UCSD
Mathematics Department

The Tropical Vertex

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Directory

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Goals. (1) Elaborate on the process of adding rays at collisions.
(2) Find an A-model (enumerative) interpretation for this process.

[Work in progress, joint with Rahul Pandharipande and Bernd Siebert.]

1. The Tropical Vertex Group (B-model)

Fix the following data:

$$M = \mathbb{Z}^2, \quad N = \mathrm{Hom}(M, \mathbb{Z}),$$

$$M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}, \quad N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$$

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$$H(R) \subseteq \mathrm{Aut}(\mathbb{k}[M] \hat{\otimes}_{\mathbb{k}} R).$$

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We will sometimes write

$$\mathbb{k}[M] = \mathbb{k}[x^{\pm 1}, y^{\pm 1}],$$

so an element

$$z^m \in \mathbb{k}[M]$$

can be written as

$$x^a y^b$$

if

$$m = (a, b).$$

Definition. The tropical vertex group $H(R)$ is the subgroup of $\mathrm{Aut}(\Bbbk[M]\hat{\otimes}_{\Bbbk}R)$ generated by automorphisms of the form

$$z^m \mapsto z^m f^{\langle n_0, m\rangle}$$

where

- $n_0 \in N$
- $f \in \Bbbk[z^{m_0}]\hat{\otimes}_{\Bbbk}R \subseteq \Bbbk[M]\hat{\otimes}_{\Bbbk}R$ for some non-zero $m_0 \in M$.
- $f - 1 \in z^{m_0}\mathfrak{m}$.
- $\langle n_0, m_0\rangle = 0$.

Remarks.

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- Elements of $H(R)$ are symplectomorphisms, preserving the symplectic form
$$\Omega = \frac{dx}{x} \wedge \frac{dy}{y}$$
- z^{m_0} is left invariant by the automorphism
$$z^m \mapsto z^m f^{\langle n_0, m\rangle}$$

Typical example. With $R = \Bbbk[[t]]$,

$$\begin{aligned} x &\mapsto x \\ y &\mapsto y(1+tx) \end{aligned}$$

is a typical example of one of the generators of $H(R)$. Here

$$\begin{aligned} m_0 &= (1,0) \\ n_0 &= (0,1) \\ f &= 1+tx \end{aligned}$$

2. Scattering diagrams

Definition. A *ray* is a pair $(\mathfrak{d}, f_{\mathfrak{d}})$ where $\mathfrak{d} \subseteq M_{\mathbb{R}}$ is given by $\mathfrak{d} = m_0' - \mathbb{R}_{\geq 0} m_0$ for some $m_0' \in M_{\mathbb{R}}$ and non-zero $m_0 \in M$, and

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satisfies

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Definition. A *line* is a pair $(\mathfrak{d}, f_{\mathfrak{d}})$ where $\mathfrak{d} \subseteq M_{\mathbb{R}}$ is given by $\mathfrak{d} = m_0' - \mathbb{R} m_0$ for some $m_0' \in M_{\mathbb{R}}$ and non-zero $m_0 \in M$, and

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satisfies

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Definition. A *scattering diagram* $\mathfrak{D}$ is a set of lines and rays such that for any $n > 0$, there are only a finite number of elements $(\mathfrak{d}, f_{\mathfrak{d}})$ with

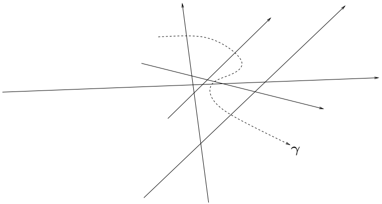
$$f_{\mathfrak{d}} \not\equiv 1 \mod \mathfrak{m}^n.$$

Consider any path

$$\gamma : [0,1] \to M_{\mathbb{R}}$$

which

- is transversal to every element of $\mathfrak{D}$ it intersects;
- does not pass through the endpoint of any ray or the intersection of any two elements;
- only passes through any given ray a finite number of times.



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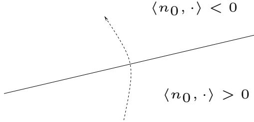
First, when γ crosses an element $(\mathfrak{d}, f_{\mathfrak{d}})$, we obtain an element of $H(R)$ given by

$$z^m \mapsto z^m f_{\mathfrak{d}}^{\langle m, n_0 \rangle},$$

where $n_0 \in N$ is primitive with $\langle n_0, m_0 \rangle = 0$ chosen with the following sign convention:



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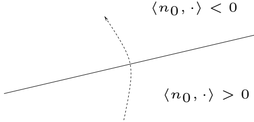


This defines an element $\theta_{\gamma,\mathfrak{d}} \in H(R)$.

The path-ordered product is then defined by

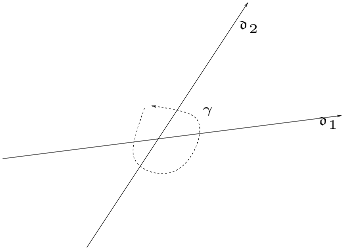
$$\theta_{\mathfrak{D},\gamma} = \prod \theta_{\mathfrak{d},\gamma},$$

where the product runs over all $\mathfrak{d}$ crossed by γ, in the order traversed by γ.

Example: Commutators I

$$\mathfrak{D} = \{(\mathfrak{d}_1, f_1), (\mathfrak{d}_2, f_2)\},$$

where $\mathfrak{d}_1$, $\mathfrak{d}_2$ are lines through the origin.



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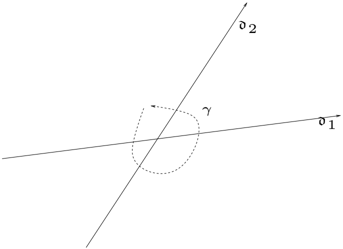


Then

$$\theta_{\mathfrak{D},\gamma} = \theta_2^{-1} \circ \theta_1^{-1} \circ \theta_2 \circ \theta_1,$$

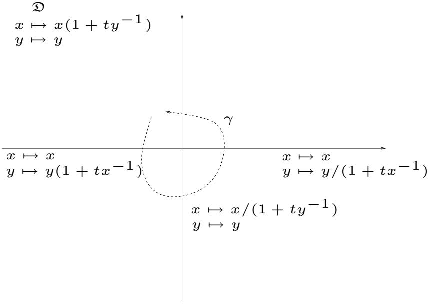
where θ_1 and θ_2 are the elements of $H(R)$ associated to the first two crossings.

Kontsevich-Soibelman Lemma. Given a scattering diagram $\mathfrak{D}$, there is a scattering diagram $\mathfrak{D}'$ containing $\mathfrak{D}$ such that $\mathfrak{D}' \setminus \mathfrak{D}$ consists only of rays, and

$$\theta_{\mathfrak{D}',\gamma} = id$$

for every closed loop γ for which $\theta_{\mathfrak{D}',\gamma}$ is defined.

Example: Commutators II



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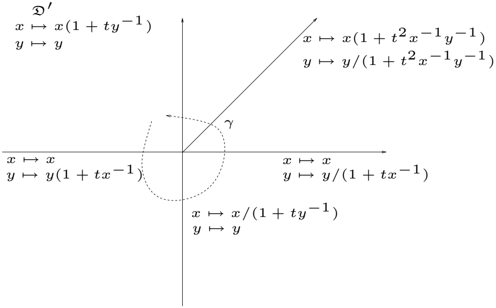
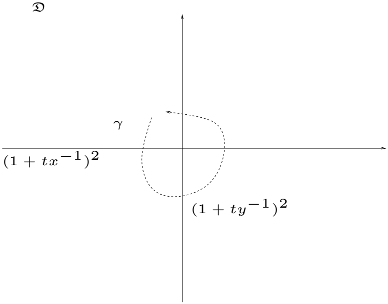


Example: Commutators III



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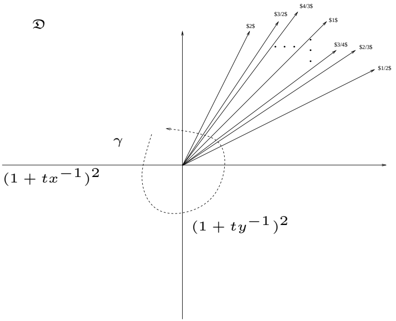
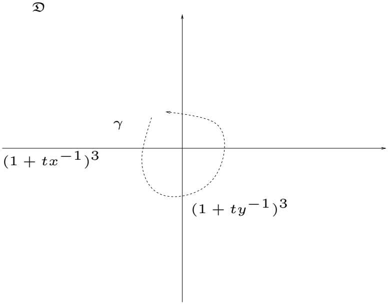


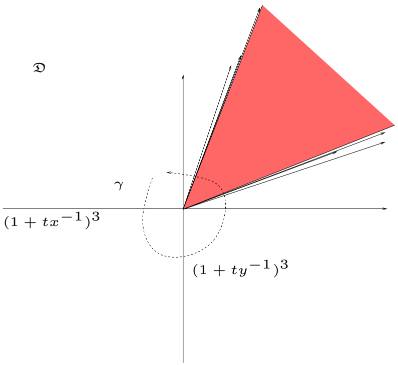
Lines of slope $(n+1)/n$, $n \geq 1$: $(1+t^{2n+1}x^{-n}y^{-n-1})^2$
Lines of slope $n/(n+1)$, $n \geq 1$: $(1+t^{2n+1}x^{-n-1}y^{-n})^2$
Line of slope 1:

$$(1-t^2x^{-1}y^{-1})^{-4} = \frac{(1+t^2x^{-1}y^{-1})^4}{(1-t^4x^{-2}y^{-2})^{2\cdot 2}}.$$

Example: Commutators IV



Example: Commutators IV



Have rays of slope $3, 8/3, 21/8, \ldots$ converging to $(3+\sqrt{5})/2$.
Have rays of slope $1/3, 3/8, 8/21, \ldots$ converging to $(3-\sqrt{5})/2$.
Have rays of all rational slopes between $(3-\sqrt{5})/2$ and $(3+\sqrt{5})/2$.

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For example, the function attached to the line of slope 1 is

$$\left(\sum_{n=0}^{\infty}\frac{1}{3n+1}\binom{4n}{n}t^{2n}x^{-n}y^{-n}\right)^{9}$$

$$=\frac{(1+t^2x^{-1}y^{-1})^9\cdot(1+t^6x^{-3}y^{-3})^{3\cdot 54}\cdots}{(1-t^4x^{-2}y^{-2})^{2\cdot 18}\cdot(1-t^8x^{-4}y^{-4})^{4\cdot 252}\cdots}$$

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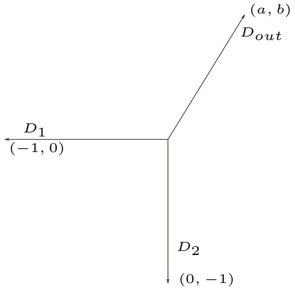
Remark. This description is based on computer calculations and may not yet have been verified.

3. The tropical vertex, A-model

Consider a weighted projective space X given by the fan:



The three labelled rays correspond to three toric divisors, D_1, D_2, and D_{out}.

Pick two integers $d_1, d_2 > 0$ and general sets of points

$$\begin{aligned} S_1 &\subseteq D_1, \\ S_2 &\subseteq D_2 \end{aligned}$$

with

$$\#S_1 = d_1, \#S_2 = d_2,$$

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Definition. Let N_d be the number of maps $\varphi : \mathbb{P}^1 \to X$ (up to reparametrization) satisfying the following properties:

1. Whenever $\varphi(p) \in D_i$, $i = 1, 2$, then $\varphi(p) \in S_i$ and φ is transversal to D_i at $\varphi(p)$.
2. There is a unique $q \in \mathbb{P}^1$ such that $\varphi(q) \in D_{out}$.
3. The intersection multiplicity of $\varphi(\mathbb{P}^1)$ with D_{out} at $\varphi(q)$ is d.

Remark. This can be defined more precisely as a sum of Gromov-Witten invariants over certain homology classes on the blow-up $\tilde{X}$ of X along the set $S_1 \cup S_2$ with the reduced scheme structure. Furthermore, we need to keep in mind there may be multiple cover contributions.

Examples. $d_1 = d_2 = 1$, $(a, b) = (1, 1)$.

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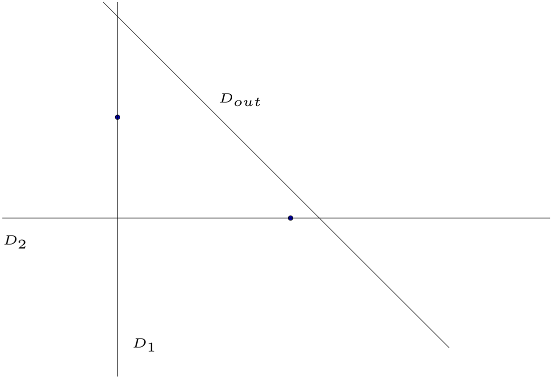

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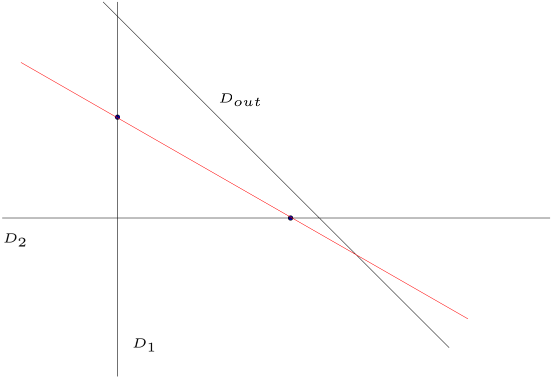
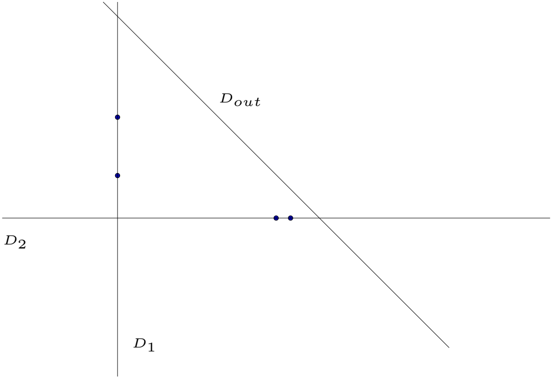


$N_1 = 1,\ N_d = 0,\ d \geq 2.$

Examples. $d_1 = d_2 = 2$, $(a, b) = (1, 1)$.



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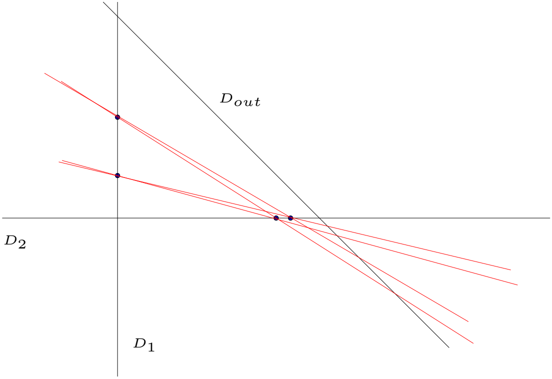

$N_1 = 4$

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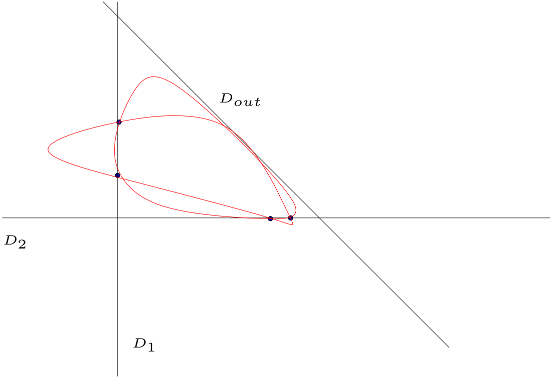

$N_1 = 4 \qquad N_2 = 2,\ N_d = 0,\ d \geq 3$

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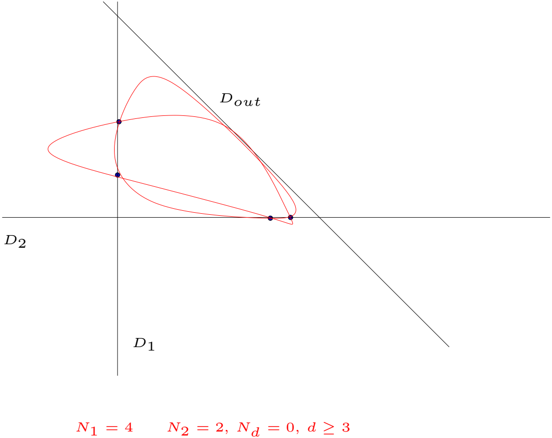
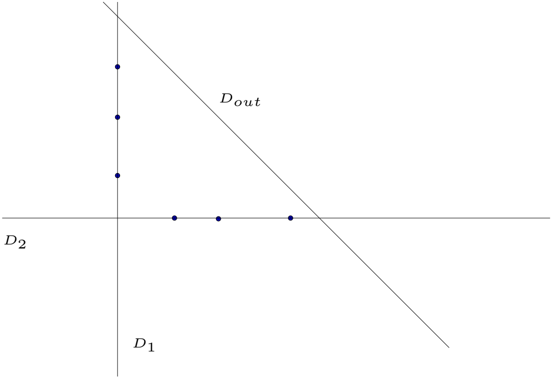


Compare with the function attached to the ray of slope 1:

$$\frac{(1 + t^2 x^{-1} y^{-1})^4}{(1 - t^4 x^{-2} y^{-2})^{2 \cdot 2}}$$

Examples. $d_1 = d_2 = 3$, $(a, b) = (1, 1)$.



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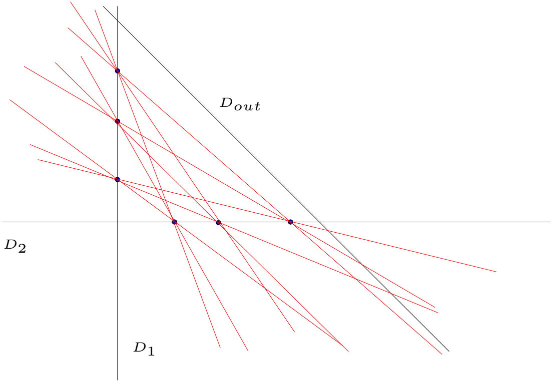

$N_1 = 9$

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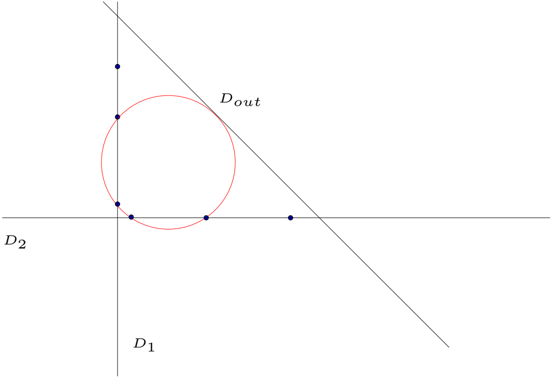

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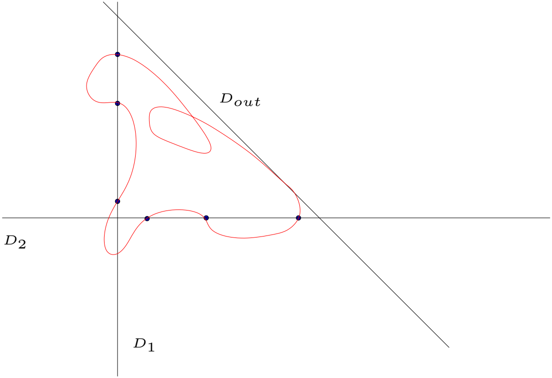

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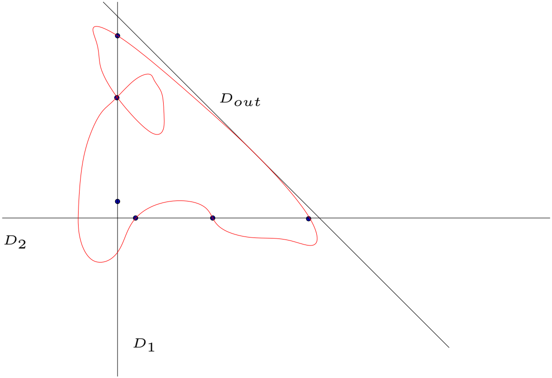

$N_1 = 9,\ N_2 = 3 \times 3 \times 2 = 18,\ N_3 = 18 + 36 = 54, \ldots$

$2 \times 3 \times 2 \times 3 = 36$ such cubics

Summary. $d_1 = d_2 = 3$, $(a, b) = (1, 1)$.

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Compare with

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Conjecture. Let $\mathfrak{D}$ be the scattering diagram consisting of two lines with attached functions $(1+tx^{-1})^{d_1}$ and $(1+ty^{-1})^{d_2}$ and let $\mathfrak{D}'$ be the scattering diagram obtained from the Kontsevich-Soibelman Lemma. Then the function f_{out} attached to the ray generated by a primitive vector (a,b) satisfies

$$\log f_{out} = \sum_{d=1}^{\infty} dN_d t^{d(a+b)} x^{-da} y^{-db}.$$

In addition, there are multiple cover formulae: the contribution to multiple covers of a "nice" curve which is d-tangent to D_{out} is

$$\sum_{d=1}^{\infty} kd \binom{(d-1)k-1}{k-1} \frac{t^{dk(a+b)} x^{-kda} y^{-kdb}}{k^2}.$$

If $d = 1$, we interpret $\binom{-1}{k-1} = (-1)^{k-1}$, giving

$$\sum_{k=1}^{\infty} k(-1)^{k-1} \frac{t^{k(a+b)} x^{-ka} y^{-kb}}{k^2}.$$

We can rewrite the formula for f_{out} as

$$f_{out} = \prod_{d=1}^{\infty} G_d(t^{a+b}x^{-a}y^{-b})^{I_d}$$

where

$$G_d(q) = \left(\sum_{k=0}^{\infty} \frac{1}{(d-2)k+1} \binom{(d-1)k}{k} q^{kd} \right)^{d}.$$

Here

$$\begin{aligned} G_1(q) &= 1+q \\ G_2(q) &= \frac{1}{(1-q^2)^2} \end{aligned}$$