.3 and 1.6 provide an almost complete tentative description of the structure of fundamental groups of branch curve complements in high degrees. In relation with the property $(\ast)$ introduced in §3, they also provide a framework to explain various observations and conjectures made in [14] and [12].

The obtained results seem to indicate that fundamental groups of branch curve complements cannot be used as invariants to symplectically distinguish homeomorphic manifolds. This is in sharp contrast with the braid monodromy data, which completely determines the symplectomorphism type of $(X, \omega)$ [2]; how to introduce effectively computable invariants retaining more of the information contained in the braid monodromy remains an open question.

2. Braid monodromy and stabilized fundamental groups

Let $D_k$ be the branch curve of a covering map $f_k : X \rightarrow \mathbb{CP}^2$ as in Theorem 1.1. Braid monodromy invariants are defined by considering a generic projection $\pi : \mathbb{CP}^2 - \{\text{pt}\} \rightarrow \mathbb{CP}^1$: the pole of the projection lies away from $D_k$, and a generic fiber of $\pi$ intersects $D_k$ in $d = \deg D_k$ distinct points, the only exceptions being fibers through cusps or nodes of $D_k$, or fibers that are tangent to $D_k$ at one of its smooth points ("vertical tangencies"). Moreover we can assume that the special points (cusps, nodes and vertical tangencies) of $D_k$ all lie in different fibers of $\pi$.

By restricting ourselves to an affine subset $\mathbb{C}^2 \subset \mathbb{CP}^2$, choosing a base point and trivializing the fibration $\pi$, we can view the monodromy of $\pi|_{D_k}$ as a group homomorphism from $\pi_1(\mathbb{C} - \{q_i\})$ (where $q_i$ are the images by $\pi$ of the special points of $D_k$) to the braid group $B_d$. More precisely, the monodromy around a vertical tangency is a half-twist (a braid that exchanges two of the $d$ intersection points of the fiber with $D_k$ by rotating them around each other counterclockwise along a certain path); the monodromy around a positive (resp. negative) node is the square (resp. the inverse of the square) of a half-twist; the monodromy around a cusp is the cube of a half-twist [13, 2].

It is sometimes convenient to choose an ordered system of generating loops for $\pi_1(\mathbb{C} - \{q_i\})$ (one loop going around each $q_i$), and to express the monodromy as a braid factorization, i.e. a decomposition of the central braid $\Delta^2$ (the monodromy around the fiber at infinity, due to the non-triviality of the fibration $\pi$ over $\mathbb{CP}^1$) into the product of the monodromies along the chosen generating loops. However, this braid factorization is only well-defined up to simultaneous conjugation of all factors (i.e., a change in the choice of the identification of the fibers with $\mathbb{R}^2$) and Hurwitz equivalence (i.e., a rearrangement of the factors due to a different choice of the system of generating loops).

The braid monodromy determines in a very explicit manner the fundamental groups $\pi_1(\mathbb{C}^2 - D_k)$ and $\pi_1(\mathbb{CP}^2 - D_k)$. Indeed, consider a generic fiber $\ell \simeq \mathbb{C} \subset \mathbb{CP}^2$ of the projection $\pi$ (e.g. the fiber containing the base point), intersecting $D_k$ in $d$ distinct points. The free group $\pi_1(\ell - (\ell \cap D_k)) = F_d$ is generated by a system of $d$ loops going around the various points in $\ell \cap D_k$. The inclusion map $i : \ell - (\ell \cap D_k) \rightarrow \mathbb{C}^2 - D_k$ induces a surjective homomorphism $i_* : F_d \rightarrow \pi_1(\mathbb{C}^2 - D_k)$. 
Definition 2.1. The images of the standard generators of the free group $F_d$ and their conjugates are called geometric generators of $\pi_1(\mathbb{C}^2 - D_k)$; the set of all geometric generators will be denoted by $\Gamma_k$.

By the Zariski-Van Kampen theorem, $\pi_1(\mathbb{C}^2 - D_k)$ is realized as a quotient of $F_d$ by relations corresponding to the various special points (vertical tangencies, nodes, cusps) of $D_k$; these relations express the fact that the action of the braid monodromy on $F_d$ induces a trivial action on $\pi_1(\mathbb{C}^2 - D_k)$. To each factor in the braid factorization one can associate a pair of elements $\gamma_1, \gamma_2 \in \Gamma_k$ (small loops around the two portions of $D_k$ that meet at the special point), well-determined up to simultaneous conjugation. The relation corresponding to a tangency is $\gamma_1 \sim \gamma_2$; for a node (of either orientation) it is $[\gamma_1, \gamma_2] \sim 1$; for a cusp it becomes $\gamma_1 \gamma_2 \gamma_1 \sim \gamma_2 \gamma_1 \gamma_2$. Taking into account all the special points of $D_k$ (i.e. considering the entire braid monodromy), we obtain a presentation of $\pi_1(\mathbb{C}^2 - D_k)$. Moreover, $\pi_1(\mathbb{CP}^2 - D_k)$ is obtained from $\pi_1(\mathbb{C}^2 - D_k)$ just by adding the extra relation $g_1 \ldots g_d \sim 1$, where $g_i$ are the images of the standard generators of $F_d$ under the inclusion.

It follows from this discussion that the creation or cancellation of a pair of nodes in $D_k$ may affect $\pi_1(\mathbb{C}^2 - D_k)$ and $\pi_1(\mathbb{CP}^2 - D_k)$ by adding or removing commutation relations between geometric generators. Although it is reasonable to expect that negative nodes can always be cancelled in the branch curves given by Theorem 1.1, the currently available techniques are insufficient to prove such a statement. Instead, a more promising approach is to compensate for these changes in the fundamental groups by considering certain quotients where one stabilizes the group by adding commutation relations between geometric generators. The resulting group is in some sense more natural than $\pi_1(\mathbb{C}^2 - D_k)$ from the symplectic point of view, and as a side benefit it is often easier to compute (see §7). Moreover, it also turns out that, in many cases, no information is lost in the stabilization process (see §3).

In order to define the stabilized group $G_k$, first observe that, because the branching index of $f_k$ above a smooth point of $D_k$ is always 2, the geometric monodromy representation morphism $\theta_k : \pi_1(\mathbb{CP}^2 - D_k) \to S_n$ describing the topology of the covering above $\mathbb{CP}^2 - D_k$ maps all geometric generators to transpositions in $S_n$. As seen above, to each nodal point of $D_k$ one can associate geometric generators $\gamma_1, \gamma_2 \in \Gamma_k$, one for each of the two intersecting portions of $D_k$, so that the corresponding relation in $\pi_1(\mathbb{C}^2 - D_k)$ is $[\gamma_1, \gamma_2] \sim 1$. Since the branching occurs in disjoint sheets of the cover, the two transpositions $\theta_k(\gamma_1)$ and $\theta_k(\gamma_2)$ are necessarily disjoint (i.e. they are distinct and commute). Therefore, adding or removing pairs of nodes amounts to adding or removing relations given by commutators of geometric generators associated to disjoint transpositions.

Definition 2.2. Let $K_k$ (resp. $\tilde{K}_k$) be the normal subgroup of $\pi_1(\mathbb{C}^2 - D_k)$ (resp. $\pi_1(\mathbb{CP}^2 - D_k)$) generated by all commutators $[\gamma, \gamma']$ where $\gamma, \gamma' \in \Gamma_k$ are such that $\theta_k(\gamma)$ and $\theta_k(\gamma')$ are disjoint transpositions. The stabilized fundamental group is defined as $G_k = \pi_1(\mathbb{C}^2 - D_k)/K_k$, resp. $\tilde{G}_k = \pi_1(\mathbb{CP}^2 - D_k)/\tilde{K}_k$.

Certain natural subgroups of $G_k$ and $\tilde{G}_k$ will play an important role in the following sections. Define the linking number homomorphism $\delta_k : \pi_1(\mathbb{C}^2 - D_k) \to \mathbb{Z}$ by $\delta_k(\gamma) = 1$ for every $\gamma \in \Gamma_k$; similarly one can define $\tilde{\delta}_k : \pi_1(\mathbb{CP}^2 - D_k) \to \mathbb{Z}_d$. When $D_k$ is irreducible (which is the general case), these can also be thought
of as abelianization maps from the fundamental groups to the homology groups $H_1(\mathbb{C}^2 - D_k, \mathbb{Z}) \simeq \mathbb{Z}$ and $H_1(\mathbb{C}P^2 - D_k, \mathbb{Z}) \simeq \mathbb{Z}_d$.

**Lemma 2.3.** $\text{Ker} \delta_k \simeq \text{Ker} \delta_k^*$.

**Proof.** Since $\pi_1(\mathbb{C}P^2 - D_k) = \pi_1(\mathbb{C}^2 - D_k)/(g_1 \ldots g_d)$ and $\delta_k(g_1 \ldots g_d) = d$, it is sufficient to show that the product $g_1 \ldots g_d$ belongs to the center of $\pi_1(\mathbb{C}^2 - D_k)$. Observe that the relation in $\pi_1(\mathbb{C}^2 - D_k)$ coming from a special point of $D_k$ can be rewritten in the form $g \sim b_g g \forall g \in F_d$, where $b \in B_d$ is the braid monodromy around the given special point, acting on $F_d$. In particular, if we consider the braid monodromy as a factorization $\Delta^2 = \prod b_i$, we obtain that $g \sim (\prod b_i)_* g = (\Delta^2)_* g$ for any element $g$. However, the action of the braid $\Delta^2$ on $F_d$ is exactly conjugation by $g_1 \ldots g_d$; we conclude that $g_1 \ldots g_d$ commutes with any element of $\pi_1(\mathbb{C}^2 - D_k)$, hence the result.

The homomorphisms $\delta_k$ and $\delta_k^*$ are obviously surjective. Moreover, $\theta_k$ is also surjective, because of the connectedness of $X$: the subgroup $\text{Im} \theta_k \subseteq S_n$ is generated by transpositions and acts transitively on $\{1, \ldots, n\}$, so it is equal to $S_n$. However, the image of $\theta_k^* = (\theta_k, \delta_k) : \pi_1(\mathbb{C}^2 - D_k) \to S_n \times \mathbb{Z}$ is the index 2 subgroup $\{(\sigma, i) : \text{sgn}(\sigma) \equiv i \text{ mod } 2\}$, and similarly for $\theta_k^* = (\theta_k, \delta_k) : \pi_1(\mathbb{C}P^2 - D_k) \to S_n \times \mathbb{Z}_d$ (note that $d$ is always even). Since $K_k \subseteq \text{Ker} \theta_k^*$, we can make the following definition:

**Definition 2.4.** Let $H^0_k = \text{Ker} \theta_k^* \simeq \text{Ker} \theta_k^*$. The reduced subgroup of $G_k$ is $G^0_k = H^0_k/K_k$. We have the following exact sequences:

$$1 \to G^0_k \to G_k \to S_n \times \mathbb{Z} \to \mathbb{Z}_2 \to 1,$$

$$1 \to G^0_k \to \tilde{G}_k \to S_n \times \mathbb{Z}_d \to \mathbb{Z}_2 \to 1.$$

Theorem 1.2 is now obvious from the definitions and from Theorem 1.1: since creating a pair of nodes amounts to adding a relation of the form $[\gamma_1, \gamma_2] \sim 1$ where $[\gamma_1, \gamma_2] \in K_k$ (resp. $\tilde{K}_k$), by construction it does not affect the groups $G_k, \tilde{G}_k$ and $G^0_k, \tilde{G}^0_k$, which are therefore symplectic invariants for $k$ large enough.

### 3. $\tilde{B}_n$-Groups and Their Stabilizations

Denote by $B_n$ (resp. $P_n, P_{n,0}$) the braid group on $n$ strings (resp. the subgroups of pure braids and pure braids of degree 0), and denote by $X_1, \ldots, X_{n-1}$ the standard generators of $B_n$. Recall that $X_i$ is a half-twist along a segment joining the points $i$ and $i+1$, and that the relations among these generators are $[X_i, X_j] = 1$ if $|i-j| \geq 2$ and $X_iX_{i+1}X_i = X_{i+1}X_iX_{i+1}$.

Let $\tilde{B}_n$ be the quotient of $B_n$ by the commutator of half-twists along two paths intersecting transversely in one point: $\tilde{B}_n = B_n/[X_2, X_3^{-1}X_1^{-1}X_2X_1X_3]$. The maps $\sigma : B_n \to S_n$ (induced permutation) and $\delta : B_n \to \mathbb{Z}$ (degree) factor through $\tilde{B}_n$, so one can define the subgroups $\tilde{P}_n = \text{Ker} \sigma$ and $\tilde{P}_{n,0} = \text{Ker} (\sigma, \delta)$. The structure of $\tilde{B}_n$ and its subgroups is described in detail in §1 of [9]; unlike $P_n$ and $P_{n,0}$ which are quite complicated, these groups are fairly easy to understand: $\tilde{P}_{n,0}$ is solvable, its commutator subgroup is $[\tilde{P}_{n,0}, \tilde{P}_{n,0}] \simeq \mathbb{Z}_2$ and its abelianization is $\text{Ab}(\tilde{P}_{n,0}) \simeq \mathbb{Z}^{n-1}$ (it can in fact be identified naturally with the reduced regular representation $R_n$ of $S_n$). More precisely, we have:
Lemma 3.1 (Moishezon). Let $x_i$ be the image of $X_i$ in $\hat{B}_n$, and define $s_1 = x_1^2$, $\eta = [x_1^i, x_2^i]$, $u_i = [x_1^{i-1}, x_2^{i+1}]$ for $1 \leq i \leq n - 2$, and $u_{n-1} = [x_1^{n-2}, x_n]$. Then $P_{n,0}$ is generated by $u_1, \ldots, u_{n-1}$, and $P_n$ is generated by $s_1, u_1, \ldots, u_{n-1}$.

The relations among these elements are $[u_i, u_j] = 1$ if $|i - j| \geq 2$, $[u_i, u_{i+1}] = \eta$, $[s_1, u_i] = 1$ if $i \neq 2$, and $[s_1, u_2] = \eta$. The element $\eta$ is central in $\hat{B}_n$, has order 2 ($i.e.$, $\eta^2 = 1$), and generates the commutator subgroups $[\hat{P}_{n,0}, \hat{P}_{n,0}] = [\hat{P}_n, \hat{P}_n] \simeq \mathbb{Z}_2$ (in particular, for any two adjacent half-twists $x$ and $y$ we have $[x^2, y^2] = \eta$). As a consequence, $\text{Ab}(\hat{P}_n) \simeq \mathbb{Z}^n$ and $\text{Ab}(\hat{P}_{n,0}) \simeq \mathbb{Z}^{n-1}$.

Moreover, the action of $\hat{B}_n$ on $\hat{P}_n$ by conjugation is given by the following formulas: $x_i^{-1}s_1x_i = s_1$ if $i \neq 2$, $x_2^{-1}s_1x_2 = s_1u_2^{-1}$; $x_i^{-1}u_jx_i = u_j$ if $|i - j| \geq 2$, $x_i^{-1}u_jx_i = u_1u_j$ if $|i - j| = 1$, and $x_i^{-1}u_ix_i = u_i^{-1}\eta$.

Proof. Most of the statement is a mere reformulation of Definition 8 and Theorem 1 in §1.5 of [9]. The only difference is that we define $u_i$ directly in terms of the generators of $\hat{B}_n$, while Moishezon defines $u_1 = (x_2x_1^2x_2^{-1})x_2^{-2} = x_1^{-1}x_2^{-2}x_1x_2^{-2}$ and constructs the other $u_i$ by conjugation. In fact, $u_i = x_i^2y_i^2$ whenever $x$ and $y$ are two adjacent half-twists having respectively $i$ and $i + 1$ among their end points and such that $xy^{-1} = x_i$; our definition of $u_i$ corresponds to the choice $x = x_i^{-1}x_{i+1}^{-1}x_i$ and $y = x_{i+1}$ for $i \leq n - 2$, and $x = x_{n-1}x_n^{-2}x_n^{-1}$ for $i = n - 1$. Also note that Moishezon’s formula for $x_2^{-3}s_1x_2$ is inconsistent, due to a mistake in equation (1.25) of [9]; the formula we give is corrected.

Intuitively speaking, the reason why $\hat{B}_n$ is a fairly small group is that, due to the extra commutation relations, very little is remembered about the path supporting a given half-twist, namely just its two endpoints and the total number of times that it circles around the $n - 2$ other points. This can be readily checked on simple examples (e.g., half-twists exchanging the first two points along a path that encircles only one of the $n - 2$ other points: since these differ by conjugation by half-twists along paths presenting a single transverse intersection, they represent the same element in $\hat{B}_n$). More generally, we have the following fact:

Lemma 3.2. The elements of $\hat{B}_n$ corresponding to half-twists exchanging the first two points are exactly those of the form $x_1u_1^k\eta^{k^{-1}/2}$ for some integer $k$.

Proof. Any half-twist exchanging the first two points can be put in the form $\gamma x_1^{-1}\gamma^{-1}$, where $\gamma \in \hat{P}_n$ can be expressed as $\gamma = s_1^\alpha u_1^\beta \cdots u_{n-1}^\beta \eta^\gamma$. Using Lemma 3.1, we have $x_1^{-1}\gamma x_1 = s_1^\alpha (u_1^{-1}\eta)^\beta (u_1u_2)^\beta_2 u_3^\beta_3 \cdots u_{n-1}^\beta_\gamma \eta^\gamma$. Since $(u_1u_2)^\beta_2 = \eta^{2(\beta_2 - 1)/2}u_1^\beta_2 u_2^\beta_2$, we can rewrite this equality as $x_1^{-1}\gamma x_1 = u_1^{-2\beta_1} \eta^{\beta_3 u_1^\beta_4 \eta^{(\beta_2 - 1)/2}} \gamma = u_1^k \eta^{k^{-1}/2}$, where $k = \beta_2 - 2\beta_1$. Multiplying by $x_1$ on the left and $\gamma^{-1}$ on the right we obtain $\gamma x_1^{-1} = x_1 u_1^k \eta^{k^{-1}/2}$.

Lemma 3.3. Let $x, y \in \hat{B}_n$ be elements corresponding to half-twists along paths with mutually disjoint endpoints. Then $[x, y] = 1$.

Proof. The result is trivial when the paths corresponding to $x$ and $y$ are disjoint or intersect only once. In general, after conjugation we can assume that $x = \gamma x_1\gamma^{-1}$ for some $\gamma \in \hat{P}_n$, and $y = x_3$. By Lemma 3.2, $x = x_1 u_1^\gamma \eta^{k^{-1}/2}$ for some integer $k$. Since $x_1$, $u_1$ and $\eta$ all commute with $x_3$, we conclude that $[x, y] = 1$ as desired.

Lemma 3.4. Let $x, y \in \hat{B}_n$ be elements corresponding to half-twists along paths with one common endpoint. Then $xy = yx$. 

Proof. After conjugation we can assume that $x = x_1$ and $y = \gamma x_2 \gamma^{-1}$ for some $\gamma \in P_n$. By the classification of half-twists in $\hat{B}_n$ (Lemma 3.2), there exists an integer $k$ such that $y = s_1^k x_1^2 s_2^k = x_2(s_1 u_2^{-1})^{-k} s_1^k = s_1^{-k} x_i^2 s_1^k$. Therefore $x y x = x_1 s_1^{-k} x_2 s_1^k x_1 = s_1^{-k} (x_1 x_2 x_1) s_1^k = s_1^{-k} (x_2 x_1 x_2) s_1^k = y x y$.

It must be noted that Lemmas 3.3 and 3.4 have also been obtained by Robb [12].

Lemma 3.5. The group $\hat{B}_n$ admits automorphisms $\epsilon_i$ such that $\epsilon_i(x_i) = x_i u_i$ and $\epsilon_i(x_j) = x_j$ for every $j \neq i$. Moreover, $\epsilon_i(u_i) = u_i \eta$ and $\epsilon_i(u_j) = u_j$ for $j \neq i$.

Proof. By Lemmas 3.3 and 3.4, the half-twists $x_1, \ldots, x_{i-1}, (x_i u_i), x_{i+1}, \ldots, x_{n-1}$ satisfy exactly the same relations as the standard generators of $\hat{B}_n$. So $\epsilon_i$ is a well-defined group homomorphism from $\hat{B}_n$ to itself, and it is injective. The formulas for $\epsilon_i(u_i)$ and $\epsilon_i(u_j)$ are easily checked. The surjectivity of $\epsilon_i$ follows from the identity $\epsilon_i(x_i u_i^{-1} \eta) = x_i$.

The following definition is motivated by the very particular structure of the fundamental groups of branch curve complements computed by Moishezon for generic projections of $\mathbb{CP}^1 \times \mathbb{CP}^1$ and $\mathbb{CP}^2$ [9, 10], which seems to be a feature common to a much larger class of examples (see §4):

Definition 3.6. Define $\hat{B}_n^{(2)} = \{(x, y) \in \hat{B}_n \times \hat{B}_n, \sigma(x) = \sigma(y) \text{ and } \delta(x) = \delta(y)\}$. We say that the group $\pi_1(\mathbb{C}^2 - D_k)$ satisfies property ($\ast$) if there exists an isomorphism $\psi$ from $\pi_1(\mathbb{C}^2 - D_k)$ to a quotient of $\hat{B}_n^{(2)}$ such that, for any geometric generator $\gamma \in \Gamma_k$, there exist two half-twists $x, y \in \hat{B}_n$ such that $\sigma(x) = \sigma(y) = \theta_k(\gamma)$ and $\psi(\gamma) = (x, y)$.

In other words, $\pi_1(\mathbb{C}^2 - D_k)$ satisfies property ($\ast$) if there exists a surjective homomorphism from $\hat{B}_n^{(2)}$ to $\pi_1(\mathbb{C}^2 - D_k)$ which maps pairs of half-twists to geometric generators, in a manner compatible with the $S_n$-valued homomorphisms $\sigma$ and $\theta_k$.

Remark 3.7. If $\pi_1(\mathbb{C}^2 - D_k)$ satisfies property ($\ast$), then the kernel of the homomorphism $\theta_k : \pi_1(\mathbb{C}^2 - D_k) \rightarrow S_n \times \mathbb{Z}$ is a quotient of $\hat{P}_n \times \hat{P}_n$ and therefore a solvable group; in particular its commutator subgroup is a quotient of $(\mathbb{Z}_2)^2$, and its abelianization is a quotient of $(\mathbb{Z}^2 \otimes R_n) \simeq (\mathbb{Z} \oplus \mathbb{Z})^n - 1$.

As an immediate consequence of Definition 3.6 and Lemma 3.3, we have:

Proposition 3.8. If $\pi_1(\mathbb{C}^2 - D_k)$ satisfies property ($\ast$), then the stabilization operation is trivial, i.e. $K_k = \{1\}$, $G_k = \pi_1(\mathbb{C}^2 - D_k)$, and $G_0^0 = \text{Ker} \theta_k^+$.

Proof. Let $\gamma, \gamma' \in \Gamma_k$ be such that $\theta_k(\gamma)$ and $\theta_k(\gamma')$ are disjoint transpositions. Consider the isomorphism $\psi$ given by Definition 3.6: there exist half-twists $x, x', y, y' \in \hat{B}_n$ such that $\psi(\gamma) = (x, y)$ and $\psi(\gamma') = (x', y')$. Since $\theta_k(\gamma) = \sigma(x) = \sigma(y)$ and $\theta_k(\gamma') = \sigma(x') = \sigma(y')$ are disjoint transpositions, $x$ and $x'$ have disjoint endpoints, and similarly for $y$ and $y'$. Therefore, by Lemma 3.3 we have $[x, x'] = 1$ and $[y, y'] = 1$, so that $\psi(\gamma), \psi(\gamma') = 1$, and therefore $[\gamma, \gamma'] = 1$. We conclude that $K_k = \{1\}$, which ends the proof.

Let $D_{p,q}$ be the branch curve of a generic polynomial map $\mathbb{CP}^1 \times \mathbb{CP}^1 \rightarrow \mathbb{CP}^2$ of bidegree $(p, q)$, $p, q \geq 2$. As will be shown in §4, it follows from the computations in [9] that $\pi_1(\mathbb{C}^2 - D_{p,q})$ satisfies property ($\ast$). This property also holds for the complement of the branch curve of a generic polynomial map from $\mathbb{CP}^2$ to itself in
degree $\geq 3$, as follows from the calculations in [10] (see also [15]), and in various other examples as well (see §4). It is an interesting question to determine whether this remarkable structure of branch curve complements extends to generic high-degree projections of arbitrary algebraic surfaces; this would tie in nicely with a conjecture of Teicher about the virtual solvability of these fundamental groups [14], and would also imply Conjecture 1.3.

4. Examples

As follows from pp. 696–700 of [5], if the symplectic manifold $X$ happens to be Kähler, then all approximately holomorphic constructions can actually be carried out using genuine holomorphic sections of $L^\otimes k$ over $X$, and as a consequence the $\mathbb{CP}^2$-valued maps given by Theorem 1.1 coincide up to isotopy with projective maps defined by generic holomorphic sections of $L^\otimes k$; therefore, in the case of complex projective surfaces all calculations can legitimately be performed within the framework of complex algebraic geometry.

The fundamental groups of complements of branch curves have already been computed for generic projections of various complex projective surfaces. In many cases, these computations only hold for specific linear systems, and do not apply to the high degree situation that we wish to consider.

Nevertheless, it is worth mentioning that, if $D \subset \mathbb{CP}^2$ is the branch curve of a generic linear projection of a hypersurface of degree $n$ in $\mathbb{CP}^3$, then it has been shown by Moishezon that $\pi_1(\mathbb{C}^2 - D) \simeq B_n$ [7]. In fact, in this specific case there is a well-defined geometric monodromy representation morphism $\theta_B$ with values in the braid group $B_n$, rather than in the symmetric group $S_n$ as usual, because the $n$ preimages of any point in $\mathbb{CP}^2 - D$ lie in a fiber of the projection $\mathbb{CP}^3 - \{\text{pt}\} \to \mathbb{CP}^2$, which after trivialization over an affine subset can be identified with $\mathbb{C}$. Moishezon’s computations then show that $\theta_B : \pi_1(\mathbb{C}^2 - D) \to B_n$ is an isomorphism. An attempt to quotient out $B_n$ by commutators as in the definition of stabilized fundamental groups yields $\tilde{B}_n$; in this case the stabilization operation is non-trivial. However this situation is specific to the linear system $O(1)$, and one expects the fundamental groups of branch curve complements to behave differently when one instead considers projections given by sections of $O(k)$ for $k \gg 0$.

Moishezon’s result about hypersurfaces in $\mathbb{CP}^3$ has been extended by Robb to the case of complete intersections (still considering only linear projections to $\mathbb{CP}^2$ rather than arbitrary linear systems) [12]. The result is that, if $D$ is the branch curve for a complete intersection of degree $n$ in $\mathbb{CP}^m$ ($m \geq 4$), then the group $\pi_1(\mathbb{C}^2 - D)$ is isomorphic to $\tilde{B}_n$. It is worth noting that, in this example, the stabilization operation is trivial. In fact, the groups $\pi_1(\mathbb{C}^2 - D)$ can be shown to have property $(\ast)$ (observe that $\tilde{B}_n$ is the quotient of $\tilde{B}_n^{(2)}$ by its subgroup $1 \times \tilde{P}_{n,0}$).

Conjecture 1.6 holds for $k = 1$ in these two families of examples: we have $\text{Ab} G^0 \simeq \mathbb{Z}^{n-1}$ and $[G^0, G^0] \simeq \mathbb{Z}_2$ in both cases, while $\mathbb{Z}^2/\Lambda_1 \simeq \mathbb{Z}$ because the canonical class is proportional to the hyperplane class which is primitive.

More interestingly for our purposes, the calculations have also been carried out in the case of arbitrarily positive linear systems by Moishezon for two fundamental examples: $\mathbb{CP}^1 \times \mathbb{CP}^1$ [9], and $\mathbb{CP}^2$ [10] (unpublished, see also [15] for a summary).

**Theorem 4.1** (Moishezon). Let $D_{p,q}$ be the branch curve of a generic polynomial map $\mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^2$ of bidegree $(p,q)$, $p,q \geq 2$. Then the group $\pi_1(\mathbb{C}^2 - D_{p,q})$ satisfies property $(\ast)$, and its subgroup $H^{p,q}_0 = \text{Ker} \theta^+_{p,q}$ has the following structure:
Ab $H_{p,q}^0$ is isomorphic to $(\mathbb{Z}_2 \oplus \mathbb{Z}_{p-q})^{n-1}$ if $p$ and $q$ are even, and $(\mathbb{Z}_{2(p-q)})^{n-1}$ if $p$ or $q$ is odd (here $n = 2pq$); the commutator subgroup $[H_{p,q}^0, H_{p,q}^0]$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ when $p$ and $q$ are even, and $\mathbb{Z}_2$ if $p$ or $q$ is odd.

In fact, Moishezon identifies $\pi_1(C^2 - D_{p,q})$ with a quotient of the semi-direct product $\tilde{B}_n \times \tilde{P}_{n,0}$, where $\tilde{B}_n$ acts from the right on $\tilde{P}_{n,0}$ by conjugation [9]. However it is easy to observe that the map $\kappa : \tilde{B}_n \times \tilde{P}_{n,0} \to \tilde{B}_n^{(2)}$ defined by $\kappa(x, u) = (x, xu)$ is a group isomorphism (recall the group structure on $\tilde{B}_n \times \tilde{P}_{n,0}$ is given by $(x, u)(x', u') = (xx', xx'u'u')$. The factor $\tilde{P}_{n,0}$ of the semi-direct product corresponds to the normal subgroup $1 \times \tilde{P}_{n,0}$ of $\tilde{B}_n^{(2)}$, while the factor $\tilde{B}_n$ corresponds to the diagonally embedded subgroup $\tilde{B}_n = \{(x, x)\} \subset \tilde{B}_n^{(2)}$.

Moreover, by carefully going over the various formulas identifying a set of geometric generators for $\pi_1(C^2 - D_{p,q})$ with certain specific elements in $\tilde{B}_n \times \tilde{P}_{n,0}$ (Propositions 8 and 10 of [9]; cf. also §1.4, Definition 24 and Remarks 28–29 of [9]), or equivalently in $\tilde{B}_n^{(2)}$ after applying the isomorphism $\kappa$, it is relatively easy to check that each geometric generator corresponds to a pair of half-twists with the expected end points in $\tilde{B}_n^{(2)}$ (see also §6 for more details). Therefore, property (*) and Conjecture 1.3 hold for these groups.

Conjecture 1.6 also holds for $\mathbb{CP}^1 \times \mathbb{CP}^1$. Indeed, $H_2(\mathbb{CP}^1 \times \mathbb{CP}^1, \mathbb{Z})$ is generated by classes $\alpha$ and $\beta$ corresponding to the two factors; the hyperplane section class is $L = p\alpha + q\beta$, while the ramification curve is $R = 3L + K = (3p - 2)\alpha + (3q - 2)\beta$. Therefore, the subgroup $\Lambda_{p,q}$ of $\mathbb{Z}^2$ is generated by $(\alpha \cdot L, \alpha \cdot R) = (q, 3q - 2)$ and $(\beta \cdot L, \beta \cdot R) = (p, 3p - 2)$. An easy computation shows that the quotient $\mathbb{Z}^2/\Lambda_{p,q} = \mathbb{Z}^2/\langle \langle q, 3q - 2 \rangle, \langle p, 3p - 2 \rangle \rangle \simeq \mathbb{Z}^2/\langle \langle q, 2 \rangle, \langle p, 2 \rangle \rangle$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_{p-q}$ when $p$ and $q$ are even, and to $\mathbb{Z}_{2(p-q)}$ otherwise.

It is worth noting that this nice description for $p, q \geq 2$ completely breaks down in the insuffiently ample case $p = 1$, where it follows from computations of Zariski [17] that $\pi_1(C^2 - D_{1,q}) \simeq B_{2q}$. So both Conjecture 1.3 and Conjecture 1.6 require a sufficient amount of ampleness in order to hold ($p, q \geq 2$).

**Theorem 4.2** (Moishezon). Let $D_k$ be the branch curve of a generic polynomial map $\mathbb{CP}^2 \to \mathbb{CP}^2$ of degree $k \geq 3$. Then the group $\pi_1(C^2 - D_k)$ satisfies property (*), and its subgroup $H_k^0 = \text{Ker} \theta_k^+ \subset H_k^0$ has the following structure: Ab $H_k^0$ is isomorphic to $(\mathbb{Z} \oplus \mathbb{Z}_3)^{n-1}$ if $k$ is a multiple of 3, and to $\mathbb{Z}^{n-1}$ otherwise (here $n = k^2$); the commutator subgroup $[H_k^0, H_k^0]$ is trivial for $k$ even and isomorphic to $\mathbb{Z}_2$ for $k$ odd.

In this case too, Moishezon in fact identifies $\pi_1(C^2 - D_k)$ with a quotient of $\tilde{B}_n \times \tilde{P}_{n,0}$ [10] (see also [15]). Property (*) and Conjecture 1.3 hold for $\mathbb{CP}^2$ when $k \geq 3$, but for $k = 2$ the group $\pi_1(C^2 - D_2)$ is much larger.

Since $H_2(\mathbb{CP}^2, \mathbb{Z})$ is generated by the class of a line, $\Lambda_k$ is the subgroup of $\mathbb{Z}^2$ generated by $(k, 3k - 3)$, and $\mathbb{Z}^2/\Lambda_k$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_3$ when $k$ is a multiple of 3 and to $\mathbb{Z}$ otherwise. Therefore Conjecture 1.6 holds for $\mathbb{CP}^2$ when $k \geq 3$.

Results for certain projections of Del Pezzo and K3 surfaces have also been announced by Robb in [12].

**Theorem 4.3** (Robb). Let $X$ be either a cubic hypersurface in $\mathbb{CP}^3$ or a $(2, 2)$ complete intersection in $\mathbb{CP}^4$, and let $D_k$ be the branch curve of a generic algebraic map $X \to \mathbb{CP}^2$ given by sections of $O(kH)$, where $H$ is the hyperplane section and $k \geq 2$. Then the subgroup $H_k^0 = \text{Ker} \theta_k^+$ of $\pi_1(C^2 - D_k)$ has abelianization $\text{Ab} H_k^0 \simeq \mathbb{Z}^{n-1}$. 
Theorem 4.4 (Robb). Let $X$ be a K3 surface realized either as a degree 4 hypersurface in $\mathbb{CP}^3$, a $(3,2)$ complete intersection in $\mathbb{CP}^4$ or a $(2,2,2)$ complete intersection in $\mathbb{CP}^5$, and let $D_k$ be the branch curve of a generic algebraic map $X \to \mathbb{CP}^2$ given by sections of $O(kH)$, where $H$ is the hyperplane section and $k \geq 2$. Then the subgroup $H^0_k = \text{Ker } \theta^+_k$ of $\pi_1(\mathbb{C}^2 - D_k)$ has abelianization $\text{Ab } H^0_k \simeq (\mathbb{Z} \oplus \mathbb{Z}_k)^{n-1}$.

Although to our knowledge no detailed proofs of Theorems 4.3 and 4.4 have appeared yet, it appears very likely from the sketch of argument given in [12] that property $(\ast)$ and Conjecture 1.3 will hold for these examples as well. In any case we can compare Robb’s results with the answers predicted by Conjecture 1.6.

In the case of the Del Pezzo surfaces, the hyperplane class $H$ is primitive, and $K = -H$ (so $R_k = (3k - 1)H$), so that the subgroup $\Lambda_k \subset \mathbb{Z}^2$ is generated by $(k, 3k - 1)$, and $\mathbb{Z}^2/\Lambda_k \simeq \mathbb{Z}$, which is in agreement with Theorem 4.3. In the case of the K3 surfaces, the hyperplane class $H$ is again primitive, but $K = 0$ and $R_k = 3kH$, so that $\Lambda_k$ is now generated by $(k, 3k)$, and $\mathbb{Z}^2/\Lambda_k \simeq \mathbb{Z} \oplus \mathbb{Z}_k$, in agreement with Theorem 4.4.

The following result for the Hirzebruch surface $F_1 = \mathbb{P}(O_{\mathbb{CP}^1} \oplus O_{\mathbb{CP}^1}(1))$ is new to our knowledge; however partial results about this surface have been obtained by Moishezon, Robb and Teicher [11, 16], and an ongoing project of Teicher and coworkers is expected to yield another proof of the same result.

Theorem 4.5. Let $D_{p,q}$ be the branch curve of a generic algebraic map $F_1 \to \mathbb{CP}^2$ given by three sections of the linear system $O(pF + qE)$, where $F$ is the class of a fiber, $E$ is the exceptional section, and $p > q \geq 2$. Then the group $\pi_1(\mathbb{C}^2 - D_{p,q})$ satisfies property $(\ast)$, and its subgroup $H^0_{p,q} = \text{Ker } \theta^+_{p,q}$ has the following structure: $\text{Ab } H^0_{p,q} \simeq (\mathbb{Z}_{3q-2p})^{n-1}$, where $n = (2p - q)q$, and the commutator subgroup $[H^0_{p,q}, H^0_{p,q}]$ is isomorphic to $\mathbb{Z}_2$ if $p$ is odd and $q$ even, and trivial in all other cases.

The proof relies on the observation that $F_1$ is the blow-up of $\mathbb{CP}^2$ at one point. Recalling the interpretation of a symplectic (or Kähler) blow-up as the collapsing of an embedded ball, it is easy to check that $F_1$ can be degenerated to a union of planes in a manner similar to $\mathbb{CP}^2$, only with some components missing; most of the calculations performed by Moishezon in [10] for $\mathbb{CP}^2$ can then be re-used in this context, with the only changes occurring along the exceptional curve $E$. More details are given in §6.2.

As a consequence of property $(\ast)$, Conjecture 1.3 holds for this example. So does Conjecture 1.6: indeed, $H_2(F_1, \mathbb{Z})$ is generated by $F$ and $E$. Recalling that $F \cdot F = 0$, $F \cdot E = 1$, $E \cdot E = -1$, and letting $L_{p,q}$ denotes $pF + qE$ and $R_{p,q} = 3L_{p,q} + K = (3p - 3)F + (3q - 2)E$, we obtain that $\Lambda_{p,q} \subset \mathbb{Z}^2$ is generated by $(F \cdot L_{p,q}, F \cdot R_{p,q}) = (q, 3q - 2)$ and $(E \cdot L_{p,q}, E \cdot R_{p,q}) = (p - q, 3p - 3q - 1)$. Therefore $\mathbb{Z}^2/\Lambda_k \simeq \mathbb{Z}^2/(q, 3q - 2), (p - q, 3p - 3q - 1) \simeq \mathbb{Z}_{3q-2p}$.

A much wider class of examples, including an infinite family of surfaces of general type, can be investigated if one brings approximately holomorphic techniques into the picture, although this makes it only possible to obtain results about the stabilized fundamental groups of branch curve complements (cf. §2) rather than the actual fundamental groups.

Theorem 4.6. For given integers $a, b \geq 1$ and $p, q \geq 2$, let $X_{a,b}$ be the double cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$ branched along a smooth algebraic curve of degree $(2a, 2b)$, and let $L_{p,q}$ be the linear system over $X_{a,b}$ defined as the pullback of $O_{\mathbb{CP}^1 \times \mathbb{CP}^1}(p,q)$ via the double cover. Let $D_{p,q}$ be the branch curve of a generic approximately holomorphic
perturbation of an algebraic map $X_{a,b} \to \mathbb{CP}^2$ given by three sections of $L_{p,q}$. Then the stabilized fundamental group $G^0_{p,q}(X_{a,b}) = \pi_1(\mathbb{CP}^2 - D_{p,q})/K_{p,q}$ satisfies property (\star), and its reduced subgroup $G^0_{p,q}(X_{a,b}) = \text{Ker} \theta^+_p/K_{p,q}$ has the following structure: $\text{Ab} G^0_{p,q}(X_{a,b}) \simeq (\mathbb{Z}^2/((p,a - 2),(q,b - 2)))^{n-1}$, where $n = 4pq$, and the commutator subgroup $[G^0_{p,q}(X_{a,b}), G^0_{p,q}(X_{a,b})]$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ if $a, b, p, q$ are all even, trivial if $a$ or $b$ is odd and $a + p$ or $b + q$ is odd, and isomorphic to $\mathbb{Z}_2$ in all other cases.

More precisely, the setup that we consider starts with a holomorphic map from $X_{a,b}$ to $\mathbb{CP}^2$ that factors through the double cover $X_{a,b} \to \mathbb{CP}^1 \times \mathbb{CP}^1$. Such a map is of course not generic in any sense; however there is a natural explicit way to perturb it in the approximately holomorphic category (see §7), giving rise to the branch curves $D_{p,q}$ that we consider. The map can also be perturbed in the holomorphic category, which at least for $p$ and $q$ large enough yields a branch curve that is equivalent to $D_{p,q}$ up to creations and cancellations of pairs of nodes. So, on the level of stabilized groups, our result does give an answer that is relevant from both the symplectic and algebraic points of view. Moreover, it is expected that, at least for $p$ and $q$ large enough, the fundamental groups themselves (rather than their stabilized quotients) should satisfy property (\star).

Theorem 4.6 implies that Conjecture 1.6 holds for the manifolds $X_{a,b}$. Indeed, $X_{a,b}$ can also be described topologically as follows: in $\mathbb{CP}^1 \times \mathbb{CP}^1$ consider $2a$ curves of the form $\mathbb{CP}^1 \times \{pt\}$ and $2b$ curves of the form $\{pt\} \times \mathbb{CP}^1$, and blow up their $4ab$ intersection points to obtain a manifold $Y_{a,b}$ containing disjoint rational curves $C_1, \ldots, C_{2a}$ (of square $-2b$) and $C'_1, \ldots, C'_{2b}$ (of square $-2a$). Then $X_{a,b}$ is the double cover of $Y_{a,b}$ branched along $C_1 \cup \cdots \cup C_{2a} \cup C'_1 \cup \cdots \cup C'_{2b}$. Now, consider the preimages $\overline{C}_i = \pi^{-1}(C_i)$ and $\overline{C}'_i = \pi^{-1}(C'_i)$, and let $L_{p,q} = p\pi^*\alpha + q\pi^*\beta$ and $R_{p,q} = 3L_{p,q} + K_{X_{a,b}} = (3p + a - 2)\pi^*\alpha + (3q + b - 2)\pi^*\beta$, where $\alpha$ and $\beta$ are the homology generators corresponding to the two factors of $\mathbb{CP}^1 \times \mathbb{CP}^1$. We have $(\overline{C}_i \cdot L_{p,q}, \overline{C}_i \cdot R_{p,q}) = (q, 3q + b - 2)$ and $(\overline{C}'_i \cdot L_{p,q}, \overline{C}'_i \cdot R_{p,q}) = (p, 3p + a - 2)$. It is easily shown that these two elements of $\mathbb{Z}^2$ generate the subgroup $\Lambda_{p,q}$; therefore $\mathbb{Z}^2/\Lambda_{p,q} = \mathbb{Z}^2/((q, 3q + b - 2), (p, 3p + a - 2), (q, b - 2))$.

The techniques involved in the proof of Theorem 4.6, which will be discussed in §7, extend to double covers of other examples for which the answer is known, possibly including iterated double covers of $\mathbb{CP}^1 \times \mathbb{CP}^1$. One example of particular interest is that of double covers of Hirzebruch surfaces branched along disconnected curves, for which we make the following conjecture:

**Conjecture 4.7.** Given integers $m, a \geq 1$, let $X_{2m,a}$ be the double cover of the Hirzebruch surface $\mathbb{F}_2$ branched along the union of the exceptional section $\Delta_\infty$ and a smooth algebraic curve in the homology class $(2a - 1)[\Delta_0]$ (where $\Delta_0$ is the zero section, of square $2m$). Given integers $p, q \geq 2$ such that $p > 2mq$, let $L_{p,q}$ be the linear system over $X_{2m,a}$ defined as the pullback of $O_{\mathbb{F}_2}(pF + q\Delta_\infty)$ via the double cover. Let $D_{p,q}$ be the branch curve of a generic approximately holomorphic perturbation of an algebraic map $X_{2m,a} \to \mathbb{CP}^2$ given by three sections of $L_{p,q}$. Then the reduced stabilized fundamental group $G^0_{p,q}(X_{2m,a}) = \text{Ker} \theta^+_p/K_{p,q}$ has abelianization $\text{Ab} G^0_{p,q}(X_{2m,a}) \simeq (\mathbb{Z}^2/((p - 2mq, m - 2), (2q, 2a - 4)))^{n-1}$.
5. Stabilized fundamental groups and homological data

Consider a compact symplectic 4-manifold $X$ such that $H_1(X, \mathbb{Z}) = 0$ and a branched covering map $f_k : X \to \mathbb{C}P^2$ determined by three sections of $L^\otimes k$, with branch curve $D_k \subset \mathbb{C}P^2$ and geometric monodromy representation morphism $\theta_k : \pi_1(\mathbb{C}^2 - D_k) \to S_n$. The purpose of this section is to construct a natural morphism $\psi_k : \text{Ker } \theta_k \to (\mathbb{Z}^2/\Lambda_k) \otimes \mathbb{R}_n \simeq (\mathbb{Z}^2/\Lambda_k)^n$ (where $\mathbb{R}_n \simeq \mathbb{Z}^n$ is the regular representation of $S_n$) and use its properties to prove Theorem 1.5.

Fix a base point $p_0$ in $\mathbb{C}^2 - D_k$, and let $p_1, \ldots, p_n$ be its preimages by $f_k$. Let $\gamma \in \pi_1(\mathbb{C}^2 - D_k)$ be a loop in the complement of $D_k$ such that $\theta_k(\gamma) = \text{Id}$. Since the monodromy of the branched cover $f_k$ along $\gamma$ is trivial, $f_k^{-1}(\gamma)$ is the union of $n$ disjoint closed loops in $X$. Denote by $\gamma_i$ the lift of $\gamma$ that starts at the point $p_i$. Since $H_1(X, \mathbb{Z}) = 0$, there exists a surface (or rather a 2-chain) $S_i \subset X$ such that $\partial S_i = \gamma_i$. Since $\gamma \subset \mathbb{C}^2 - D_k$, the loop $\gamma_i$ intersects neither the ramification curve $R_k$ nor the preimage $L_k$ of the line at infinity in $\mathbb{C}P^2$. Therefore, there exist well-defined intersection numbers $\lambda_i = S_i \cdot L_k$ and $\rho_i = S_i \cdot R_k \in \mathbb{Z}$. However, there are various possible choices for the surface $S_i$, and the relative cycle $[S_i]$ is only well-defined up to an element of $H_2(X, \mathbb{Z})$. Therefore, the pair $(\lambda_i, \rho_i) \in \mathbb{Z}^2$ is only defined up to an element of the subgroup $\Lambda_k$.

**Definition 5.1.** With the above notations, we denote by $\psi_k : \text{Ker } \theta_k \to (\mathbb{Z}^2/\Lambda_k)^n$ the morphism defined by $\psi_k(\gamma) = ((S_i \cdot L_k, S_i \cdot R_k))_{1 \leq i \leq n}$.

In fact, there is no canonical ordering of the preimages of $p_0$, and $\psi_k$ more naturally takes values in $(\mathbb{Z}^2/\Lambda_k) \otimes \mathbb{R}_n$, as evidenced by Lemma 5.2 below.

Definition 5.1 can naturally be extended to the case $H_1(X, \mathbb{Z}) \neq 0$ by instead considering the morphism $\hat{\psi}_k : \text{Ker } \theta_k \to H_1(X - L_k - R_k, \mathbb{Z})$ which maps a loop $\gamma$ to the homology classes of its lifts $\gamma_i$ in $X - L_k - R_k$. However, the properties to be expected of this morphism in general are not entirely clear, due to the lack of available non-simply connected examples (even though the techniques in §6–7 could probably be applied to the 4-manifold $\Sigma \times \mathbb{C}P^1$ for any Riemann surface $\Sigma$).

We now investigate the various properties of $\psi_k$.

**Lemma 5.2.** For every $\gamma \in \text{Ker } \theta_k$ and $g \in \pi_1(\mathbb{C}^2 - D_k)$, $\psi_k(g^{-1} \gamma g) = \theta_k(g) \cdot \psi_k(\gamma)$, where $S_n$ acts on $(\mathbb{Z}^2/\Lambda_k)^n$ by permuting the factors (i.e., $\psi_k$ is equivariant).

**Proof.** Denoting by $\sigma$ the permutation $\theta_k(g)$, observe that the lifts of $g^{-1} \gamma g$ are freely homotopic to those of $\gamma$, and more precisely that the lift of $g^{-1} \gamma g$ through $p_{\sigma(i)}$ is freely homotopic to the lift of $\gamma$ through $p_i$. Therefore, the $\sigma(i)$-th component of $\psi_k(g^{-1} \gamma g)$ is equal to the $i$-th component of $\psi_k(\gamma)$.

**Lemma 5.3.** $K_k \subset \text{Ker } \psi_k$, i.e. $\psi_k$ factors through the stabilized group.

**Proof.** Recall from Definition 2.2 that $K_k$ is generated by commutators $[\gamma_1, \gamma_2]$ of geometric generators that are mapped to disjoint transpositions by $\theta_k$. If $\gamma_1$ is a geometric generator, then $n - 2$ of its lifts to $X$ are contractible closed loops in $X - L_k - R_k$, while the two other lifts are not closed; and similarly for $\gamma_2$. However, if $\theta_k(\gamma_1)$ and $\theta_k(\gamma_2)$ are disjoint, then all the lifts of $[\gamma_1, \gamma_2]$ are contractible loops in $X - L_k - R_k$; therefore $[\gamma_1, \gamma_2] \in \text{Ker } \psi_k$.

It is worth noting that, similarly, if $\gamma_1$ and $\gamma_2$ are geometric generators mapped by $\theta_k$ to adjacent (non-commuting) transpositions, then $(\gamma_1 \gamma_2 \gamma_1)(\gamma_2 \gamma_1 \gamma_2)^{-1} \in \text{Ker } \psi_k$. 


(only one of the lifts of this loop is possibly non-trivial, but its algebraic linking numbers with $L_k$ and $R_k$ are both equal to zero).

**Lemma 5.4.** For any $\gamma \in \text{Ker}\theta_k$, the $n$-tuple $\psi_k(\gamma) = ((\lambda_i, \rho_i))_{1 \leq i \leq n}$ has the property that $(\sum \lambda_i, \sum \rho_i) \equiv (0, \delta_k(\gamma)) \mod \Lambda_k$.

**Proof.** $\gamma \in \pi_1(\mathbb{C}^2 - D_k)$ is homotopically trivial in $\mathbb{C}^2$, so there exists a topological disk $\Delta \subset \mathbb{C}^2$ such that $\partial \Delta = \gamma$. Now observe that $\partial(f_k^{-1}(\Delta)) = \sum \gamma_i$; therefore $(\sum \lambda_i, \sum \rho_i)$ is equal (mod $\Lambda_k$) to the algebraic intersection numbers of $f_k^{-1}(\Delta)$ with $L_k$ and $R_k$. We have $f_k^{-1}(\Delta) \cdot L_k = 0$ since $f_k^{-1}(\Delta) \subset f_k^{-1}(\mathbb{C}^2) = X - L_k$, and $f_k^{-1}(\Delta) \cdot R_k = \Delta \cdot D_k = \delta_k(\gamma)$. □

**Lemma 5.5.** For any geometric generator $\gamma \in \Gamma_k$, $\psi_k(\gamma^2) = ((\lambda_i, \rho_i))_{1 \leq i \leq n}$ is given by $(\lambda_i, \rho_i) = (0, 1)$ if $i$ is one of the two indices exchanged by the transposition $\theta_k(\gamma)$, and $(\lambda_i, \rho_i) = (0, 0)$ otherwise.

**Proof.** All lifts of $\gamma^2$ are homotopically trivial, except for two of them which are freely homotopic to each other and circle once around the ramification curve $R_k$. □

**Lemma 5.6.** There exist two geometric generators $\gamma_1, \gamma_2 \in \Gamma_k$ such that $\theta_k(\gamma_1) = \theta_k(\gamma_2)$ and $\psi_k(\gamma_1 \gamma_2) = ((-1, 0), (1, 2), (0, 0), \ldots, (0, 0))$.

**Proof.** Consider a generic line $\ell \subset \mathbb{CP}^2$ intersecting $D_k$ transversely in $d = \deg D_k$ points, and let $\Sigma = f_k^{-1}(L)$. The restriction $f_k|\Sigma : \Sigma \to \ell \subset \mathbb{CP}^1$ is a connected simple branched cover of degree $n$ with $d$ branch points, with monodromy described by the morphism $\theta_k \circ \iota_* : \pi_1(\ell - \{d \text{ points}\}) \to S_n$. It is a classical fact that the moduli space of all connected simple branched covers of $\mathbb{CP}^1$ with fixed degree and number of branch points is connected, i.e. up to a suitable reordering of the branch points we can assume that the monodromy of $f_k|\Sigma$ is described by any given standard $S_n$-valued morphism.

So we can find an ordered system of generators $\gamma_1, \ldots, \gamma_d$ of the free group $\pi_1(\ell \cap (\mathbb{C}^2 - D_k))$ such that $\theta_k(\gamma_1) = \theta_k(\gamma_2) = (12)$ and all the other transpositions $\theta_k(\gamma_i)$ for $i \geq 3$ are elements of $S_{n-1} = \text{Aut}(2, \ldots, n)$. The loop $\gamma_1 \gamma_2$ then belongs to $\text{Ker}\theta_k$, and admits only two non-trivial lifts $g_1$ and $g_2$ in $\Sigma$, those which start in the first two sheets of the branched cover. The loops $g_1$ and $g_2$ bound a topological annulus $A$ which intersects $R_k$ in two points (projecting to the first two intersection points of $\ell$ with $D_k$). This annulus separates $\Sigma$ into two components, a “large” component consisting of the sheets numbered from 2 to $n$, and a disk $\Delta$ corresponding to the first sheet of the cover, which does not intersect $R_k$ but contains one of the $n$ preimages of the intersection point of $\ell$ with the line at infinity in $\mathbb{CP}^2$. The lift $g_1$ bounds $\Delta$ with reversed orientation; since $\Delta \cdot R_k = 0$ and $\Delta \cdot L_k = 1$, the first component of $\psi_k(\gamma_1 \gamma_2)$ is $(-1, 0)$. The lift $g_2$ bounds $\Delta \cup A$; since $A \cdot R_k = 2$ and $A \cdot L_k = 0$, the second component of $\psi_k(\gamma_1 \gamma_2)$ is $(1, 2)$. □

**Proof of Theorem 1.5.** By Lemma 5.4, $\psi_k$ maps the kernel of $\theta_k^+ : \pi_1(\mathbb{C}^2 - D_k) \to S_n \times \mathbb{Z}$ into the subgroup $\Gamma = \{(\lambda_i, \rho_i), \sum \lambda_i = \sum \rho_i = 0\} \cong (\mathbb{Z}/\Lambda_k) \otimes \mathcal{R}_n$ of $(\mathbb{Z}/\Lambda_k)^n$. By Lemma 5.3, $\psi_k$ factors through the quotient $\text{Ker}\theta_k^+ / K_k = G_k^0(X, \omega)$, and gives rise to a map $\phi_k : G_k^0(X, \omega) \to \Gamma \cong (\mathbb{Z}/\Lambda_k) \otimes \mathcal{R}_n \cong (\mathbb{Z}/\Lambda_k)^{n-1}$. Since $\Gamma$ is abelian, $[G_k^0, G_k^0] \subset \text{Ker}\phi_k$, so $\phi_k$ factors through the abelianization $\text{Ab}G_k^0(X, \omega)$, as announced in the statement of Theorem 1.5.
We now show that \( \phi_k \) is surjective, i.e. that \( \psi_k \) maps Ker \( \theta_k^+ \) onto \( \Gamma \). First, let \( \gamma \) and \( \gamma' \) be two geometric generators of \( \pi_1(\mathbb{C}^2 - D_k) \) corresponding to adjacent transpositions in \( S_n \); then \( \gamma^2 \gamma'^{-2} \in \text{Ker } \theta_k^+ \), and Lemma 5.5 implies that \( \psi_k(\gamma^2 \gamma'^{-2}) \) has only two non-zero entries, one equal to \((0, 1)\) and the other equal to \((0, -1)\). Recalling from \S2 that \( \theta_k \) is surjective, and using Lemma 5.2, by considering suitable conjugates of \( \gamma^2 \gamma'^{-2} \) we can find elements \( g_{ij} \) of Ker \( \theta_k^+ \) such that \( \psi_k(g_{ij}) \) has only two non-zero entries, \((0, 1)\) at position \( i \) and \((0, -1)\) at position \( j \).

Next, consider the geometric generators \( \gamma_1, \gamma_2 \) given by Lemma 5.6: the element \( \gamma_1 \gamma_2^{-1} \) belongs to Ker \( \theta_k^+ \), and \( \psi_k(\gamma_1 \gamma_2^{-1}) = ((-1, -1), (1, 1), (0, 0), \ldots, (0, 0)). \) Therefore \( \psi_k(g_{12} \gamma_1 \gamma_2^{-1}) = ((-1, -1), (1, 1), (0, 0), \ldots, (0, 0)). \) So, using the surjectivity of \( \theta_k \) and Lemma 5.2, we can find elements \( g_{ij} \) of Ker \( \theta_k^+ \) such that \( \psi_k(g_{ij}) \) has only two non-zero entries, \((1, 0)\) at position \( i \) and \((-1, 0)\) at position \( j \). We now conclude that \( \psi_k(\text{Ker } \theta_k^+) = \Gamma \) by observing that the \( 2n - 2 \) elements \( \psi_k(g_{in}) \) and \( \psi_k(g'_{in}) \), \( 1 \leq i \leq n - 1 \), generate \( \Gamma \).

We finish this section by mentioning two conjectures related to Conjecture 1.6. First of all, we mention that Conjecture 1.6 implies a result about the fundamental groups of Galois covers associated to branched covers of \( \mathbb{CP}^2 \). More precisely, given a complex surface \( X \) and a generic projection \( X \to \mathbb{CP}^2 \) of degree \( n \) with branch curve \( D_k \), the associated Galois cover \( \tilde{X}_k \) is obtained by compactification of the \( n \)-fold fibered product of \( X \) with itself above \( \mathbb{CP}^2 \): the complex surface \( \tilde{X}_k \) is a degree \( n! \) cover of \( \mathbb{CP}^2 \) branched along \( D_k \). Moishezon and Teicher have constructed many interesting examples of complex surfaces by this method, and computed their fundamental groups (see e.g. [13], [16], [11]). Given an ordered system of geometric generators \( \gamma_1, \ldots, \gamma_d \) of \( \pi_1(\mathbb{C}^2 - D_k) \), the fundamental group \( \pi_1(\tilde{X}_k) \) is known to be isomorphic to the quotient of Ker \( \theta : \pi_1(\mathbb{C}^2 - D_k) \to \pi_1(X) \) by the subgroup generated by \( \gamma_1, \ldots, \gamma_d, \prod \gamma_i \) (see e.g. [16], \S4).

By Lemma 5.5, the elements \( \gamma_i^2 \) and their conjugates map under \( \psi_k \) to elements of \( (\mathbb{Z}^2/\Lambda_k)^n \) with only two non-trivial entries \((0, 1)\); therefore, assuming Conjecture 1.6, quotienting by all squares of geometric generators leads to quotienting the image of \( \psi_k \) by \( \{(0, p_i) ; \sum p_i \text{ is even}\} \subset (\mathbb{Z}^2/\Lambda_k)^n \). Because of Lemma 5.4, and observing that \( \delta_k \) takes only even values on Ker \( \theta_k \), we are left with only the first factor in each summand \( \mathbb{Z}^2/\Lambda_k \). Moreover, one easily checks that \( \psi_k(\prod \gamma_i) = ((1, 0), (1, 0), \ldots, (1, 0)) \equiv ((1, 0), \ldots, (1, 0), (1 - n, d)) \mod \Lambda_k \); and by Lemma 5.4, the sum of the first factors is always zero, so we end up with a group isomorphic to \( (\mathbb{Z}^n_k)^{n - 2} \), where \( ks \) is the divisibility of \( L_k \) in \( H_2(X, \mathbb{Z}) \). Moreover, if we also assume that property \((*) \) holds in addition to Conjecture 1.6, it can easily be checked that the commutator subgroup \( [G^0_k, G^0_k] \) is contained in the subgroup generated by the \( \gamma_i^2 \). Therefore, we have the following conjecture, satisfied by the examples in \S4:

**Conjecture 5.7.** If \( X \) is a simply connected complex surface and \( k \) is large enough, then the fundamental group of the Galois cover \( \tilde{X}_k \) associated to a generic projection \( f_k : X \to \mathbb{CP}^2 \) defined by sections of \( L^\otimes k \) is \( \pi_1(X_k) = (\mathbb{Z}^n_k)^{n - 2} \), where \( ks \) is the divisibility of \( L_k \) in \( H_2(X, \mathbb{Z}) \) and \( n_k = \deg f_k \).

Also, a careful observation of the examples in \S4 suggests the following possible structure for the commutator subgroup \( [G^0_k, G^0_k] \), which is worth mentioning in spite of the rather low amount of supporting evidence:

**Conjecture 5.8.** If the symplectic manifold \( X \) is simply connected and \( k \) is large enough, then the commutator subgroup \( [G^0_k, G^0_k] \) is isomorphic to \( \Gamma_1 \times \Gamma_2 \), where
6. Moishezon-Teicher techniques for ruled surfaces

6.1. Overview of Moishezon-Teicher techniques. Moishezon and Teicher have developed a general strategy, consisting of two main steps [8, 9, 13], in order to compute the group \( \pi_1(\mathbb{C}^2 - D) \) when \( D \) is the branch curve of a generic projection to \( \mathbb{CP}^2 \) of a given projective surface \( X \subset \mathbb{CP}^N \). First, one computes the braid factorization (see §2) associated to the curve \( D \). This calculation involves a degeneration of the surface \( X \) to a singular configuration \( X_0 \) consisting of a union of planes intersecting along lines in \( \mathbb{CP}^N \), and a careful analysis of the “regeneration” process which produces the generic branch curve \( D \) out of the singular configuration [8]. As explained in §2, the braid factorization explicitly provides, via the Zariski-Van Kampen theorem, a (rather complicated) presentation of the group \( \pi_1(\mathbb{C}^2 - D) \). In a second step, one attempts to obtain a simpler description by reorganizing the relations in a more orderly fashion and by constructing morphisms between subgroups of \( \pi_1(\mathbb{C}^2 - D) \) and groups related to ~\( B_n \). This process is carried out in [9] for the case \( X \cong \mathbb{CP}^1 \times \mathbb{CP}^1 \), and in subsequent papers for other examples.

6.1.1. Degenerations and braid monodromy calculations. The starting point of the calculation is a degeneration of the projective surface \( X \subset \mathbb{CP}^N \) to an arrangement \( X_0 \) of planes in \( \mathbb{CP}^N \) intersecting along lines. The degeneration process in the case of manifolds like \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) and \( \mathbb{CP}^2 \) is described in detail in [8]. For example, in the case of \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) embedded by the linear system \( O(p, q) \), one first degenerates the surface \( X \) of degree \( 2pq \) to a sum of \( q \) copies of \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) embedded by \( O(p, 1) \) (each of degree \( 2p \)) inside \( \mathbb{CP}^N \); then each of these surfaces is degenerated into \( p \) quadric surfaces (\( \mathbb{CP}^1 \times \mathbb{CP}^1 \) embedded by \( O(1, 1) \)); finally, each of the \( pq \) quadric surfaces is degenerated into a union of two planes intersecting along a line. The resulting arrangement can be represented by the diagram in Figure 1.

![Figure 1](image-url)

Each triangle in the diagram represents a plane. Each edge separating two triangles represents an intersection line \( L_i \) between the corresponding planes; note that the outer edges of the diagram are not part of the configuration. The branch curve for the projection \( X_0 \to \mathbb{CP}^2 \) is an arrangement of lines in \( \mathbb{CP}^2 \) (the projections of the various intersection lines \( L_i \)); however, in the regeneration process each of these lines acquires multiplicity 2, and the vertices where two or more lines intersect in \( X_0 \) turn into certain standard local configurations.
Therefore the braid factorization for $D$ can be computed by looking at the local contributions of the various vertices in the diagram. Since the regeneration process turns a local configuration into a branch curve of degree $2m$, where $m$ is the number of edges meeting at the given vertex, the local contribution of a vertex is naturally described by a word in the braid group $B_{2m}$. Moreover, because projecting $X_0$ to $\mathbb{CP}^2$ creates extra intersection points between the projections of the lines $L_i$ whenever they do not intersect in $X_0$ (i.e. when they do not correspond to edges with a common vertex in the diagram), the branch curve $D$ contains a number of additional nodes besides the local vertex configurations.

The major difficulty is to arrange the various local configurations and the additional nodes into a single braid factorization describing the curve $D$: given a linear projection $\pi : \mathbb{CP}^2 - \{\text{pt}\} \to \mathbb{CP}^1$, one needs to fix a base point in $\mathbb{CP}^1$ and to choose an ordered system of loops in $\mathbb{CP}^1 - \text{crit} \pi|_D$ in order to obtain a braid factorization. This choice determines in particular how the local braid monodromy (in $B_{2m}$) for each vertex of the grid is embedded into the braid monodromy of $D$ (in $B_d$, $d = \deg D$). A careless setup leads to local embeddings $B_{2m} \to B_d$ that may be extremely difficult to determine.

An important observation of Moishezon is that the construction has sufficient flexibility to allow the images in $\mathbb{CP}^2$ of the various lines and intersection points to be chosen freely. This makes it possible to use the following very convenient setup [8]. First choose an ordering of the vertices in the diagram describing $X_0$; for example, for $\mathbb{CP}^1 \times \mathbb{CP}^1$ Moishezon chooses an ordering first by row, then by column, starting from the lower-left corner of the diagram: 00, 10, 20, ..., 01, 11, ..., pq. This determines a lexicographic ordering of the edges of the diagram: observing that each line $L_i$ passes through two vertices $v_i$ and $v_i'$ ($v_i < v_i'$), the ordering is given by $L_i < L_j$ iff either $v_i' < v_j'$, or $v_i' = v_j'$ and $v_i < v_j$. It is then possible to choose a configuration where the projections of the lines $L_i$ are given by equations with real coefficients, with slopes increasing according to the chosen lexicographic ordering, so that the intersection of the arrangement of lines in $\mathbb{CP}^2$ with a real slice $\mathbb{R}^2$ looks as in Figure 2.

![Figure 2](image)

The choice of the slopes of the lines ensures that the intersection points of $D$ with the reference fiber of $\pi$ (chosen to be $\{x = A\}$ for some real number $A \gg 0$) are ordered in the natural way along the real axis, thus yielding a natural set of geometric generators $\{\gamma_i, \gamma_i'\}$ for $\pi_1(\mathbb{C}^2 - D)$, as shown on the right of Figure 2; recall that each line $L_i$ has multiplicity 2 and hence yields two generators, and note
that the correct ordering of these generators counterclockwise around the base point
is $\gamma_d/2, \gamma_d/2, \ldots, \gamma_1', \gamma_1$. Moreover, the various vertices of the diagram describing $X_0$
appear, in sequence, for increasing values of $x$ (from left to right).

Since all the contributions to the braid monodromy of $D$ are now localized along
the real $x$-axis, it is a fairly straightforward task to choose a set of generating
loops in the base $\mathbb{C}P^1$ of the fibration $\pi$ and enumerate accordingly the various
contributions to the braid monodromy of $D$ (standard configurations at the vertices
of the diagram and extra nodes coming from pairs of edges without a common
vertex). Going through the list of vertices in decreasing sequence (“from right to
left”) yields the simplest formula (Proposition 1 of [8]):

**Proposition 6.1** (Moishezon). With the above setup, the braid monodromy of $D$

is given by the factorization $\prod_{i=\nu}^1 (C_i \cdot F_i)$, where $\nu$ is the number of vertices in the
diagram, $C_i$ is a product of contributions from nodal intersections between parts of $D$
corresponding to non-adjacent edges, and $F_i$ is the braid monodromy corresponding
to the $i$-th vertex, obtained as the image of a standard local configuration under
the embedding $B_{2m} \hookrightarrow B_d$ which maps the standard half-twists generating $B_{2m}$, to
half-twists along arcs that remain below the real axis.

Proposition 6.1 makes it fairly simple to obtain a presentation of $\pi_1(\mathbb{C}^2 - D)$
in terms of the “global” generators $\{\gamma_i, \gamma_i'\}$: the nature of the local embeddings
$B_{2m} \hookrightarrow B_d$ implies that the relations coming from each vertex are obtained from
standard “local” relations (determined by the local braid monodromy) simply by
renaming each of the $2m$ local geometric generators into the corresponding global
generator. Additionally, the extra nodes yield various commutation relations among
geometric generators.

The local configurations for the various types of vertices have been analyzed by
Moishezon in [8], leading to explicit formulas for the local contributions to the braid
factorization. The easiest case is that of “2-points” such as the corner points 00
and $pq$ in the diagram for $\mathbb{C}P^1 \times \mathbb{C}P^1$. The only line that passes through the vertex
locally regenerates to a conic in $\mathbb{C}^2$, presenting a single vertical tangency near the
origin; hence the local braid monodromy is a single half-twist in $B_2$, giving rise to an
equality relation between the two corresponding geometric generators of $\pi_1(\mathbb{C}^2 - D)$.

The next case is that of “3-points” such as those occurring on the boundary of the
diagram for $\mathbb{C}P^1 \times \mathbb{C}P^1$. During the first step of “regeneration”, which turns $X_0$
into a union of $pq$ quadric surfaces, the lines corresponding to the diagonal edges
are replaced by conics (the branch curve of a bidegree $(1,1)$ map from $\mathbb{C}P^1 \times \mathbb{C}P^1$
to $\mathbb{C}P^2$). For the vertices along the top and right sides of the diagram (labelled
$pj$ or $iq$), the partially regenerated configuration in $\mathbb{C}P^2$ therefore consists of a
portion of conic tangent to a line, with the line having the greatest slope; after
further regeneration, the line acquires multiplicity 2 and the tangent intersection is
replaced by three cusps. The local contribution to braid monodromy can therefore
be expressed by the product $Z_1^3 \cdot Z_1^3 \cdot Z_1^3 \cdot Z_1^3 \cdot Z_1^3$, where the various factors are
powers of half-twists along the paths represented in Figure 3 (cf. [8] and equation
(2.4) in [9]). The first three factors correspond to cusps arising from the tangent
intersection between the conic and the line, while the last factor corresponds to the
vertical tangency of the conic.

The 3-points on the bottom and left sides of the diagram give rise to a very similar
local configuration, except for the ordering of the various components. Finally, the
interior vertices of the diagram for $\mathbb{C}P^1 \times \mathbb{C}P^1$ are all of the same type (“6-points”
in Moishezon’s terminology); a careful analysis of their regeneration yields a certain braid factorization in $B_{12}$, accounting for the 6 vertical tangencies, 24 nodes and 24 cusps in the local model, as described in [8]. The local contributions to the relations defining $\pi_1(C^2 - D)$ have also been calculated by Moishezon for these various standard models in §2 of [9] (see also below).

6.1.2. Fundamental group calculations. The setup described in §6.1.1 provides an explicit presentation of $\pi_1(C^2 - D)$ in terms of geometric generators $\{\gamma_i, \gamma'_i\}$, $i = 1, \ldots, \frac{d}{2}$. By Proposition 6.1, the relations consist on one hand of standard relations given by local models for the various vertices of the diagram describing the degenerated surface $X_0$, and on the other hand of commutation relations coming from non-adjacent edges of the diagram. The goal is then to simplify this presentation and ultimately identify $\pi_1(C^2 - D)$ with a certain quotient of $B_n^{(2)}$ (or $B_n \ltimes P_{n,0}$).

In the remainder of this section, we describe the recipes used by Moishezon for the case $X = \mathbb{CP}^1 \times \mathbb{CP}^1$, following §3 of [9]; these methods also apply to other complex surfaces admitting similar degenerations, such as $X = \mathbb{CP}^2$ [10] or $X = \mathbb{F}_1$ (§6.2).

A first observation of Moishezon is that, after a slight change in the choice of generators, many of the local relations at the vertices can be expressed in terms of half of the generators only. More precisely, for each value of $i$, define a twisting action $\rho_i$ on the two generators $\gamma_i, \gamma'_i$ by the formula $\rho_i(\gamma_i) = \gamma'_i$ and $\rho_i(\gamma'_i) = \gamma'_i \gamma_i \gamma'_i^{-1}$. Choose integers $l_i$ satisfying the following compatibility conditions: if $i < j$ are the labels of the two diagonal edges meeting at a 6-point vertex of the diagram, then $l_j = l_i - 1$; if $i < j$ are the labels of the two vertical edges meeting at a 6-point, then $l_j = l_i + 1$; finally, if $i < j$ are the labels of the two horizontal edges meeting at a 6-point, then $l_j = l_i$. Now let $e_i = \rho_i^{-1}(\gamma_i)$ and $e'_i = \rho_i^{-1}(\gamma'_i)$. Because of the invariance properties of the local models [8], the local relations corresponding to 2-points and 3-points have the same expressions in terms of $\{e_i, e'_i\}$ as in terms of $\{\gamma_i, \gamma'_i\}$, independently of the amount of twisting, and those for 6-points are also independent of the $l_i$ as long as the compatibility relations hold. On the other hand, if $i_1 < \cdots < i_6$ are the labels of the edges meeting at a 6-point ($i_1$ and $i_6$ are the two diagonal edges), then it is possible to eliminate either $e_{i_1}$ or $e_{i_6}$ from the list of generators, because the local relations imply that

$$e_{i_6} = (e_{i_5} e_{i_3} e_{i_2}^{-1} e_{i_4}^{-1})^{-1} e_{i_5} (e_{i_3} e_{i_2} e_{i_4}^{-1} e_{i_5})^{-1}.$$  

The second important observation of Moishezon is that, in many cases (assuming the diagram is “large enough”, i.e. in the case of a bidegree $(p, q)$ linear system on $\mathbb{CP}^1 \times \mathbb{CP}^1$ that $p, q \geq 2$), the relations coming from cusps and nodes of $D$ can all be reformulated into a very nice pattern (cf. Lemma 14 of [9]). If the two edges $i$ and $j$ bound a common triangle in the diagram, then the local relations at their
common vertex imply that
\[(6.2) \quad e_i e_i e_i = e_j e_j e_j, \quad e_i e'_i e_i = e'_j e_j e'_j, \quad e_i e_j e'_i = e_j e'_i e_j, \quad \text{and} \quad e'_i e'_j e'_i = e'_j e'_i e'_j.\]
Otherwise, if there is no triangle having \(i\) and \(j\) as edges, or equivalently if the two transpositions \(\theta(e_i) = \theta(e'_i)\) and \(\theta(e_j) = \theta(e'_j) \in S_n\) are disjoint, then we have
\[(6.3) \quad [e_i, e_j] = [e'_i, e'_j] = [e_i, e'_j] = [e'_i, e_j] = 1.\]
Looking at \(e_1, \ldots, e_4\), among which there are only \(n - 1\) independent generators (by (6.1), many of the \(e_i\) corresponding to diagonal edges can be expressed in terms of the others), a first consequence of the relations (6.2–6.3) is the following (Proposition 8 of [9]):

**Lemma 6.2** (Moishezon). In the case of the linear system \(O(p, q)\) on \(\mathbb{CP}^1 \times \mathbb{CP}^1\), the subgroup \(\mathcal{B}\) of \(\pi_1(\mathbb{C}^2 - D)\) generated by \(e_1, \ldots, e_{d/2}\) is isomorphic to a quotient of \(\hat{B}_n\) \((n = 2pq)\). More precisely, there exists a surjective morphism \(\hat{\alpha} : \hat{B}_n \to \mathcal{B}\) with the property that each \(e_i\) is the image of a half-twist in \(\hat{B}_n\), and \(\theta \circ \hat{\alpha} = \pi\) (i.e. the end points of the half-twists agree with the transpositions \(\theta(e_i)\)).

We now need to add to this description the other generators \(e'_i\), or equivalently the elements \(a_i = e'_i e_i^{-1}\). In the case of \(\mathbb{CP}^1 \times \mathbb{CP}^1\), we relabel these elements as \(d_{ij}\) for the diagonal edge in position \(ij\) (\(1 \leq i \leq p, 1 \leq j \leq q\), see Figure 1), \(v_{ij}\) for the vertical edge in position \(ij\) (\(1 \leq i < p, 1 \leq j \leq q\)), and \(h_{ij}\) for the horizontal edge in position \(ij\) (\(1 \leq i \leq p, 1 \leq j < q\)). We are especially interested in \(a_2 = v_{11}\). Moishezon’s next observation is that, as a consequence of relations (6.2–6.3) and of the local relations of the lower-left-most 6-point in the diagram, the subgroup generated by \(v_{11}\) and the conjugates \(g^j v_{11} g, g \in \mathcal{B}\), is naturally isomorphic to a quotient of \(\hat{P}_{n, 0}\) ([9], Definition 5 and Lemma 17). Moreover, the subgroup of \(\pi_1(\mathbb{C}^2 - D)\) generated by the \(e_i\) and by \(v_{11}\) is similarly isomorphic to a quotient of the semi-direct product \(\hat{B}_n \ltimes \hat{P}_{n, 0}\), or equivalently (as seen in §4) \(\hat{B}_{n, 0}^{(2)}\).

The most important relations in \(\pi_1(\mathbb{C}^2 - D)\) are those coming from the vertical tangencies of \(D\), which we now list for the various types of vertices. If the edge labelled \(i\) passes through a 2-point, then the local relation \(e_i = e'_i\) can be rewritten in the form \(a_i = 1\). If \(i < j\) are the labels of the two edges meeting at a 3-point, then we have \(e_i = e_j^{-1} e'_j e'_i e_j\), or equivalently \(e'_i = e_i^{-1} e'_i e_i e'_i\). Using (6.2) this relation can be rewritten as
\[(6.4) \quad a_{ij} = e_i^{-1} e'_i e'_j e_j^{-1} e_i e_j^{-1} = e_i^{-2} (e_i e_j) a_i (e_j^{-1} e_i^{-1}) e_j e_i e'_i e_i^{-1}.\]
Finally, if \(i_1 < \cdots < i_6\) are the labels of the edges meeting at a 6-point (according to the ordering rules, \(i_1\) and \(i_6\) are diagonal, \(i_2\) and \(i_5\) are vertical, and \(i_3\) and \(i_4\) are horizontal), then, besides (6.1), we also have
\[(6.5) \quad \begin{align*}
a_{i_6} &= (e_{i_5} e_{i_2} e_{i_4} e_{i_1}^{-1} e_{i_5}^{-1})^{-1} a_{i_1} (e_{i_5} e_{i_2} e_{i_4}^{-1} e_{i_5}^{-1}) \\
a_{i_5} &= (e_{i_4}^{-1} e_{i_2} e_{i_4}^{-1} e_{i_6}^{-1})^{-1} a_{i_2} (e_{i_4}^{-1} e_{i_2} e_{i_4}^{-1} e_{i_6}^{-1}) \\
a_{i_4} &= (e_{i_3}^{-1} e_{i_2} e_{i_6}^{-1} e_{i_6}^{-1})^{-1} a_{i_2} (e_{i_3}^{-1} e_{i_2} e_{i_6}^{-1} e_{i_6}^{-1})
\end{align*}\]
\[(6.6) \quad \begin{align*}
a_{i_3} &= (e_{i_6} e_{i_1})^{-1} a_{i_1} a_{i_1} (e_{i_6} e_{i_1}) (e_{i_6} e_{i_1}) \\
a_{i_2} &= (e_{i_6} e_{i_1})^{-1} a_{i_1} a_{i_1} (e_{i_6} e_{i_1}) (e_{i_6} e_{i_1})
\end{align*}\]
A first consequence of relations (6.4–6.6) is that, going inductively through the various vertices of the grid, all \(a_i\) can be expressed in terms of the \(e_1, \ldots, e_{d/2}\) and
of $a_2 = v_{11}$. Therefore $\pi_1(\mathbb{C}^2 - D)$ is generated by the $e_i$ and by $v_{11}$; hence it is isomorphic to a quotient of $\tilde{B}_n^{(2)}$. In other words, we have a surjective homomorphism $\alpha : \tilde{B}_n^{(2)} \to \pi_1(\mathbb{C}^2 - D)$, extending the morphism $\bar{\alpha} : \tilde{B}_n \to \mathcal{B}$ of Lemma 6.2.

From this point on, the results in §3 make it possible to present Moishezon’s argument in a simpler and more illuminating way. Observe that by Lemma 6.2 each $e_i$ is the image by $\alpha$ of a half-twist in the diagonally embedded subgroup $\tilde{B}_n \subset \tilde{B}_n^{(2)}$. Moreover, it is a general fact about irreducible plane curves that all geometric generators are conjugate to each other in $\pi_1(\mathbb{C}^2 - D)$; therefore each of the geometric generators $e_i$ is the image of a pair of half-twists in $\tilde{B}_n^{(2)}$. Alternatively this can be seen directly from the above-listed relations; these relations also imply that each $a_i$ belongs to the normal subgroup of pure degree 0 elements $\alpha(P_{n_0} \times P_{n_0})$, and therefore that the half-twists corresponding to the geometric generators $e_i'$ have the correct end points as prescribed by the $S_n$-valued monodromy representation morphism $\theta$. Therefore $\pi_1(\mathbb{C}^2 - D)$ has the property (*) defined in §3.

In view of Lemmas 3.3 and 3.4, at this point in the argument we can discard all the relations in $\pi_1(\mathbb{C}^2 - D)$ coming from nodes and cusps of $D$ since they automatically hold in quotients of $\tilde{B}_n^{(2)}$, and focus on the relations (6.4–6.6) instead.

By Lemma 3.2, pairs of half-twists in $\tilde{B}_n^{(2)}$ with fixed end points can be classified by two integers. More precisely, fix an ordering of the $n$ sheets of the branched cover $f$, e.g. from left to right and from bottom to top in the diagram. This provides an ordering of the end points of the half-twists corresponding to $e_i$ and $e_i'$; we can find an element $g \in \tilde{B}_n^{(2)}$ such that $e_i = \alpha(g^{-1}(x_1, x_1)g)$, with ordering of the end points preserved. Then by Lemma 3.2 there exist integers $k$ and $l$ such that $e_i' = \alpha(g^{-1}(x_1u_1^{-k}x^k(-k-1)/2, x_1u_1^{-l}x^{-l(-l-1)/2})g)$, i.e. $a_i = \alpha(g^{-1}(u_1^k\gamma^k(-k-1)/2, u_1^l\gamma^{-l(-l-1)/2}))$. One easily checks by Lemma 3.1 that reversing the ordering of the end points changes $k$ into $-k$ and $l$ into $-l$.

Since $\alpha$ is a priori not injective, the integers $k$ and $l$ are not necessarily unique, and there may exist another pair of integers $(k', l') = (k + \kappa, l + \lambda)$ with the same property, i.e. such that $\mu = (u_1^k\gamma^{k'(k-1)/2-k(k-1)/2}, u_1^l\gamma^{l(-l-1)/2-l(-l-1)/2}) \in \text{Ker } \alpha$. If $\kappa$ is odd, then the normal subgroup generated by $\mu$ contains the commutator of $\mu$ with $(u_2, 1)$, which is equal to $(\eta, 1)$; so $(\eta, 1) \in \text{Ker } \alpha$. If $\kappa$ is even, then $\eta^{k'(k-1)/2-k(k-1)/2} = \eta^{\kappa'/2} = \eta^{\kappa(k-1)/2}$ (recall that $\eta^2 = 1$). Similarly, if $\lambda$ is odd then $(1, \eta) \in \text{Ker } \alpha$, otherwise $\eta^l(l-1)/2-l(l-1)/2 = \eta^{\lambda(l-1)/2}$. In both cases we arrive to the conclusion that $\bar{\mu} = (u_1^k\eta^{\kappa(k-1)/2}, u_1^l\eta^{\lambda(l-1)/2}) \in \text{Ker } \alpha$. In fact, $\mu$ and $\bar{\mu}$ generate the same normal subgroups, so we also have the converse implication.

Therefore the set of all possible values for $(\kappa, \lambda)$ forms a subgroup $\Lambda \subset \mathbb{Z}^2$; in fact $\Lambda = \{(\kappa, \lambda), (u_1^k\eta^{(\kappa-1)/2}, u_1^l\eta^{\lambda(k-1)/2}) \in \text{Ker } \alpha\}$, and the pair of integers $(k, l)$ is only defined mod $\Lambda$. So, to $e_i$ and $e_i'$ we can associate an element $\bar{a}_i = (k, l) \in \mathbb{Z}^2/\Lambda$. This element $\bar{a}_i$ contains all the relevant information about $e_i$ and $e_i'$ apart from the end points. Indeed, because of Lemma 3.5, up to composition of $\alpha$ with an automorphism of $\tilde{B}_n^{(2)}$ we can assume $e_i$ to be the image by $\alpha$ of any given pair of half-twists with the correct end points. And, by Lemma 3.2, if two half-twists $x, y \in \tilde{B}_n$ have the same end points, then $x^2y^{-2} \in \{1, \eta\}$, so up to a factor of $\eta$ the product $e_i'e_i = a_i^2e_i^2$ is determined by $a_i$; that ambiguity can in fact be lifted by arguing that $e_i$ and $e_i'$ are images of half-twists.

The subgroup $\Lambda$ can be determined by looking at the relations in $\pi_1(\mathbb{C}^2 - D)$ coming from vertical tangencies of $D$, which determine the kernel of $\alpha$. We now
reformulate these relations in terms of the $\tilde{a}_i$. First, at a 2-point, the relation $a_i = 1$ becomes $\tilde{a}_i = (0,0)$. What happens at a 3-point depends on the ordering of the sheets of $f$ (i.e., of the triangles of the diagram): the relation (6.4) becomes

$$\pm \tilde{a}_i + \pm \tilde{a}_j = (1,1),$$

where the first sign is $+$ if the triangle $T$ which has both $i$ and $j$ among its edges comes after the other triangle bounded by the edge $i$ and otherwise, and the second sign is $+$ if $T$ comes after the other triangle bounded by the edge $j$ and otherwise. In the case of a 6-point with the standard ordering used by Moishezon, (6.5) and (6.6) become

$$\tilde{a}_{i_0} = \tilde{a}_{i_1}, \quad \tilde{a}_{i_4} = \tilde{a}_{i_5}, \quad \tilde{a}_{i_8} = \tilde{a}_{i_9}, \quad \tilde{a}_{i_1} - \tilde{a}_{i_2} + \tilde{a}_{i_3} = 0.$$

In the case of $\mathbb{C}P^1 \times \mathbb{C}P^1$, denoting by $d_{ij}$, $\tilde{v}_{ij}$ and $\tilde{h}_{ij}$ the elements of $\mathbb{Z}^2/A$ corresponding to $d_{ij}$, $v_{ij}$ and $h_{ij}$, the relations become (listing the vertices from left to right and bottom to top): $d_{1,1} = (0,0)$, $\tilde{v}_{i_1} - \tilde{d}_{i+1,1} = (1,1)$, $\tilde{h}_{i_1} + \tilde{d}_{i+1,1} = (1,1)$; $d_{i+1,j+1} = \tilde{d}_{i,j}$, $\tilde{v}_{i,j+1} = \tilde{v}_{i,j}$, $\tilde{h}_{i,j} + \tilde{d}_{i,j} - \tilde{v}_{i,j} + \tilde{h}_{i,j} = 0$; $-d_{p,j} - \tilde{h}_{p,j} = (1,1)$, $\tilde{d}_{i,q} - \tilde{v}_{i,q} = (1,1)$, $\tilde{d}_{p,q} = (0,0)$. Moreover, by construction $\tilde{v}_{11} = (0,1)$ (because $v_{11}$ was identified to a generator of $\tilde{P}_{n,0}$).

Working inductively from the lower-left corner of the diagram, these equations yield the formulas

$$d_{i,j} = (j-i,0), \quad \tilde{v}_{i,j} = (1-i,1), \quad \tilde{h}_{i,j} = (1-j,1)$$

(compare with Proposition 10 of [9], recalling that the identification between $\tilde{B}_n \times \mathbb{Z}$ and $B_n(2)$ coming from vertical tangencies correspond to equality relations between pairs of half-twists in $\tilde{B}_n(2)$, by the above remarks $\ker \alpha$ is the normal subgroup of $\tilde{B}_n(2)$ generated by a certain number of elements of the form $(u^2 \eta (\alpha-1)/2, u^2 \eta (\lambda-1)/2)$, and therefore it is completely determined by the subgroup $\Lambda \subset \mathbb{Z}^2$. In our case, $\ker \alpha$ is the normal subgroup of $\tilde{B}_n(2)$ generated by $(u^2 \eta (\alpha-1)/2, u^2 \eta (\lambda-1)/2)$ and $(u^2 \eta (\alpha-1)/2, u^2 \eta (\lambda-1)/2)$.

Because all relations in $\pi_1(\mathbb{C}P^2 - D)$ coming from vertical tangencies correspond to equality relations between pairs of half-twists in $\tilde{B}_n(2)$, we have two cases to consider. First, if $p$ is odd, then by considering the commutator of $(u^2 \eta (\alpha-1)/2, u^2 \eta (\lambda-1)/2)$ with $(u_2,1)$ we obtain that $(\eta,1) \in \ker \alpha$ (and similarly if $q$ is odd); but one easily checks that $(1,\eta) \notin \ker \alpha$. On the other hand, if $p$ and $q$ are both even, then no non-trivial element of $C = \{1,\eta\} \times \{1,\eta\}$ belongs to $\ker \alpha$. Therefore, $[H_{p,q}^0, H_{p,q}^0] \simeq (\mathbb{Z}/(C \cap \ker \alpha))$ is isomorphic to $\mathbb{Z}_2$ if $p$ or $q$ is odd, and to $\mathbb{Z}_2 \times \mathbb{Z}_2$ if $p$ and $q$ are even. Moreover, we have $\text{Ab} H_{p,q}^0 \simeq \tilde{P}_{n,0} \times \tilde{P}_{n,0} / (C, \ker \alpha) \simeq (\mathbb{Z}^2/A)^{n-1}$, which one easily shows to be isomorphic to $(\mathbb{Z}_2 \oplus \mathbb{Z}_{p,q})^{n-1}$ or $(\mathbb{Z}_2(2,p,q))^{n-1}$ depending on the parity of $p$ and $q$. This completes the proof of Theorem 4.1. The computations for $\mathbb{C}P^2$ (Theorem 4.2) and other algebraic surfaces admitting similar degenerations can be carried out by the same method; for example, the case of the Hirzebruch surface $\mathbb{F}_1$ is treated in §6.2 below.

6.2. The Hirzebruch surface $\mathbb{F}_1$. In this section, we prove Theorem 4.5 using the method outlined in the preceding section. Consider the projective embedding
of $\mathbb{P}_1$ defined by sections of the linear system $O(pF+qE)$, $p > q \geq 2$ (recall $F$ is the fiber and $E$ is the exceptional section). This projective surface can be degenerated in the same manner as the Veronese surface of which it is a blow-up (the projective embedding of $\mathbb{CP}^2$ defined by sections of $O(p)$), following the procedure described in §3 of [8]. This surface of degree $n = (2p-q)q$ can be first degenerated into a sum of $q$ Hirzebruch surfaces, of degrees respectively $2p-1$, $2p-3$, $\ldots$, $(2p-q)+1$. Each of these Hirzebruch surfaces can then be degenerated into the union of a plane and a certain number of quadric surfaces, which in turn can each be degenerated to two planes. The resulting diagram is pictured in the right half of Figure 4.

One uses the same setup as in §6.1.1, ordering the vertices from left to right and bottom to top, and the edges accordingly. The braid monodromy is given by Proposition 6.1. It follows from Moishezon’s work that all vertices correspond to well-known configurations: the two vertices $qq$ and $pq$ are 2-points, while the other boundary vertices are 3-points and the interior vertices are 6-points.

As in §6.1.2, one replaces the natural set of geometric generators $\{\gamma_i, \gamma'_i\}$ by twisted generators $e_i = \rho_{l_i}^i(\gamma_i)$ and $e'_i = \rho_{l'_i}^i(\gamma'_i)$, where the integers $l_i$ satisfy the required compatibility conditions, in order to have (6.1) at all 6-points. Moreover, relations (6.2) and (6.3) hold for all pairs of edges ((6.2) if the edges bound a common triangle, (6.3) otherwise), by the same argument as for $\mathbb{CP}^2$: the proof of Lemma 1 of [10] (see also Lemma 14 of [9]) applies almost without modification.

Eliminating redundant diagonal edges as allowed by (6.1), we are left with exactly $n-1$ independent generators among the $e_i$. In the case of $\mathbb{CP}^1 \times \mathbb{CP}^1$, relations (6.2) and (6.3) imply that the subgroup $\mathcal{B}$ generated by the $e_i$ is isomorphic to a quotient of $\hat{B}_n$, and Lemma 6.2 extends to the case of the Hirzebruch surface $\mathbb{F}_1$.

As previously, we let $a_i = e'_i e_i^{-1}$, and we relabel these elements as $d_{ij}$, $v_{ij}$ and $h_{ij}$. We are now interested in $a_1 = v_{11}$: one can again show that the subgroup generated by $v_{11}$ and the conjugates $g^{-1}v_{11}g$, $g \in \mathcal{B}$ is isomorphic to a quotient of $\hat{P}_{n,0}$, by Lemma 5 of [10] (the argument is the same for $\mathbb{F}_1$ as for $\mathbb{CP}^2$); the subgroup of $\pi_1(\mathbb{C}^2 - D)$ generated by the $e_i$ and by $a_1$ is again isomorphic to a quotient of $\hat{B}_n \times \hat{P}_{n,0} \simeq \hat{B}_n^{(2)}$.

Relations (6.4–6.6) imply that, going through the various 3-points and 6-points of the diagram, all the $a_i$ can be expressed in terms of $e_1, \ldots, e_{d/2}$ and $a_1 = v_{11}$; therefore $\pi_1(\mathbb{C}^2 - D)$ is generated by $e_1, \ldots, e_{d/2}$ and $a_1$, so that we again obtain a surjective morphism $\alpha: \hat{B}_n^{(2)} \to \pi_1(\mathbb{C}^2 - D)$. As in the case of $\mathbb{CP}^1 \times \mathbb{CP}^1$, the various geometric generators are images by $\alpha$ of pairs of half-twists with correct end points, so that property (*) holds once more. Using the classification of half-twists
in $\tilde{B}_n$ (Lemma 3.2), we can consider pairs of integers $\tilde{a}_i$ instead of the elements $a_i$; once again, the $\tilde{a}_i$ are only defined modulo a certain subgroup $\Lambda \subset \mathbb{Z}^2$.

The various relations between the $\tilde{a}_i$ are now the following: $\tilde{v}_{i,j} - \tilde{h}_{i+1,j} = (1, 1)$; $\tilde{d}_{i,j} = \tilde{v}_{i,j}$, $\tilde{h}_{i+1,j} = \tilde{h}_{i,j}$, $\tilde{d}_{i,j} - \tilde{v}_{i,j} + \tilde{h}_{i,j} = 0$; $-\tilde{d}_{p,j} - \tilde{h}_{p,j} = (1, 1)$, $\tilde{v}_{q,j} = (0, 0)$, $\tilde{d}_{p,q} = (1, 1)$, $\tilde{d}_{p,q} = (0, 0)$. Moreover, $\tilde{v}_{1,1} = (0, 1)$. Therefore, $\tilde{d}_{i,j} = (2j - 2i + 1, j - i + 1)$, $\tilde{v}_{i,j} = (2j - 2i - 2)$ and $\tilde{h}_{i,j} = (1 - 2j, 1 - j)$ (compare with Proposition 4 of [10]), and we are left with two additional relations: $(2p - 2, p - 2) = (1, 1)$ and $(2 - 2q, 2 - q) = (0, 0)$. Therefore, $\Lambda$ is the subgroup of $\mathbb{Z}^2$ generated by $(2p - 3, p - 3)$ and $(2q - 2, q - 2)$, and $\text{Ker} \alpha$ is the normal subgroup of $\tilde{B}_n^{(2)}$ generated by $(u_1^{2p-3}, u_2^{2q-3})$ and $(u_1^{2q-2}, u_2^{2q-3})$.

Considering the commutator of the first generator with $(u_2, 1)$, we obtain that $(\eta, 1) \in \text{Ker} \alpha$. Moreover, if either $p$ is even or $q$ is odd, then considering the commutator of one of the generators with $(1, u_2)$, we obtain that $(1, \eta) \in \text{Ker} \alpha$. On the contrary, if $p$ is odd and $q$ is even then $H_1^{(2)} \cong \mathbb{Z}/2$ and $H_2^{(2)} \cong \mathbb{Z}/3$.

7. **Double covers of $\mathbb{CP}^1 \times \mathbb{CP}^1$**

In this section, we sketch the proof of Theorem 4.6, which combines the methods described in §6 with ideas similar to those in [3].

7.1. **Generic perturbations of iterated branched covers.** Let $C$ be a smooth algebraic curve of degree $(2a, 2b)$ in $Y = \mathbb{CP}^1 \times \mathbb{CP}^1$, and let $X_{a,b} \to Y$ be the double cover of $Y$ branched along $C$. Then one can construct a map $f^0 : X_{a,b} \to \mathbb{CP}^2$ simply by composing the double cover $\pi : X_{a,b} \to Y$ with a generic projective map $g : Y \to \mathbb{CP}^2$ determined by sections of $O(p,q)$. The map $f^0$ is not generic: its ramification curve is the union of the ramification curve of $\pi$ and the preimage by $\pi$ of the ramification curve of $g$, and so the branch curve $D^0$ of $f^0$ is the union of $g(C)$ (with multiplicity 1) and the branch curve $D_g$ of $g$ (with multiplicity 2).

This situation is extremely similar to that considered in [3] for the composition of a generic map from a symplectic 4-manifold to $\mathbb{CP}^2$ with a quadratic map from $\mathbb{CP}^2$ to itself. The local behavior of the map $f^0$ is generic everywhere except at the intersection points of $C$ with the ramification curve of $g$; assuming that $C$ and $g$ are chosen generically, a local model for $f^0$ near these points is $(x, y) \mapsto (-x^2 + y, -y^2)$, for which a generic local perturbation is given e.g. by $(x, y) \mapsto (-x^2 + y, -y^2 + cx)$ where $c$ is a small positive constant (cf. also [3]). There are several ways in which the map $f^0$ can be perturbed and made generic. If the linear system $\pi^*O(p,q)$ is sufficiently ample, then $f^0$ can be deformed within the holomorphic category into a generic projective map which no longer factors through the double cover $\pi$. Another possibility, if $p$ and $q$ are sufficiently large, is to use approximately holomorphic methods (Theorem 1.1) to deform $f^0$ into a map with generic local models (cf. [3]).

In both cases, the effect of the perturbation on the topology of the branch curve of $f^0$ is pretty much the same. First, the local model near an intersection point of $C$ with the ramification curve of $g$ is perturbed as described above (up to isotopy), which transforms a tangent intersection of $g(C)$ with the branch curve of $g$ in $\mathbb{CP}^2$ into a standard configuration with three cusps [3]. Secondly, the two copies
of the branch curve of \( g \), which make up the multiplicity two component of \( D^0 \), are separated and made transverse to each other; this deformation of \( D_g \) is performed either within the holomorphic category or resorting to approximately holomorphic perturbations. In the second case, the perturbation process can be performed in a very flexible manner, which in some cases may create negative intersections; restricting oneself to algebraic perturbations is a convenient way to avoid this phenomenon, but makes the global perturbation harder to describe explicitly. In any case, up to isotopy and creation or cancellation of pairs of intersections between the two deformed copies of the branch curve of \( g \), the topology of the resulting generic branch curve \( D \) is uniquely determined and can be computed easily from that of \( D^0 \). In fact, the approximately holomorphic perturbation process can always be carried out, even for small values of \( p \) and \( q \) for which neither the holomorphic construction nor Theorem 1.1 are able to yield generic projective maps; in this situation, we can still study the topology of the curve \( D \), but Theorem 4.6 only describes a “virtual” generic projective map.

As in §6, the study of the curve \( D \) relies on a degeneration process: one first degenerates the curve \( C \) in \( Y = \mathbb{CP}^1 \times \mathbb{CP}^1 \) into a union of two sets of parallel lines, \( 2a \) along one factor and \( 2b \) along the other factor. Parallel lines are then merged, so that the resulting configuration \( C_0 \subset Y \) consists of only two components, a \((1,0)\)-line of multiplicity \( 2a \) and a \((0,1)\)-line of multiplicity \( 2b \). Finally, one degenerates the projective embedding of \( Y \) given by the linear system \( O(p,q) \) into an arrangement \( Y_0 \) of planes intersecting along lines, as in §6.1. The fully degenerated branch curve is a union of lines, some of which correspond to the intersections between the planes in \( Y_0 \) (each contributing with multiplicity 4, since the branch curve of \( g \) is counted with multiplicity 2), while the others are the images of the \( p + q \) components into which \( C_0 \) degenerates (some of these components contribute with multiplicity \( 2a \), others with multiplicity \( 2b \)).

The curve \( D \) can be recovered from this arrangement of lines by the converse “regeneration” process, which first yields the union \( D_g \cup g(C_0) \) (by deforming \( Y_0 \) into the smooth surface \( Y \)), then \( D_g \cup g(C) = D^0 \) (by separating the multiple components of \( C_0 \) and smoothing the resulting curve), and finally \( D \) (by performing the prescribed local perturbation at the intersection points of the two ramification curves and by perturbing the two copies of \( D_g \) in a generic way).

### 7.2. Braid monodromy calculations.

The braid monodromy for the curve \( D_g \cup g(C_0) \) (and for the subsequent regenerations \( D^0 \) and \( D \)) can be computed using the same methods as in §6.1.1. The diagram describing the degenerated configuration is as represented on Figure 5, which differs from Figure 1 only by the addition of edges corresponding to \( C_0 \) along the top and right boundaries of the diagram.

Thanks to Proposition 6.1, we only need to understand the local behavior of the curves \( D_g \cup g(C_0) \), \( D^0 \) and \( D \) near the various vertices of the diagram. At all vertices except those through which \( C_0 \) passes (top and right sides of the diagram), the local description of \( D_g \cup g(C_0) \) and \( D^0 \) is exactly the same as that of \( D_g \), which has already been discussed in §6.1: the various vertices are standard 2-points, 3-points and 6-points as in Moishezon’s work [9]. Moreover, the local configuration for \( D \) at such a vertex simply consists of two copies of the local configuration for \( D_g \), shifted apart from each other by a generic translation. The two components, which correspond to the two preimages of the ramification curve of \( g \) under the branched cover \( \pi \), may intersect at nodal points of either orientation; we won’t
be overly concerned by the details of these intersections, since the various possible configurations only differ by isotopies and creations or cancellations of pairs of nodes, which do not affect the stabilized fundamental group in any way.

We now consider a vertex along the top boundary of the diagram, at position \( iq \) with \( 1 \leq i \leq p - 1 \). The local configuration for \( D_g \cup g(C_0) \) at such a point is as shown on Figure 6. The parts labelled 1, 1', 2, 2' correspond to \( D_g \), and form a standard 3-point (cf. §6.1.1 and Figure 3), presenting three cusp singularities near the point A. The parts labelled 3 and 4 correspond to \( g(C_0) \), obtained by “regeneration” of the two lines associated to the horizontal edges of the diagram passing through the vertex. The curve \( g(C_0) \) presents tangent intersections with the two lines 2 and 2' near the point B, and with the conic 1, 1' at the point C. The two intersections of the line labelled 4 with the conic 1, 1' in \( \mathbb{CP}^2 \) remain as nodes since the corresponding curves fail to intersect in \( Y \).

The local description of the curve \( D^0 = D_g \cup g(C) \) is obtained from that of \( D_g \cup g(C_0) \) by separating \( C_0 \) into \( 2b \) parallel components; this yields \( 2b \) copies of the lines labelled 3 and 4 in Figure 6, and the local configuration near the points B and C becomes as shown in the right half of Figure 6 (the pictures correspond to the case \( b = 2 \)). Finally, in order to obtain \( D \) we must perturb \( D^0 \) in the manner explained in §7.1: the multiplicity two component \( D_g \subset D^0 \) (corresponding to the parts labelled 1, 1', 2, 2' in Figure 6) is separated into two distinct copies (in particular the point A is duplicated), while each tangent intersection of \( g(C) \) with \( D_g \) (such as those near points B and C) gives rise to three cusps. It is then possible to write explicitly the local braid monodromy for \( D \), with values in \( B_{4b+8} \) by enumerating carefully the \( 4b + 2 \) vertical tangencies, \( 18b + 6 \) cusps, and nodes of the local model.
a role in the argument, namely the six cusps near point $A$ and one of the 12 of the half-twists represented in Figure 7. Actually, the truly important information is contained in the vertical tangencies, which correspond to the half-twists of some of the line components to which $\pi$ has to be calculated again, with results very similar to those above. In fact, it can easily be checked that, up to a Hurwitz equivalence, the only effect of the change of ordering on the local braid monodromy is the simultaneous conjugation of all contributions by a braid that exchanges the groups of points labelled $2, 2', 2''$ and $3_1, \ldots, 3_{2a}$ by moving them around each other counterclockwise.

The last vertex that remains to be investigated is the corner vertex at position $pq$. The local configuration for $D^0 = D_g \cup \pi(C)$ is obtained from that represented by moving them around each other counterclockwise.
in Figure 9 (left) by smoothing the $4ab$ mutual intersections between the lines labelled $2_1, \ldots, 2_{2a}$ and $3_1, \ldots, 3_{2b}$. Indeed, the local configuration for $D_g$ is simply a conic (labelled 1, 1’ in Figure 9), while $g(C_0)$ consists of two lines tangent to that conic, and $g(C)$ is obtained by “thickening” these two lines into respectively $2a$ and $2b$ components ($2_1, \ldots, 2_{2a}$ corresponding to the vertical edge of the diagram, and $3_1, \ldots, 3_{2b}$ corresponding to the horizontal edge of the diagram) and smoothing their mutual intersections. The curve $D$ is then obtained from $D^0$ by separating the multiplicity 2 component $D_g$ into two distinct copies, while each tangent intersection of $D_g$ with $g(C)$ gives rise to three cusps.

The braid monodromy for the corner vertex can be deduced explicitly from this description. We are particularly interested in the stabilized quotient $G$ of the geometric generators $\gamma_i, \gamma_i'$ and $\tilde{\gamma}_i, \tilde{\gamma}_i'$ the four generators corresponding to the $i$-th interior edge. Moreover, each edge along the top boundary of the diagram contributes $2b$ generators (denoted by $z_{i,1}, \ldots, z_{i,2b}$ for the horizontal edge in position $iq$, where $1 \leq i \leq p$), and similarly each edge along the right boundary contributes $2a$ generators $(y_{j,1}, \ldots, y_{j,2a}$ for the vertical edge in position $p_j$, where $1 \leq j \leq q$).

We are in fact interested in the stabilized quotient $G$ of $\pi_1(\mathbb{C}^2 - D)$ (see Definition 2.2), which can be expressed in terms of the same generators by adding suitable commutation relations. Let $\Gamma$ be the subgroup of $G$ generated by the $\gamma_i, \gamma_i'$, and let $\tilde{\Gamma}$ be the subgroup generated by the $\tilde{\gamma}_i, \tilde{\gamma}_i'$. By definition, the elements of $\Gamma$ always commute with those of $\tilde{\Gamma}$, because the images by the geometric monodromy representation $\theta$ of the geometric generators $\gamma_i, \gamma_i'$ and $\tilde{\gamma}_i, \tilde{\gamma}_i'$ act on two disjoint sets of $n/2 = 2pq$ sheets of the branched cover $f$.

As in §6, we introduce twisted generators $e_i, e_i'$ and $\tilde{e}_i, \tilde{e}_i'$ for $\Gamma$ and $\tilde{\Gamma}$, by choosing integers $l_i$ satisfying the same compatibility conditions at the inner vertices as in §6, and setting as previously $e_i = \rho_i^l(\gamma_i), e_i' = \rho_i^l(\gamma_i'), \tilde{e}_i = \tilde{\rho}_i^l(\tilde{\gamma}_i)$ and $\tilde{e}_i' = \tilde{\rho}_i^l(\tilde{\gamma}_i')$, with the obvious definition for $\rho_i$ and $\tilde{\rho}_i$. Even though this could be avoided by proving a suitable invariance property, we will assume that $l_i = 1$ for every diagonal edge in the top-most row or in the right-most column of the diagram (so $e_i = \gamma_i, \tilde{e}_i = \tilde{\gamma}_i$),

![Figure 9](image-url)
and $l_j = 0$ for every vertical edge in the top-most row and every horizontal edge in the right-most column (so $e_j = \gamma_j$, $e_j' = \tilde{\gamma}_j$). Finally, as in §6.1 we let $a_i = e_i' e_i^{-1}$ and $\tilde{a}_i = e_i' \tilde{e}_i^{-1}$, and we relabel these elements as $d_{ij}, v_{ij}, h_{ij}$ (resp. $\tilde{d}_{ij}, \tilde{v}_{ij}, \tilde{h}_{ij}$) according to their position in the diagram.

**Lemma 7.1.** The subgroup $\mathcal{B}_\Gamma \subset \Gamma$ generated by the $e_i$ and the subgroup $\tilde{\mathcal{B}}_\Gamma \subset \tilde{\Gamma}$ generated by the $\tilde{e}_i$ are naturally isomorphic to quotients of $\mathcal{B}_{g/2}$. Moreover, the subgroups $\Gamma$ and $\tilde{\Gamma}$ of $G$ are naturally isomorphic to quotients of $\tilde{\mathcal{B}}_{g/2}$, with geometric generators corresponding to pairs of half-twists. Furthermore, $\Gamma$ is generated by the elements of $\mathcal{B}_\Gamma$ and $v_1$, and $\tilde{\Gamma}$ is generated by the elements of $\tilde{\mathcal{B}}_\Gamma$ and $\tilde{v}_1$.

**Proof.** We first look at relations corresponding to the interior vertices of the diagram (Figure 5) and to the vertices along the bottom and left boundaries. Since the local description of $D$ at these vertices simply consists of two superimposed copies of $D_g$, and since the generators of $\Gamma$ commute with those of $\tilde{\Gamma}$, one easily checks that the local configurations yield relations among the $e_i, e_i'$ that are exactly identical to those discussed in §6 in the case of $\mathbb{CP}^1 \times \mathbb{CP}^1$; additionally, an identical set of relations also holds among the $\tilde{e}_i, \tilde{e}_i'$.

Next we consider the local configuration at a vertex along the top boundary of the diagram, and more precisely the cusp singularities present near the point labelled $A$ on Figure 6, as pictured on Figure 7. Denoting by $i$ and $j$ respectively the labels of the diagonal and vertical edges meeting at the given vertex, the relations corresponding to these six cusps are

\begin{align}
\gamma_i' \gamma_j' \gamma_i' &= \gamma_j \gamma_i \gamma_j, \\
\gamma_i' \gamma_j' \gamma_i' &= \gamma_j' \gamma_i' \gamma_j, \\
\gamma_i' (\gamma_j^{-1} \gamma_j' \gamma_j) \gamma_i' &= (\gamma_j^{-1} \gamma_j' \gamma_j) \gamma_i' (\gamma_j^{-1} \gamma_j' \gamma_j), \\
\gamma_i' \gamma_j' \gamma_i' &= \gamma_j \gamma_i \gamma_j, \\
\gamma_i' \gamma_j' \gamma_i' &= \gamma_j' \gamma_i' \gamma_j, \\
\gamma_i' (\gamma_j^{-1} \gamma_j' \gamma_j) \gamma_i' &= (\gamma_j^{-1} \gamma_j' \gamma_j) \gamma_i' (\gamma_j^{-1} \gamma_j' \gamma_j).
\end{align}

It can easily be checked that these relations satisfy a property of invariance under twisting similar to that of 3-points. In fact, replacing the various generators by their images under arbitrary powers of the twisting actions $\rho_i, \tilde{\rho}_i, \rho_j, \tilde{\rho}_j$ amounts to a conjugation of the relations (7.1) by braids belonging to the local monodromy (either the entire local monodromy, or two of the six cusps near $A$, or combinations thereof), and thus always yields valid relations.

Therefore, the twisted generators $e_i, e_i', e_j, e_j'$ of $\Gamma$ satisfy the relations (6.2), and similarly for $\tilde{e}_i, \tilde{e}_i', \tilde{e}_j, \tilde{e}_j'$ in $\tilde{\Gamma}$. One easily checks that a similar conclusion holds for pairs of inner edges meeting at a vertex along the right boundary of the diagram (recall that the local braid monodromy only differs by a simple conjugation). Finally, because we are looking at the stabilized fundamental group, the commutation relations discussed in §6 automatically hold in $\Gamma$ and $\tilde{\Gamma}$.

So, except for the equality relations arising from vertical tangencies at the vertices along the top and right boundaries of the diagram, all the relations described in §6.1 for the case of $\mathbb{CP}^1 \times \mathbb{CP}^1$ simultaneously hold in $\Gamma$ and in $\tilde{\Gamma}$. Therefore, the structure of $\Gamma$ and $\tilde{\Gamma}$ can be studied by the same argument as in the case of $\mathbb{CP}^1 \times \mathbb{CP}^1$ ([9], see also §6), which yields the desired result.

**Lemma 7.2.** The equality $z_{r,i} = z_{r,1}$ holds for every $1 \leq r \leq p$, $1 \leq i \leq 2b$; similarly, $y_{r,i} = y_{r,1}$ for every $1 \leq r \leq q$, $1 \leq i \leq 2a$. Moreover, the $y_{r,i}$ and the $z_{r,i}$ are all conjugates of $y_{0,1}$ under the action of elements of $\mathcal{B}_\Gamma$ and $\tilde{\mathcal{B}}_\Gamma$.

**Proof.** First consider the corner vertex at position $pq$, and more precisely the half-twists $\tau_{ij}$ arising from the vertical tangencies of the local model near this vertex.
(Figure 9). Denoting by $\mu$ the label of the diagonal edge in position $pq$, the half-twist $\tau_{i_1}$ yields the relation $(y_{q,1}^{-1} \ldots y_{q,2a}^{-1} y_{p,1}^{-1} \ldots y_{p,1-i}^{-1} z_{p,i}(z_{p,i-1}^{-1} \ldots z_{p,1}^{-1})(y_{q,2a} \ldots y_{q,1})) = \gamma_{\mu}^{i_1} \gamma_{\nu}^{i_1} y_{q,i}^{-1} \gamma_{\mu}^{-i_1} \gamma_{\nu}^{-i_1}$. It follows that the quantity $(z_{p,1}^{-1} \ldots z_{p,1-i}^{-1})z_{p,i}(z_{p,i-1}^{-1} \ldots z_{p,1}^{-1})$ is independent of $i$, by an easy induction on $i$ implies that $z_{p,i} = z_{p,1}^{-1}$ for all $i$. Observing that $y_{q,1}, \ldots, y_{q,2a}$ and $z_{p,1}, \ldots, z_{p,26}$ are mapped by $\theta$ to disjoint transpositions and hence commute in $G$, we in fact have $z_{p,i} = z_{p,1}^{-1} \gamma_{\mu}^{i_1} \gamma_{\nu}^{i_1} \gamma_{\mu}^{-i_1} \gamma_{\nu}^{-i_1}$ for all $i$. Since by assumption the twisting parameter $l_{\mu}$ is equal to 1, the generators $\gamma_{\mu} = e_{\mu}$ and $\gamma_{\mu}^{-1} = e_{\mu}$ belong to $B_{\Gamma}$ and $\tilde{B}_{\Gamma}$. This proves the claims made about the $z_{p,i}$.

Similarly comparing the relations corresponding to the half-twists $\tau_{i_1}$, it can be seen immediately that the quantity $(y_{q,1}^{-1} \ldots y_{q,1-i}^{-1})y_{q,i}(y_{q,1-i}^{-1} \ldots y_{q,1})$ is independent of $i$, which implies that $y_{q,i} = y_{q,1}$ for all $i$.

We now proceed by induction: assume that $z_{r+1,i} = z_{r+1,1}$ for all $i$, and that $z_{r+1,1}$ is a conjugate of $y_{q,1}$ under the action of $B_{\Gamma}$ and $\tilde{B}_{\Gamma}$. Let $\mu$ and $\nu$ be the labels of the diagonal and vertical edges meeting at the vertex in position $r$, and let $\psi_{r} = \gamma_{\nu}^{l_{\nu}} \gamma_{\mu}^{l_{\mu}}$, $\tilde{\psi}_{r} = \psi_{r}^{-1}$, $\tilde{\psi}_{r}^{*} = \tilde{\psi}_{r}^{-1}$, $\beta_{r} = \psi_{r} \gamma_{\nu}^{l_{\nu}} \psi_{r}^{-1}$, $\beta_{r}^{*} = \beta_{r}^{-1}$, $\tilde{\beta}_{r} = \tilde{\psi}_{r} \gamma_{\mu}^{l_{\mu}} \tilde{\psi}_{r}^{-1}$, $\tilde{\beta}_{r}^{*} = \tilde{\beta}_{r}^{-1}$. Defining $\zeta_{r} = \psi_{r} \gamma_{\nu}^{l_{\nu}} \gamma_{\mu}^{l_{\mu}}$, $\zeta_{r}^{*} = \zeta_{r}^{-1}$, $\tilde{\zeta}_{r} = \tilde{\psi}_{r} \gamma_{\mu}^{l_{\mu}} \tilde{\psi}_{r}^{-1}$, $\tilde{\zeta}_{r}^{*} = \tilde{\zeta}_{r}^{-1}$. Similar calculations for the other elements yield that:

$$
\zeta_{r} = \gamma_{\mu}^{-l_{\mu}}(\gamma_{\nu}^{-l_{\nu}} \gamma_{\mu}^{l_{\mu}}) \gamma_{\mu}^{l_{\mu}}, ~ \tilde{\zeta}_{r} = \gamma_{\nu}^{-l_{\nu}}(\gamma_{\mu}^{-l_{\mu}} \gamma_{\nu}^{l_{\mu}}) \gamma_{\nu}^{l_{\nu}}, ~ \zeta_{r}^{*} = \gamma_{\mu}^{-l_{\mu}} \gamma_{\nu}^{l_{\mu}}, ~ \tilde{\zeta}_{r}^{*} = \gamma_{\nu}^{-l_{\nu}} \gamma_{\mu}^{l_{\mu}}.
$$

Due to the choice of twisting parameters $l_{\mu} = 1$ and $l_{\nu} = 0$, $\zeta_{r} \in B_{\Gamma}$ and $\tilde{\zeta}_{r} \in \tilde{B}_{\Gamma}$.

Since the $z_{s,r}$ commute with the $z_{r+1,i}$ in $G$ (they are mapped to disjoint transpositions by $\theta$), and since by assumption $z_{r+1,i} = z_{r+1,1}$ for all $i$, we have:

$$(z_{r+1}^{-1} \ldots z_{r+1,i}^{-1})z_{r+1,i}(z_{r+1,i-1}^{-1} \ldots z_{r+1,1}^{-1}) = z_{r+1,1}$$

for all $i$. Therefore, the relation arising from the vertical tangency $\tau_{i_1}^*$ (Figure 8) at the vertex $rq$ can be written in the form:

$$z_{r+1,i} = \tilde{\zeta}_{r}^{*} \zeta_{r}^{*} (z_{r+1}^{-1} \ldots z_{r+1,i-1})z_{r+1,i}(z_{r+1,i-1}^{-1} \ldots z_{r+1,1}^{-1}) \zeta_{r}^{*} \tilde{\zeta}_{r}^{*}.$$

In particular, the value of $(z_{r+1}^{-1} \ldots z_{r+1,i-1})$ does not depend on $i$, which implies that $z_{r,i} = z_{r,1}$ for all $i$. Moreover, we have $z_{r,i} = \zeta_{r}^{*} \zeta_{r}^{*} \tilde{\zeta}_{r}^{*} \tilde{\zeta}_{r}^{*}$. So, by induction on decreasing values of $r$, we obtain the desired results about $z_{r,i}$.

The case of $y_{r,i}$ is handled using exactly the same argument, going inductively through the vertices along the right boundary of the diagram. Indeed, observe that the local braid monodromy at one of these vertices simply differs from that at a vertex along the top boundary by a conjugation which exchanges the positions of two groups of geometric generators; however, because the corresponding transpositions in $S_n$ are disjoint, these generators commute with each other in $G$, so that the relations induced by the local braid monodromy can be expressed in exactly the same form.

\begin{lemma}
The element $i_{11}$ belongs to the subgroup of $G$ generated by $\Gamma$, $B_{\Gamma}$, and $y_{q,1}$.
\end{lemma}

\begin{proof}
Consider the local relations for the vertex at position $1q$, and more precisely the equality relation corresponding to the half-twist labelled $\tau_{i_1}^*$ in Figure 8; with
the same notations as in the proof of Lemma 7.2, we have $z_{2,1} = \zeta_1^{-1}\zeta_1^{-1}z_{1,1}\zeta_1\zeta_1$. Moreover, the cusp point with monodromy $\kappa_1^0$ pictured on Figure 7 yields the relation $\zeta_1 z_{1,1}\zeta_1 = z_{1,1}\zeta_1 z_{1,1}$. It follows that $z_{2,1} = \zeta_1^{-1}z_{1,1}\zeta_1^{-1}\zeta_1$. Therefore, using formula (7.2) for $\zeta_1$, we obtain $\zeta_\nu' = \zeta_\nu''\zeta_\nu''z_{1,1}^{-1}\zeta_1 z_{2,1}^{-1}\zeta_1^{-1}z_{1,1}^{-1}\zeta_\nu''^{-1}z_\nu''^{-1}$, where $\mu$ and $\nu$ are the labels of the two interior edges meeting at the considered vertex.

Observe that, since $l_\nu = 0$ and $l_\mu = 1$, the generators $\tilde{\gamma}_\nu = \tilde{e}_\nu$ and $\tilde{\gamma}_\mu = \tilde{e}_\mu$ belong to $B_\Gamma$. Moreover, it is obvious from (7.2) that $\zeta_1 \in \Gamma$. Using the result of Lemma 7.2 to express $z_{1,1}$ and $z_{2,1}$ in terms of $y_{q,1}$, it follows that $\tilde{\gamma}_\nu'' = \tilde{\gamma}_\nu''$ belongs to the subgroup of $G$ generated by $\Gamma$, $B_\Gamma$, and $y_{q,1}$. Therefore, $\tilde{v}_{1,q} = \tilde{e}_{\nu''}^{-1}$ belongs to this subgroup. Finally, the local relations analogous to (6.5) for the $\tilde{v}_{1,r}$ imply that $\tilde{v}_{1,q}$ and $\tilde{v}_{1,r+1}$ are conjugates of each other under the action of elements of $B_\Gamma$. Therefore, by induction $\tilde{v}_{1,1}$ can be expressed in terms of $\tilde{v}_{1,q}$ and elements of $B_\Gamma$, which completes the proof.

Lemma 7.4. The subgroup $B$ of $G$ generated by $B_\Gamma$, $B_\Gamma$, and $y_{q,1}$ is naturally a quotient of $\tilde{B}_n$, with geometric generators corresponding to half-twists.

Proof. We construct a surjective map $\alpha : \tilde{B}_n \rightarrow B$ as follows (recall that $n = 4pq$). First observe that the subgroup of $\tilde{B}_n$ generated by the half-twists $x_1, \ldots, x_{2pq-1}$ is naturally isomorphic to $\tilde{B}_{n/2}$, which by Lemma 7.1 admits a surjective homomorphism to $B_\Gamma$ mapping half-twists to geometric generators. We use this homomorphism to define $\alpha(x_i)$ for $1 \leq i \leq 2pq - 1$. Any two half-twists in $\tilde{B}_{n/2}$ are conjugate to each other; therefore, after a suitable conjugation we can assume that $\alpha(x_{2pq-1}) = e_\mu$, where $\mu$ is the label of the diagonal edge at position $pq$ in the diagram, and that the other $\alpha(x_i)$ (for $i \leq 2pq - 2$) are geometric generators mapped by $\theta$ to transpositions disjoint from $\theta(y_{q,1})$. Because of the stabilization process, this last requirement implies that $\alpha(x_i)$ commutes with $y_{q,1}$ for $i \leq 2pq - 2$.

Similarly, the subgroup of $\tilde{B}_n$ generated by $x_{2pq+1}, \ldots, x_{n-1}$ is naturally isomorphic to $\tilde{B}_{n/2}$ and admits a surjective homomorphism to $B_\Gamma$, which we use to define $\alpha(x_i)$ for $2pq + 1 \leq i \leq n - 1$. Once again, without loss of generality we can assume that $\alpha(x_{2pq+1}) = \tilde{e}_\mu$ and that the other $\alpha(x_i)$ commute with $y_{q,1}$. Finally, we define $\alpha(x_{2pq}) = y_{q,1}$.

All that remains to be checked is that $\alpha$ can be made into a group homomorphism (obviously surjective by construction), i.e. that the relations defining $\tilde{B}_n$ are also satisfied by the chosen images $\alpha(x_i)$ in $B$. Since $\alpha$ is built out of two group homomorphisms and since the elements of $B_\Gamma$ commute with those of $B_\Gamma$, the only relations to be checked are those involving $x_{2pq}$.

Consider the corner vertex at position $pq$ in the diagram: the cusp singularities arising from the regeneration of the rightmost tangent intersection of $D_q$ with $\gamma(C)$ in Figure 9 imply the relations $\gamma^\mu_\mu y_{q,1} \gamma^\mu_\mu = y_{q,1}^\mu y_{q,1}$ and $\gamma^\mu_\mu y_{q,1} \gamma^\mu_\mu = y_{q,1}^\mu y_{q,1}$. Since $l_\mu = 1$, we have $\gamma^\mu_\mu = e_\mu$ and $\gamma^\mu_\mu = \tilde{e}_\mu$, so that these relations can be rewritten as $\alpha(x_{2pq-1}) \alpha(x_{2pq}) \alpha(x_{2pq-1}) = \alpha(x_{2pq}) \alpha(x_{2pq-1}) \alpha(x_{2pq})$ and $\alpha(x_{2pq+1}) \alpha(x_{2pq}) \alpha(x_{2pq+1}) = \alpha(x_{2pq}) \alpha(x_{2pq+1}) \alpha(x_{2pq+1})$. Finally, for all $i$ such that $|i - 2pq| \geq 2$, the relation $\alpha(x_{2pq}), \alpha(x_i) = 1$ holds by construction. Therefore, $\alpha$ defines a surjective group homomorphism from $\tilde{B}_n$ to $B$, mapping half-twists to geometric generators.
Proposition 7.5. The morphism $\alpha$ extends to a surjective group homomorphism from $\tilde{B}_n^{(2)} \simeq B_n \times \tilde{P}_{n,0}$ to $G$ mapping pairs of half-twists to geometric generators. In particular, the group $G$ has property ($\ast$).

Proof. Lemma 7.2 implies that $G$ is generated by $\Gamma$, $\tilde{\Gamma}$, and $y_{0,1}$. Therefore, by Lemma 7.1, $G$ is generated by $B$, $v_{11}$, and $y_{0,1}$, while Lemma 7.3 implies that $v_{11}$ can be eliminated from the list of generators. Since Lemma 7.4 identifies $B$ with a quotient of $\tilde{B}_n$, the main remaining task is to check that the subgroup $\mathcal{P}$ generated by the $g^{-1}v_{11}g, g \in B$, is naturally isomorphic to a quotient of $\tilde{P}_{n,0}$. This can be done by proving that $\mathcal{P}$ is a primitive $\tilde{B}_n$-group (Definition 5 of [9]), as it follows from the discussion in §1 of [9] that every such group is a quotient of $\tilde{P}_{n,0}$ (compare Propositions 1, 2, 3 of [9] with the presentation of $\tilde{P}_{n,0}$ given in Lemma 3.1).

As stated in Lemma 7.1, the arguments of [9] show that the subgroup generated by the $g^{-1}v_{11}g, g \in B_1$, is a primitive $\tilde{B}_{n/2}$-group (and hence a quotient of $\tilde{P}_{n/2,0}$). The desired result about $\mathcal{P}$ then follows simply by observing that $v_{11}$ commutes with $y_{0,1}$ and with the generators of $B_1$ and using a criterion due to Moishezon (Proposition 6 of [9]): indeed, an obvious corollary of this criterion is that, upon enlarging the conjugation action from $B_{n/2}$ to $\tilde{B}_n$, it is sufficient to check that the additional half-twist generators act trivially on the given prime element ($v_{11}$).

Since $G$ is obviously generated by its subgroups $B$ and $\mathcal{P}$, and since $\mathcal{P}$ is normal, it is naturally a quotient of $\tilde{B}_n \times \tilde{P}_{n,0} \simeq \tilde{B}_n^{(2)}$. Moreover, the geometric generators of $G$ are all mutually conjugate (because the curve $D$ is irreducible), and by construction the $e_i$ (and $\bar{e}_i$) correspond to pairs of half-twists in $\tilde{B}_n^{(2)}$, so the same is true of all geometric generators. Finally, by going carefully over the construction, it is not hard to check that the end points of the half-twists $(x, y)$ corresponding to a given geometric generator $\gamma$ are always the natural ones, in the sense that $\sigma(x) = \sigma(y) = \theta(\gamma)$. Therefore, $G$ has property ($\ast$).

At this point, the only remaining task in the proof of Theorem 4.6 is to characterize the kernel of the surjective morphism $\alpha : \tilde{B}_n^{(2)} \to G$ given by Proposition 7.5. As a consequence of Lemmas 3.3 and 3.4, the commutation relations induced either by nodes in the branch curve $D$ or by the stabilization process, as well as the relations induced by the cusp points of $D$, automatically hold, so that $\text{Ker} \alpha$ is generated by equality relations between pairs of half-twists induced by the vertical tangencies of $D$. Moreover, as in §6.1.2 the classification of half-twists in $\tilde{B}_n$ (Lemma 3.2) allows us to associate to every $a_i$ (resp. $\tilde{a}_i$) a pair of integers $\hat{a}_i$ (resp. $\tilde{a}_i$), well-defined modulo the subgroup $\Lambda = \{(\kappa, \lambda), (u_1^\kappa \eta^{(\kappa-1)/2}, u_2^\lambda \eta^{(\lambda-1)/2}) \in \text{Ker} \alpha \} \subset \mathbb{Z}^2$. Recall however from §6.1.2 that this construction requires us to choose an ordering of the $n = 4pq$ sheets of the branched cover; in our case, these split into two sets of $2pq$ sheets, the first one on which the $\theta(e_i), \theta(\tilde{e}_i)$ act by permutations, and the second one on which the $\theta(e_i), \theta(\tilde{e}_i)$ act by permutations. The ordering we will consider is obtained by enumerating first the first set of $2pq$ sheets, and then the second one. In each set, the sheets are naturally in correspondence with the $2pq$ triangles of the diagram in Figure 5: the ordering we choose for each of the two sets of $2pq$ sheets is obtained as in the case of $\mathbb{CP}^1 \times \mathbb{CP}^1$ [9] by enumerating the $2pq$ triangles of the diagram from left to right and from bottom to top.

We have seen above that the relations coming from the vertical tangencies at the inner vertices of the diagram and at those along the lower and left boundaries are exactly the same as in the case of $\mathbb{CP}^1 \times \mathbb{CP}^1$, except they simultaneously apply to
the generators of $\Gamma$ and to those of $\tilde{\Gamma}$. Therefore, as in §6.1.2, these relations do not contribute to $\text{Ker } \alpha$ by themselves, but they translate into equalities between the $\tilde{a}_i$ (and similarly between the $\tilde{a}_i$), which yield the following formulas (with the obvious notations): $\tilde{a}_{i,j} = \tilde{d}_{i,j} = (j - i, 0)$, $\tilde{v}_{i,j} = \tilde{h}_{i,j} = (1 - i, 1)$, $\tilde{h}_{i,j} = (1 - j, 1)$ (compare with (6.9)).

Next, we consider the corner vertex at position $pq$, for which the braid monodromy contribution of the vertical tangencies is represented in Figure 9. Recall that some of the half-twists $\tau_{ij}$ were used in the proof of Lemma 7.2 to eliminate $y_{q,2}, \ldots, y_{q,2a}$ and $z_{p,1}, \ldots, z_{p,2b}$ from the list of generators by expressing them in terms of $y_{q,1}$; however, since these relations imply that $y_{q,1} = y_{q,1}$ and $z_{p,i} = z_{p,1}$ (cf. Lemma 7.2), all the other relations coming from the $\tau_{ij}$ become redundant. Therefore these equality relations do not make any contributions to the kernel of $\alpha$. We are left with the two half-twists $t, \tilde{t}$ of Figure 9.

Denote by $\mu$ the label of the diagonal edge passing through the corner vertex. Because $G$ has property $(*)$, and using the results of §3, we can find an element $g \in \tilde{B}_\alpha^{(2)}$ such that $z_{p,1} = \alpha(g^{-1}(x_1, x_1)g)$, $e_\mu = \gamma_\mu = \alpha(g^{-1}(x_2, x_2)g)$, and $y_{q,1} = \alpha(g^{-1}(x_3, x_3)g)$. Recalling that $d_{p,q} = (q - p, 0)$ and observing that the conjugation by $g$ preserves the ordering of the end points for $e_\mu$, by definition of $d_{p,q}$ we have $e_\mu = \alpha(g^{-1}(x_2 u_2^{-p} q y_1^{(q-p)(q-p-1)/2}, x_2)g)$, and therefore $\gamma_\mu = e_\mu e_\mu e_\mu = \alpha(g^{-1}(x_2 u_2^{-p} q y_1^{(q-p)(q-p-1)/2}, x_2)g)$. The half-twist $t$ yields the relation $\gamma_\mu = \alpha(g^{-1}(x_2 u_2^{-p} q y_1^{(q-p)(q-p-1)/2}, x_2)g)$. A similar calculation shows that the relation introduced by \( \tilde{t} \) can also be rewritten in the form $(a - b + p - q, a - b) \in \Lambda$. A similar calculation shows that the relation introduced by $\tilde{t}$ can also be rewritten in the form $(a - b + p - q, a - b) \in \Lambda$.

We now consider the vertex at position $rq$ $(1 \leq r \leq p - 1)$, and investigate in the same manner the equality relations coming from the vertical tangencies $\tau_1', \tau_1''$, $t, \tilde{t}$ represented in Figure 8. Recall that the relations induced by $\tau_1'$ were used in the proof of Lemma 7.2 to show that $z_{r,i} = \zeta_1^{-r-1} \zeta_2^{-r-1} z_{r+1,i} \zeta_2 \zeta_1$ and consequently eliminate the $z_{r,i}$ from the list of generators; these relations are therefore already accounted for. Next, we turn to the relation induced by $\tau_1''$, which taking into account that $z_{r,i} = z_{r,1}$ and $z_{r+1,i} = z_{r+1,1}$ can be written in the form $z_{r+1,1} = \zeta_2 \zeta_1 z_{r+1} z_{r,1}$. Using the expression of $z_{r,1}$ in terms of $z_{r+1,1}$, this identity can also be expressed by the commutation relation $[\zeta_1, \zeta_1, \zeta_1] = 1$. Recalling that $z_{r+1,1}$ commutes with $e_\mu$ and $e_\nu$, the relation can then be rewritten as $[z_{r+1,1}, e_\mu, e_\nu] = 1$. Taking into account the ordering of the sheets of the branched cover, an easy calculation in $\tilde{B}_\alpha^{(2)}$ shows that this relation automatically holds as a consequence of the equality $v_{r,q} = \tilde{v}_{r,q}$.

The relation induced by the half-twist $t$ (Figure 8) can be expressed as $\gamma_\mu = z_{r,1}^{-1} \zeta_{1r}^{-1} \zeta_{1r}^{-1} \zeta_{1r}^{-1} \zeta_{1r}^{-1} z_{r,1}$. Using property $(*)$ and recalling that $d_{r,q} = (q - r, 0)$ and $\tilde{v}_{r,q} = (1 - r, 1)$, we can find $g \in \tilde{B}_\alpha^{(2)}$, preserving the ordering of the end points for $e_\mu$ and $e_\nu$, such that $z_{r,1} = \alpha(g^{-1}(x_1, x_1)g)$, $\gamma_\mu = e_\mu = \alpha(g^{-1}(x_2, x_2)g)$, $\gamma_\nu = e_\nu = \alpha(g^{-1}(x_3, x_3)g)$, and $\gamma_\mu = e_\mu = \alpha(g^{-1}(x_2 u_2^{-p} q y_1^{(q-p)(q-p-1)/2}, x_2)g)$. So $z_{r,1}^{-1} \zeta_{1r}^{-1} \zeta_{1r}^{-1} \zeta_{1r}^{-1} \zeta_{1r}^{-1} z_{r,1}$
is equal to $\alpha(g^{-1}(xu_2^{-2}r^{-b}y^{2-r-b}+x^{-2}u_2^{-b}y^{(2-b)(1-b)/2}g))$. Comparing this with the expression for $\gamma_\mu$, it becomes apparent that the relation induced by $t$ is in fact equivalent to the condition $(q+b-2, b-2) \in \Lambda$. A similar calculation for the half-twist $\tilde{t}$ shows that the relation it induces can also be expressed in the form $(q+b-2, b-2) \in \Lambda$.

Finally, the case of the vertices along the right boundary of the diagram can be studied by exactly the same argument; the relations corresponding to the vertical tangencies of the local model can be expressed by the single requirement that $(p+a-2, a-2) \in \Lambda$.

Therefore, $\Lambda \subset \mathbb{Z}^2$ is the subgroup generated by $(p+a-2, a-2)$ and $(q+b-2, b-2)$, and $\text{Ker}\, \alpha$ is the normal subgroup of $\tilde{B}_n^{(2)}$ generated by the two elements $g_1 = (u_1^{p+a-2}y^{\lambda(p+a-2)}, u_1^{-2}y^{\lambda(a-2)})$ and $g_2 = (u_1^{q+b-2}y^{\lambda(q+b-2)}, u_1^{-2}y^{\lambda(b-2)})$, where $\lambda(i) = i(i-1)/2$. Observe that $G_{p,q}^0 = (\tilde{P}_{n,0} \times \tilde{P}_{n,0})/\text{Ker}\, \alpha$, and recall from Lemma 3.1 that $[\tilde{P}_{n,0}, \tilde{P}_{n,0}] = \{1, \eta\} \cong \mathbb{Z}_2$ and $\text{Ab}\, \tilde{P}_{n,0} \cong \mathbb{Z}^{n-1}$.

We first consider the commutator subgroup $[G_{p,q}^0, G_{p,q}^0] \cong C/(C \cap \text{Ker}\, \alpha)$, where $C = \{1, \eta\} \times \{1, \eta\}$. First of all, if $a \mp b$ is odd, then considering the commutator of $g_1$ with $(u_2, 1)$ we obtain that $(\eta, 1) \in \text{Ker}\, \alpha$, and similarly if $b \mp q$ is odd; otherwise, one easily checks that $(\eta, 1) \notin \text{Ker}\, \alpha$. Moreover, if $a$ is odd, then considering the commutator of $g_1$ with $(1, u_2)$ we obtain that $(1, \eta) \in \text{Ker}\, \alpha$, and similarly if $b$ is odd; when $a$ and $b$ are both even, $(1, \eta) \notin \text{Ker}\, \alpha$. Also, it is easy to check that $\text{Ker}\, \alpha$ only contains $(\eta, \eta)$ if it also contains $(\eta, 1)$ and $(1, \eta)$. The claim made in the statement of Theorem 4.6 about the structure of $[G_{p,q}^0, G_{p,q}^0]$ follows.

Finally, we have $\text{Ab}\, G_{p,q}^0 \cong (\tilde{P}_{n,0} \times \tilde{P}_{n,0})/\langle C, \text{Ker}\, \alpha \rangle \cong (\mathbb{Z}^2/\Lambda)^{n-1}$. Observing that $\mathbb{Z}^2/\Lambda = \mathbb{Z}^2/\langle (p+a-2, a-2), (q+b-2, b-2) \rangle \cong \mathbb{Z}^2/\langle (p, a-2), (q, b-2) \rangle$, this completes the proof of Theorem 4.6.

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