M. Lipyanskiy – 28/5/08 – A semi-infinite cycle construction of Floer theory

Finite dim. case:

geometric theory	$\longleftrightarrow$ Morse theory
chains $P \xrightarrow{\sigma} M$	C_* gen^d by critical points of $f: M \to \mathbb{R}$
singular homology	Morse homology
bordism	

geometric theory is often more natural ($\exists$ preferred cycles...)
& allows more generality (bordism theory etc.)

Goal: find similar analogue in ∞-dim. setting:

geometric ??? $\longleftrightarrow$ Morse-Floer theory

We'll do this in SW-Floer case: (should work in all cases)

Setup: X closed 4-manifold, $W = W_+ \oplus W_-$ spinor bundle

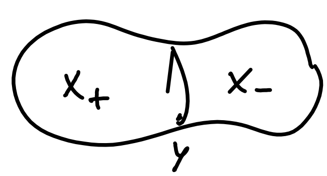
configuration space $\mathcal{C}(X) = \{(A, \varphi),\ A$ connection, $\varphi \in \Gamma(W^+)\}$

$\circlearrowleft$ gauge group $G(X) = \{X \to S^1\}$

$B(X) := \mathcal{C}(X)/G(X)$.

$M(X)$ = moduli space of solutions

If $X = X_+ \cup_Y X_-$



then

$M(X) = M(X^+) \times_{B(Y)} M(X^-)$, here $M(X^\pm)$ Hilbert manifolds (∞-dim!)

$B(Y) = B^+ \oplus B^-$ defined by $\pm$ eigenspaces of $(*d, D_B)$

$R^\pm : M(X^\pm) \to B(Y)$ boundary restriction map

$\mathcal{L} : B(Y) \to \mathbb{R}$ action functional

Fact: $\pi^+ \circ DR^+ : TM(X^+) \to T^+B$ Fredholm
$\pi^- \circ DR^+ : TM(X^+) \to T^-B$ Compact

(spectral boundary condition as in Atiyah-Patodi-Singer would then give an elliptic theory ...)

The Fact implies that $M(X) = M(X^+) \times_{B(Y)} M(X^-)$ is finite-dim!

However, compactness isn't clear ??

Goal: set up axioms for "cycles" $P \xrightarrow{b} B$ polarized Hilbert mfld

P e.g. $M(X^\pm)$, B e.g. $M(Y)$

s.t. given a $\begin{cases} \text{cycle } P \to B \\ \text{cocycle } Q \to B \end{cases}$, $P \times_B Q$ is compact.

Idea: we'll constrain action $\mathcal{L}$ bounded below on cycles, above cocycles so properness will give us compactness.

B Hilbert space, with polarization $B = B_+ \oplus B_-$

$\mathcal{L} : B \to \mathbb{R}$ (think: action functional)

Look for a notion of cycles $P \xrightarrow{b} B$, P Hilbert mfld.

Motivating observation:

- $\mathcal{L} : H \to \mathbb{R}$, $\mathcal{L}(h) = |h|^2$

  given h_i with $\mathcal{L}(h_i) < c \Rightarrow \{h_i\}$ weakly precompact

  If $h_\infty = \text{weak lim}(h_i)$, then $\mathcal{L}(h_\infty) \leq \lim \mathcal{L}(h_i)$,

  and if $\mathcal{L}(h_\infty) = \lim \mathcal{L}(h_i)$ then $h_\infty =$ strong limit.

- our $\mathcal{L}$ is more like $\mathcal{L}(h) = |h_+|^2 - |h_-|^2$ .....

Def. $B = B_+ \oplus B_-$ polarized Hilbert space, $\mathcal{L}: B \to \mathbb{R}$

P Hilbert manifold, $P \xrightarrow{\sigma} B$ is a cycle if

1) P is semi-infinite ie. $T\sigma$: $TP \to T^+B$ Fredholm
$TP \to T^-B$ compact

2) $\mathcal{L}|_{\mathrm{Im}\,\sigma} > -C$, lower semicont. (weak top on B)

3) if $p_i \in P$ with $\mathcal{L}(\sigma(p_i)) < C' \Rightarrow$
$\sigma(p_i)$ weakly precompact

4) if y = weak limit of $\sigma(p_i)$, with $\mathcal{L}(y) = \lim \mathcal{L}(\sigma(p_i))$
then p_i is precompact in P

NB: definition doesn't assume any properties of $\mathcal{L}$ wrt polarization $B_+ \oplus B_-$
but will be trivial unless $\mathcal{L}$ is "reasonable"

★ If $P \xrightarrow{\sigma} B$ is a cycle and $Q \xrightarrow{\tau} B$ is a cocycle
(same notion up to exch. signs)
then $P \times_B Q$ is <u>compact</u>.

If $Q \pitchfork P$ then $P \times_B Q$ is a compact finite dim. <u>manifold</u>

<u>Thm:</u> In above SW setting, $M(X^+) \to B(Y)$ is a cycle
$M(X^-) \to B(Y)$ is a cocycle

• <u>Bordism groups</u> $\Omega F^{\pm}(B)$ generated by (co)cycles $P \xrightarrow{\sigma} B$
with rel. $P = 0$ if $P = \partial W$.

By construction,
get an intersection pairing $\Omega F^+_*(B) \otimes \Omega F^-_*(B) \to \Omega_*(B)$

e.g. in SW-case, $M(X^+) \times_B M(X^-) = M(X)$

(Note: $\Omega F^-(B, \mathcal{L}) \simeq \Omega F^+(-B, -\mathcal{L})$).

Another example: (Ham. Floer homology)

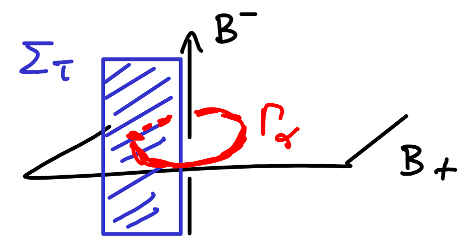
$B = L^2_{1/2}(S^1, \mathbb{C}^n)$; $H: \mathbb{C}^n \to \mathbb{R}$, assume $H(z) = c|z|^2$ for $|z| \gg 1$
and $H = 0$ near origin

$-J\partial_\theta$ induces a polarization of $L^2_{1/2}(S^1, \mathbb{C}^n)$ (Fourier modes)

$$\mathcal{L}_H(\gamma) = \underbrace{\int_0^{2\pi} \langle -J\partial_\theta \gamma, \gamma \rangle \, d\theta}_{= |\gamma^+|^2_{L^2_{1/2}} - |\gamma^-|^2_{L^2_{1/2}}} - \int H(\gamma)\, d\theta$$

- Let $\Gamma_\alpha = \{ \gamma \in B^+ \mid |\gamma| = \alpha \}$ sphere in B_+

$\Sigma_\tau = \underbrace{D^-(\tau)}_{\text{ball in } B^-} \times \{ s e^+, s \in [0,1] \}$ for some $e^+ \in B^+$, $|e^+| = 1$.



Claim: Γ_α is a cycle
Σ_t is a cocycle

We actually can see the crit pts of $\mathcal{L}_H$ b/w them via a flow:

- $M(T) = \{ S^1 \times [0,T] \xrightarrow{u} \mathbb{C}^n, \quad \partial_t u + J(\partial_\theta u - J\nabla H(u)) = 0 \}$

with L^2_1 - topology

Restriction maps at $t = 0$ & T: $R_0 \times R_T : M(T) \to B \times B$

$\rightsquigarrow$ given a cycle P, $\mathcal{F}_T(P) := \left(P \times_{B\, R_0} M(T) \xrightarrow{R_T} B \right)$

(time T flow of P)

- $M(T) \to B \times B$ is cobordant to $B \xrightarrow{\Delta} B \times B$

i.e. $\exists\, W$ st. $\partial W = M(T) - (B, \Delta)$

(cobordism = let T vary down to 0)

→ can use this to prove existence of a crit. pt of $\mathcal{L}_H$

<u>Rmk:</u> obstacle to extending this theory to sympl. mflds other than $\mathbb{C}^n$:

in gen'l, $L^2_{1/2}(S^1, M)$ isn't a Hilbert manifold.

... $\exists$ other setup (less conceptual but technically easier).....