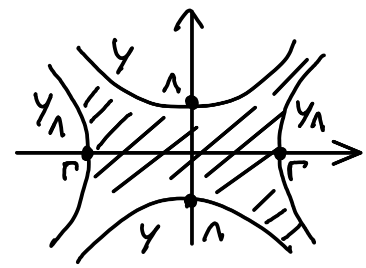
T. Ekholm - 30/5/08 - Surgery long exact sequence (continued)

Recall:

$\Lambda \subset Y$ Legendrian sphere

$\rightsquigarrow$ Legendrian surgery gives Y_Λ,

and $\Gamma \subset Y_\Lambda$ Leg. sphere



Reeb orbits in Y_Λ $\longleftrightarrow$ $\begin{cases} \text{Reeb orbits in } Y \\ \text{cyclic words of Reeb chords of } \Lambda \end{cases}$

Reeb chords of Γ $\longleftrightarrow$ words of Reeb chords of Λ

Note: c Reeb chord in Λ of odd degree $\Rightarrow$ $c^2 = 0$ in cyclic ($\rightsquigarrow$ eliminates bad orbits ...).

Mixed complex: $C(\Lambda, Y; \varepsilon)$ generated by $\begin{cases} \text{Reeb orbits in } Y \\ \text{cyclic words of Reeb chords of } \Lambda \end{cases}$

& its "stabilization" $S(C(\Lambda, Y; \varepsilon)) = C(\Lambda, Y; \varepsilon) \oplus \mathbb{Q}\langle \tau_L, 1_\Lambda \rangle$

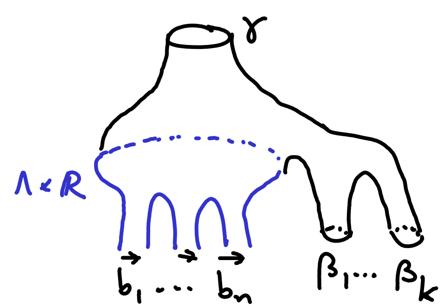
$\delta : C(\Lambda, Y; \varepsilon) \circlearrowleft$ defined by:

- for w = word of Reeb chords,

  $\delta w = \partial^c w$ cyclic differential on rel. contact homology
- for γ Reeb orbit, $\delta\gamma = \delta^{(0)}\gamma + \delta^{(1)}\gamma$ where

  $\delta^{(0)}\gamma = d\gamma$ linearized contact homology (wrt ε)

  $\delta^{(1)}\gamma$ counts curves

$\mathcal{M}(\gamma, \underline{\beta}, \underline{b}) =$ 

⋮

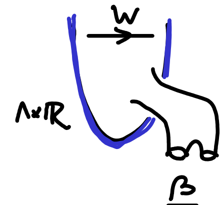
$$\delta^{(1)}\gamma = \sum_{\{\underline{\beta},\underline{b}\,/\,\dim \mathcal{M}=1\}} k(\underline{\beta})^{-1}\cdot \underbrace{\varepsilon(\underline{\beta})}_{\text{augmentation}} \cdot \left|\mathcal{M}(\gamma,\underline{\beta},\underline{b})/\mathbb{R}\right| \cdot \underbrace{\underline{b}}_{\text{nonempty cyclic word!}}$$

$k(\overline{\beta})$ = product of multiplicities of β_i

On stabilized complex, $S(C(Y,\Lambda,\varepsilon)) = C \oplus \mathbb{Q}\langle 1_\Lambda, \tau_L\rangle$:

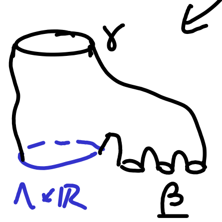
- $(S\delta)\,w = \partial^c w + (\partial w)_0\, 1_\Lambda$

  ie count not only words in $A(\Lambda,\varepsilon)^{\text{cycl.}}_{\geq 1}$ as above but also count empty word of Reeb chords



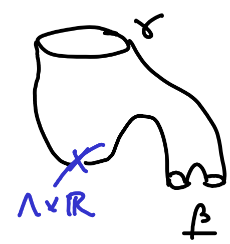
- $(S\delta)\,\tau_L = 1_\Lambda$
- $(S\delta)(\gamma) = \delta^{(0)}\gamma + \delta^{(1)}\gamma + \delta^{(2)}\gamma + \delta^{(3)}\gamma$

$\delta^{(2)}\gamma$: allow $\overline{b}$ = empty word, ie. & count with coeff: 1_Λ



$$\delta^{(3)}\gamma = \sum_{\underline{\beta}/\dim\mathcal{M}=1} k(\underline{\beta})^{-1}\varepsilon(\underline{\beta})\,\left|\mathcal{M}/\mathbb{R}\right|\, \tau_L$$

counts

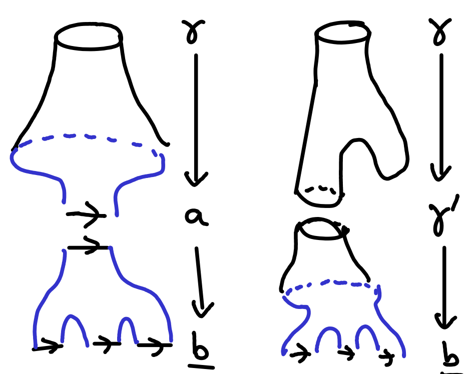


(NB: τ_L = will correspond to $L \subset Y_\Lambda$ core of handle - caps $\Lambda\times\mathbb{R}$)

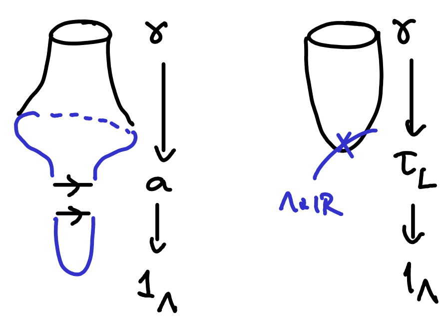
Thm: $\delta^2 = 0$, $(S\delta)^2 = 0$

Pf: look at ends of 1-dim! moduli spaces: (w/out drawing ends β_i plugged by ε's ...)

For δ:



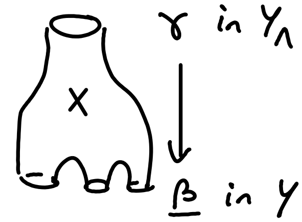
+ in $S\delta$, also:



- Also want a <u>chain map</u> $\Psi : Q_1(Y_\Lambda, \varepsilon_\Lambda) \longrightarrow C(Y,\Lambda;\varepsilon)$
$S\Psi : Q_1(Y_\Lambda, \varepsilon_\Lambda) \longrightarrow SC(Y,\Lambda;\varepsilon)$

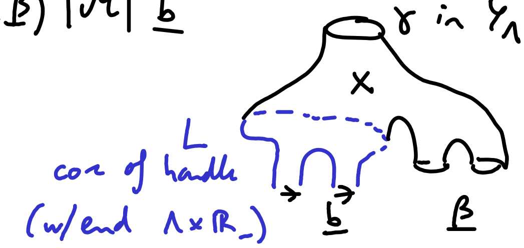
γ Reeb orbit in Y_Λ $\Rightarrow$ $\Psi(\gamma) = \underbrace{\Psi^{(0)}(\gamma)}_{\text{R. orbits}} + \underbrace{\Psi^{(1)}(\gamma)}_{\text{c-words of R-chords}}$

- $\Psi^{(0)}(\gamma) = \Phi_X(\gamma)$ SFT cobordism map for cobordism $X : Y \to Y_\Lambda$ (counts rigid holom. curves)



- $\Psi^{(1)}(\gamma) = \sum_{\dim \mathcal{M}(\gamma,\underline{b},\underline{\beta})=0} \kappa(\underline{\beta})^{-1}\varepsilon(\underline{\beta})\,|\mathcal{M}|\,\underline{b}$

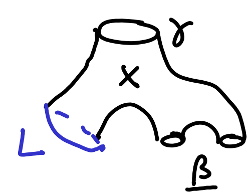
(<u>b</u> nonempty cyclic word)



And in $S\Psi$ also add

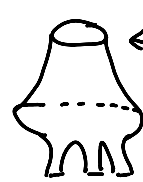
- $\Psi^{(2)}(\gamma)$: same as $\Psi^{(1)}$ but allow $\underline{b}$ = empty word, then count w/ 1_Λ.

- $\Psi^{(3)}(\gamma) = \sum_{\dim \mathcal{M}(\gamma,\underline{\beta},L)=0} \kappa(\underline{\beta})^{-1}\varepsilon(\underline{\beta})\,|\mathcal{M}|\,\tau_L$



<u>Lemma:</u> Ψ is a chain map which induces an isom. on cohomology

<u>Proof:</u> Main ingredient: (beyond "standard" SFT):
for a small enough handle $(X = \{|2x^2 - y^2| < \varepsilon^2\},\ \varepsilon \to 0)$
we have a unique disk of smallest energy



$\Rightarrow$ in action filtration, Ψ looks like:

$$\begin{array}{cc} & \xleftarrow{\text{action}} \\ \text{action} \uparrow & \begin{pmatrix} \pm 1 & & & 0 \\ & \pm 1 & & \\ & & \pm 1 & \\ ?? & & & \pm 1 & \\ & & & & \ddots \end{pmatrix} \end{array}$$

diagonal terms = those distinguished disks

below-diagonal terms = higher energy disks

$\rightsquigarrow$ get an isomorphism (canonical at level of associated graded vect. spaces) Δ

$\Rightarrow$ get:

$$0 \to \underset{\substack{\text{cyclic words of}\\ \text{chords of } \Lambda \\ (\text{subcomplex of } C(Y,\Lambda,\varepsilon))}}{A^c(Y,\Lambda,\varepsilon)} \xrightarrow{\text{incl.}} \overset{\text{mixed complex}}{C(Y,\Lambda;\varepsilon)} \xrightarrow{\text{proj.}} \underset{\substack{\text{Reeb orbits in } Y \\ (\text{quotient complex of } C)}}{Q_1(Y,\varepsilon)} \to 0$$

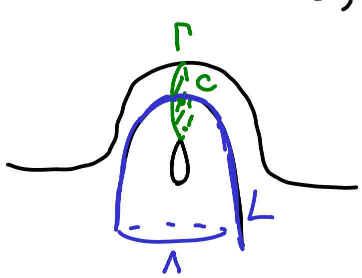
$$C(Y,\Lambda;\varepsilon) \overset{\text{Lemma}}{\cong} \underset{\text{Reeb orbits in } Y_\Lambda}{Q_1(Y_\Lambda,\varepsilon_\Lambda)}$$

short exact seq. of complexes, induces long exact sequence

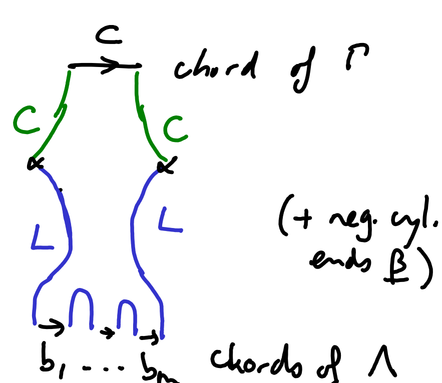
- On <u>linearized Legendrian contact homology</u>:

$$\Psi: \underset{\text{Reeb chords of } \Gamma}{A_1(\Gamma, Y_\Lambda; \varepsilon_\Lambda; \varepsilon_C)} \longrightarrow \underset{\text{non-empty words of Reeb chords}}{A^+(Y,\Lambda;\varepsilon)}$$

(linearized rel. CH, using augmentations $\varepsilon_\Lambda = \varepsilon \circ \Phi_X$ for Y_Λ & ε_C where C = co-core, $\partial C = \Gamma$ for lin. rel. CH)



Ψ counts



$$\Psi(c) = \sum_{\dim \mathcal{M}(c;\bar\beta,\bar b)=0} \kappa(\bar\beta)^{-1}\, \varepsilon(\bar\beta)\, |\mathcal{M}|\, \bar b$$

<u>Lemma</u>: Ψ is a chain map, inducing isom. on homology. ✓

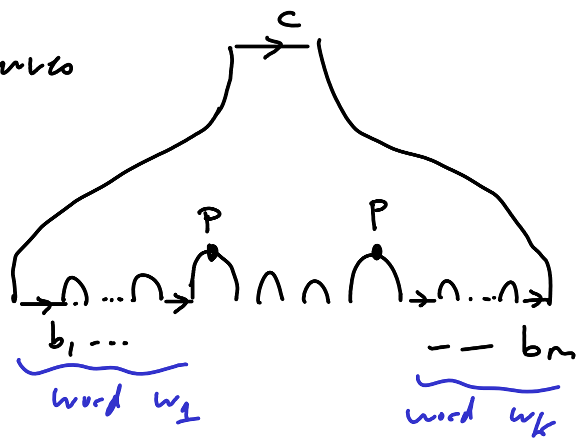
• For the <u>full rel. contact homology</u> of Γ, $\quad A(\Gamma, Y_\Lambda; \varepsilon_\Lambda)$:

1) Let $B(Y, \Lambda, \varepsilon) =$ $\mathbb{Q}$-<u>algebra</u> freely generated by words of Reeb chords of Λ

$$\left(\text{visually: } B \ni (\vec{b}_1 \dots \vec{b}_n) * (\vec{b}'_1 \dots \vec{b}'_k) * \dots \right)$$

To define the differential Δ on B, fix a point $p \in \Lambda$

& count curves



hitting $\{p\} \times \mathbb{R} \subset \Lambda \times \mathbb{R}$

• $F(c)$ = count of curves of this type, each such contributes

$$w_1 \, p \, w_2 \, p \dots w_k$$

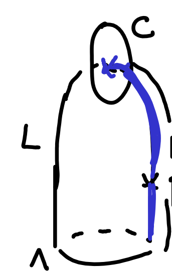
Then extend F to words by Leibniz and then

let
$$\Delta(w_1 * w_2 * \dots) = F(w_1) * w_2 * \dots \\ + (-1)^{*} w_1 * F(w_2) * \dots$$

2) • To define a chain map $A(\Gamma, Y_\Lambda, \varepsilon_\Lambda) \longrightarrow B(Y, \Lambda, \varepsilon)$

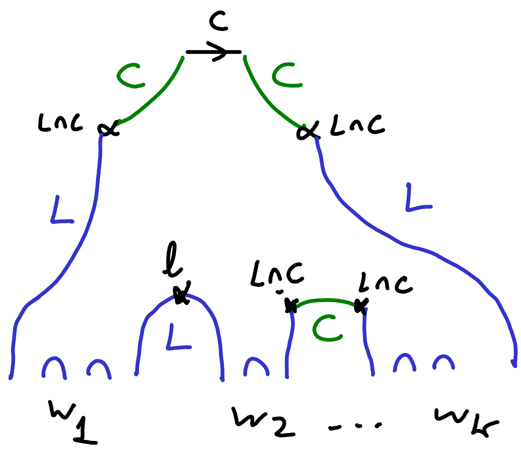
fix a half-infinite curve ℓ in L connecting

$L \cap C$ to $\{p\} \times \mathbb{R}$



and get a chain isom. ψ by counting

curves in X of the following type:



contributes to

$c \longmapsto w_1 * \cdots * w_k$