R. Pandharipande – 26/5/08 – Descendent integrals on open Riem. surfaces

(w/ J. Solomon)

$\overline{\mathcal{M}}_{g,n}$ moduli space of genus g stable curves w/ n marked points

$\psi_i \in H^2(\overline{\mathcal{M}}_{g,n})$ cotangent line classes:

$$\begin{array}{ccc} T_i^*C & \longrightarrow & \mathbb{L}_i \\ \downarrow & & \downarrow \\ [\; C \;] & \in & \overline{\mathcal{M}}_{g,n} \end{array}$$

$$\psi_i = c_1(\mathbb{L}_i)$$

→ <u>Descendent integrals</u>: $\int_{\overline{M}_{g,n}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n} \in \mathbb{Q}$ $\qquad (3g-3+n = \Sigma \alpha_i)$

<u>Motivation</u>: • intersection theory on moduli space $\overline{M}_{g,n}$.

<u>Def</u>: $\langle \tau_{\alpha_1} \cdots \tau_{\alpha_n} \rangle = \int_{\overline{M}_{g,n}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n}$ (where g = st. $3g-3+n = \Sigma \alpha_i$)

generating series: $F := \sum_{\{n_i\}} \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!} \langle \tau_0^{n_0} \tau_1^{n_1} \tau_2^{n_2} \cdots \rangle$

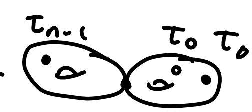
(finite, i.e. $(n_0, n_1, n_2, \cdots, 0, 0, \cdots)$)

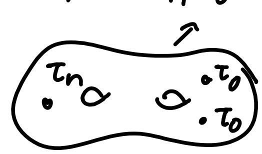
$\langle\langle \tau_{\alpha_1} \cdots \tau_{\alpha_n} \rangle\rangle := \frac{\partial}{\partial t_{\alpha_1}} \cdots \frac{\partial}{\partial t_{\alpha_n}} F$ satisfies <u>KdV hierarchy</u> (Witten '90).

($= \langle \tau_{\alpha_1} \cdots \tau_{\alpha_n} \rangle + \ldots$ ← higher terms in t_i)

<u>Recurrence relations</u>:

<u>TRR</u>: • $n \geq 1$: $(2n+1) \langle\langle \tau_n \tau_0^2 \rangle\rangle = \langle\langle \tau_{n-1} \tau_0 \rangle\rangle \langle\langle \tau_0^3 \rangle\rangle$



pictorially: 

$+ 2 \langle\langle \tau_{n-1} \tau_0^2 \rangle\rangle \langle\langle \tau_0^2 \rangle\rangle$

$+ \frac{1}{4} \langle\langle \tau_{n-1} \tau_0^4 \rangle\rangle$

(but pictorial description doesn't give a proof...)

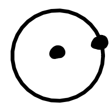
<u>Note</u>: constants are combinatorial; $\frac{1}{4}$ comes from $\int_{\overline{M}_{1,1}} \psi_1 = \frac{1}{24}$.

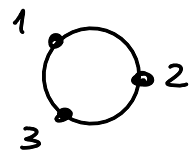
• $n=1$: $u = \langle\langle \tau_0 \tau_0 \rangle\rangle \rightsquigarrow \frac{\partial u}{\partial t_1} = u u_{t_0} + \frac{1}{12} u_{t_0 t_0 t_0}$ (KdV)

, ...

Open curves:

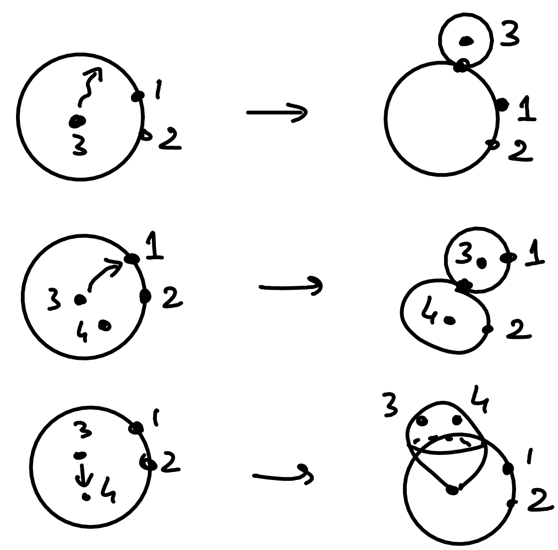
Look at $\mathcal{M}_{0,k,\ell}$ = moduli space of disks with
k boundary marked pts (ordered)
ℓ interior marked pts

- $\mathcal{M}_{0,1,1} = \{pt\}$  $\int_{\overline{\mathcal{M}}_{0,1,1}} 1 = 1$

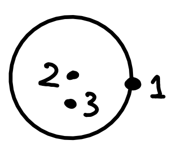
- $\mathcal{M}_{0,3,0} = \{pt\}$ 

$\int_{\overline{\mathcal{M}}_{0,3,0}} 1 = 1.$

- bubbling occurs when e.g.:

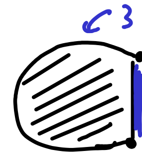


$\rightarrow$ compactification $\overline{\mathcal{M}}_{0,k,\ell}$ has boundary & corners

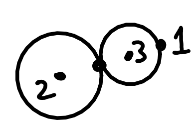
Ex: $\overline{\mathcal{M}}_{0,1,2}$: 

$\simeq$ real blow up of D^2 at a boundary point

$\leftarrow$ 3 is on $\partial D^2 \setminus \{1\}$

$\simeq$ bigon  $\leftarrow$ 3 is at 1

(place 1 at 1
2 at 0 $\rightsquigarrow 3 \in \overline{D^2}$ (if $=0$, sphere bubble
if $\in \partial D^2$, disc bubble)

except... if 3 is at 1, get interval of 

)

- $\dim \overline{\mathcal{M}}_{0,k,\ell} = \frac{3}{2}(-1) + \frac{k}{2} + \ell. \quad \rightarrow$ need k odd.

Def: $\langle \tau_{\alpha_1} \dots \tau_{\alpha_n} \sigma^k \rangle_{0,k,n} = \int_{\overline{M}_{0,k,l}} \psi_1^{\alpha_1} \dots \psi_n^{\alpha_n}$

⚠ integral of classes on a mfld with boundaries & corners !!!

Jake Solomon $\Rightarrow$ can define these integrals (by choosing a trivializing section over $\partial\overline{M}_{0,k,l}$ to get relative classes ....)

Ex: $\langle \tau_2 \sigma^5 \rangle_{0,5,1} = 8$

String equation $\quad \langle \tau_0 \prod_{i=1}^{n} \tau_{\alpha_i} \sigma^k \rangle = \sum_{j=1}^{n} \langle \tau_{\alpha_1} \dots \tau_{\alpha_{j-1}} \tau_{\alpha_{j+1}} \dots \tau_n \sigma^k \rangle$

Dilaton eqn: $\quad \langle \tau_1 \prod_{j=1}^{n} \tau_{\alpha_i} \sigma^k \rangle = (-1+k+n) \langle \tau_{\alpha_1} \dots \tau_{\alpha_n} \sigma^k \rangle$

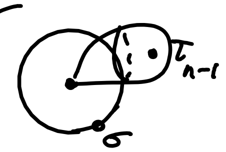
- TRR: Define as before $F_0^{disk} = \sum_k F_{0,k}$

$$F_{0,k} = \sum_{\{n_i\}} \left(\prod \frac{t_i^{n_i}}{n_i!} \right) \langle \tau_0^{n_0} \dots \sigma^k \rangle \frac{s^k}{k!}$$

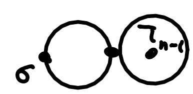
and $\langle\langle \tau_{\alpha_1} \dots \tau_{\alpha_n} \sigma^k \rangle\rangle_{open} := \dfrac{\partial^{n+k} F_0^{disk}}{\partial t_1 \dots \partial t_n \partial s^k}$

Thm:
$$\boxed{\begin{aligned}\langle\langle \tau_n \sigma \rangle\rangle_{open} &= \langle\langle \tau_{n-1} \tau_0 \rangle\rangle_{closed} \langle\langle \tau_0 \sigma \rangle\rangle_{open} \\ &+ \langle\langle \tau_{n-1} \rangle\rangle_{open} \langle\langle \sigma^2 \rangle\rangle_{open}\end{aligned}}$$

interior bubble



∂ bubble



$= s \cdot \langle \tau_{n-1} \sigma \rangle$ + higher order terms

- Higher genus: (one half of a genus g curve w/ $\mathbb{R}$ involution):

$\overline{M}_{g,k,l} \rightsquigarrow$ dim. $\frac{3g-3}{2} + \frac{k}{2} + l$, define $\langle\langle \dots \rangle\rangle$ similarly

$$\langle\langle \tau_n \sigma \rangle\rangle = \frac{1}{2n+1} \langle\langle \tau_{n-1} \tau_0 \rangle\rangle_{\text{closed}} \langle\langle \tau_0 \sigma \rangle\rangle + \frac{2}{2n+1} \langle\langle \tau_{n-1} \sigma \rangle\rangle \langle\langle \sigma \rangle\rangle + \frac{2}{2n+1} \langle\langle \tau_{n-1} \rangle\rangle \langle\langle \sigma^2 \rangle\rangle + \frac{4}{2n+1} W \langle\langle \tau_{n-1} \sigma^2 \rangle\rangle$$

$(W = \langle \tau_1 \rangle_{1,\text{open}})$.

<u>Claim</u>: this equation's consistency for $\langle\langle \ldots \rangle\rangle$ essentially determines $\langle\langle \tau_{n-1} \tau_0 \rangle\rangle_{\text{closed}}$ to be what we know it to be.

<u>Q</u>: what is this equation in terms of integrable systems?