UCSD
Mathematics Department

From affine manifolds to complex manifolds: instanton corrections from tropical disks

Mark Gross

Directory

Last Revision Date: May 25, 2008

1. Affine manifolds and mirror symmetry

Definition. An *affine manifold* B is a real manifold with charts $\psi_i : U_i \to \mathbb{R}^n$ such that $\psi_i \circ \psi_j^{-1} \in Aff(\mathbb{R}^n)$ for all i, j.

An affine manifold is *integral* if $\psi_i \circ \psi_j^{-1} \in Aff(\mathbb{Z}^n)$ for all i, j.

Given an integral affine manifold B, we can define a family of lattices $\Lambda \subseteq \mathcal{T}_B$ locally generated by $\partial/\partial y_1, \ldots, \partial/\partial y_n$, where $y_1, \ldots, y_n$ are local affine coordinates.

Similarly we define $\check{\Lambda} \subseteq \mathcal{T}_B^*$ locally generated by $dy_1, \ldots, dy_n$.

Definition.

$$\begin{aligned} X(B) &:= \mathcal{T}_B/\Lambda \\ \check{X}(B) &:= \mathcal{T}_B^*/\check{\Lambda} \end{aligned}$$

$X(B)$ carries a canonical complex structure:

If $y_1, \ldots, y_n$ are local affine coordinates on B, $x_1, \ldots, x_n$ coordinates on $\mathcal{T}_B$ so that

$$(x_1, \ldots, x_n, y_1, \ldots, y_n)$$

corresponds to the tangent vector

$$\sum_{i=1}^{n} x_i \frac{\partial}{\partial y_i},$$

at the point with coordinates

$$(y_1, \ldots, y_n),$$

then complex coordinates are given by

$$z_i = e^{2\pi\sqrt{-1}(x_i + \sqrt{-1} y_i)}.$$

These coordinates define a well-defined complex structure, independent of the choice of coordinates.

$\check{X}(B)$ carries a canonical symplectic structure:

$\mathcal{T}_B^*$ always carries a canonical symplectic form, and this symplectic form descends to a symplectic form on $\check{X}(B)$.

This gives torus fibrations

$$\begin{aligned} X(B) &\rightarrow B \\ \check{X}(B) &\rightarrow B \end{aligned}$$

Toy version of the SYZ conjecture. Mirror symmetry is the correspondence between complex manifolds $X(B)$ and symplectic manifolds $\check{X}(B)$.

Problem. There are very few interesting compact examples of mirror symmetry arising in this manner, e.g. complex tori.

To get more interesting examples, we need to consider *integral affine manifolds with singularities*.

This is a manifold B with an open subset $B_0 \subseteq B$ with an integral affine structure, with

$$\Delta := B \setminus B_0$$

codimension two in B.

Example. There exists a three-dimensional integral affine manifold with singularities B such that

$$X(B_0) \to B_0 \text{ and } \check{X}(B_0) \to B_0$$

can be compactified *topologically* to torus fibrations

$$X(B) \to B \text{ and } \check{X}(B) \to B$$

with $X(B)$ homeomorphic to the quintic three-fold and $\check{X}(B)$ homeomorphic to the mirror quintic. [G., Ruan, Haase and Zharkov].

Generally true. Given reasonable restrictions on the singularities of B, $X(B_0)$ and $\check{X}(B_0)$ can be compactified *topologically*.

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Expectation. Given reasonable restrictions on the singularities of B, $\check{X}(B_0)$ can be compactified *symplectically*. [Symington, $\dim B = 2$, Castano-Bernard, Matessi, $\dim B = 3$.]

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Expectation. Given reasonable restrictions on the singularities of B, $\check{X}(B_0)$ can be compactified *sympletically*. [Symington, $\dim B = 2$, Castano-Bernard, Matessi, $\dim B = 3$.]

Certainty. $X(B_0)$ usually cannot be compactified as a *complex manifold*.

Definition. For B an integral affine manifold, $\epsilon > 0$, define

$$X_\epsilon(B) = \mathcal{T}_B/\epsilon\Lambda.$$

Reconstruction problem. [G., Kontsevich-Soibelman] Let B_0 be an integral affine manifold with compactification B. Find a family of compact complex manifolds $X_\epsilon(B)$ for $\epsilon < \epsilon_0$ which are compactifications of small deformations of $X_\epsilon(B_0)$ of size $O(e^{-C/\epsilon})$.

Approaches. (1) Fukaya, 2001. Studied deformations of the complex structure on $X_\epsilon(B_0)$ for $\dim B = 2$ using the Kodaira-Spencer equation, and found heuristic arguments to suggest solutions "concentrate" along trees made of gradient flow lines on B_0 with leaves at the singularities.

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(2) Kontsevich and Soibelman, 2004. Produce a *non-Archimedean* K3 surface (a rigid analytic space) from an affine S^2 with 24 singular points. The description involves "corrections" coming from trees of gradient flow lines as in Fukaya's approach.

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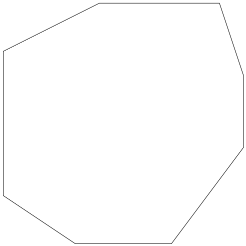
(3) G. and Siebert, (2001-)2007. Produce a flat family $\mathcal{X} \to S = \mathrm{Spec}\Bbbk[[t]]$ from B, with $\mathcal{X}_0$ a degenerate Calabi-Yau manifold. A *discrete Legendre transform* of B produces a new affine manifold $\check{B}$, and we can replace gradient flow lines with straight lines on $\check{B}$. The flat family is described using *tropical trees* on $\check{B}$. This simplifies the construction, allowing it to work in any dimension.

2. A toy example.

Let $B \subseteq \mathbb{R}^n$ be a lattice polytope, and let $\mathscr{P}$ be a decomposition of B into lattice polytopes.

Let $\varphi : B \to \mathbb{R}$ be a strictly convex piecewise linear function with respect to $\mathscr{P}$, with integer slopes.



B

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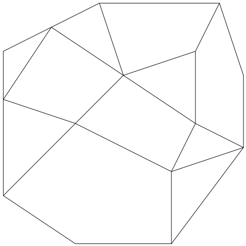


$\mathscr{P}$

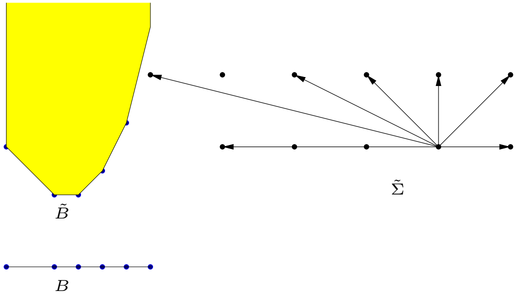
Let

$$\tilde{B} = \{(\mathbf{m}, r) \in \mathbb{R}^{n+1} | \mathbf{m} \in B, r \geq \varphi(\mathbf{m})\},$$

and let

$$\tilde{\Sigma}$$

be the normal fan to $\tilde{B}$, living in $(\mathbb{R}^{n+1})^\vee$:



We obtain a toric variety

$$X_{\tilde{\Sigma}}$$

from this fan.

The projection onto the last coordinate

$$(\mathbb{R}^{n+1})^\vee \to \mathbb{R}^\vee$$

induces a morphism

$$f : X_{\tilde{\Sigma}} \to \mathbb{A}^1$$

with

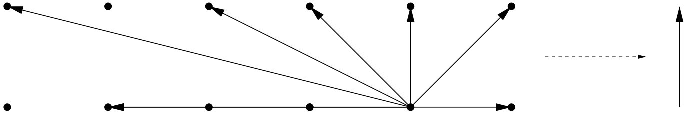
$$f^{-1}(t) \cong \mathbb{P}_B,$$

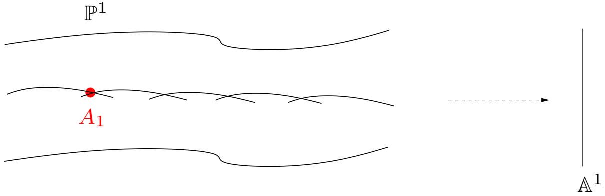
the projective toric variety defined by the polytope B, and

$$f^{-1}(0)$$

is a union of toric varieties $\mathbb{P}_\sigma$ for maximal cells $\sigma \in \mathscr{P}$.

e.g.





Different approach: Build a k-th order deformation of the central fibre:

$$\mathcal{X}_k \to \mathrm{Spec}\,\Bbbk[t]/(t^{k+1})$$

by gluing together thickenings of affine subsets of irreducible components of $\mathcal{X}_0$.

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A choice of a face $\tau_1 \subseteq \tau_2$ determines an affine open subset of this stratum.

For any pair of cells $\tau_1 \subseteq \tau_2$, we will define a ring, which gives a thickening of this open affine subset.

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For $\sigma \in \mathscr{P}$ a maximal cell, let $n_\sigma \in \mathbb{R}^{n\vee}$ be the slope of φ.

Define $\varphi_{\tau_1} : \mathbb{R}^n \to \mathbb{R}$ by

$$\varphi_{\tau_1}(m) = \max\{\langle n_\sigma, m\rangle | \tau_1 \subseteq \sigma\}.$$

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Define the monoid (semi-group)

$$P_{\tau_1} := \{(m, r) | m \in \mathbb{Z}^n, r \in \mathbb{Z}, r \geq \varphi_{\tau_1}(m)\}.$$

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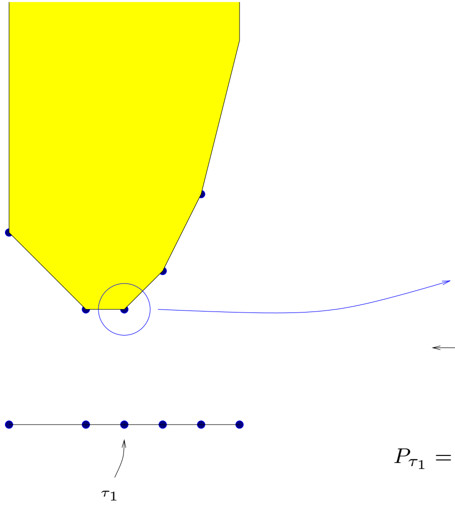
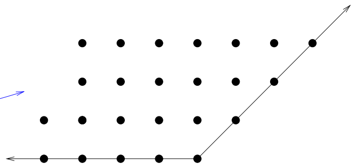
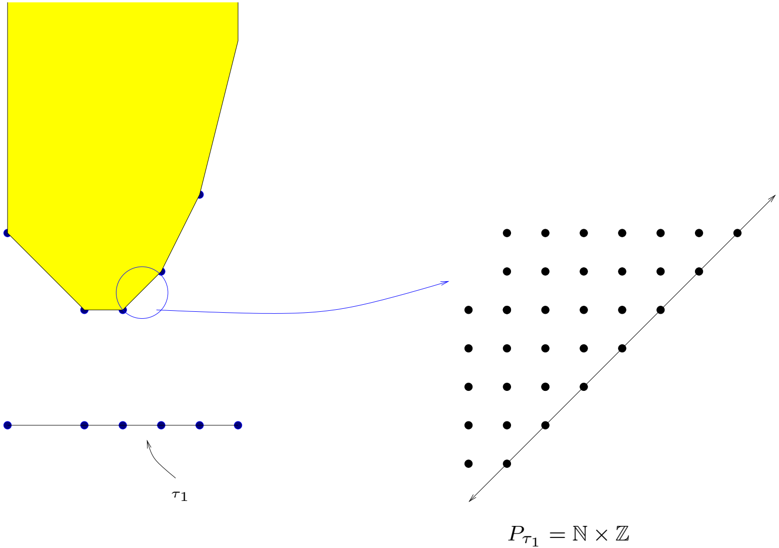
Set

$$R_{\tau_1} = \mathbb{k}[P_{\tau_1}].$$





$P_{\tau_1} = \mathbb{N}^2$


τ_1
$P_{\tau_1} = \mathbb{N} \times \mathbb{Z}$

For $\sigma \in \mathscr{P}$ a maximal cell, define

$$\mathrm{ord}_\sigma(m, r) = r - \langle n_\sigma, m \rangle,$$

the height of (m, r) above the plane defined by n_σ.

This is the order of vanishing of the monomial $z^{(m,r)}$ on the irreducible component defined by σ.

We next define monomials ideals

$$I^{>k}_{\tau_1,\tau_2} \subseteq \Bbbk[P_{\tau_1}]$$

by

$$I^{>k}_{\tau_1,\tau_2} = \langle z^{(m,r)} | (m,r) \in P_{\tau_1} \text{ and } \mathrm{ord}_\sigma(m,r) > k \text{ for some } \sigma \supseteq \tau_2 \rangle.$$

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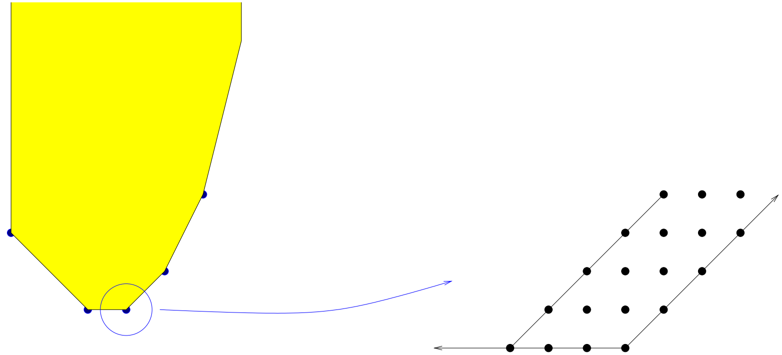
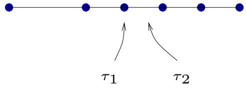
Set

$$R^k_{\tau_1,\tau_2} := \Bbbk[P_{\tau_1}]/I^{>k}_{\tau_1,\tau_2}.$$

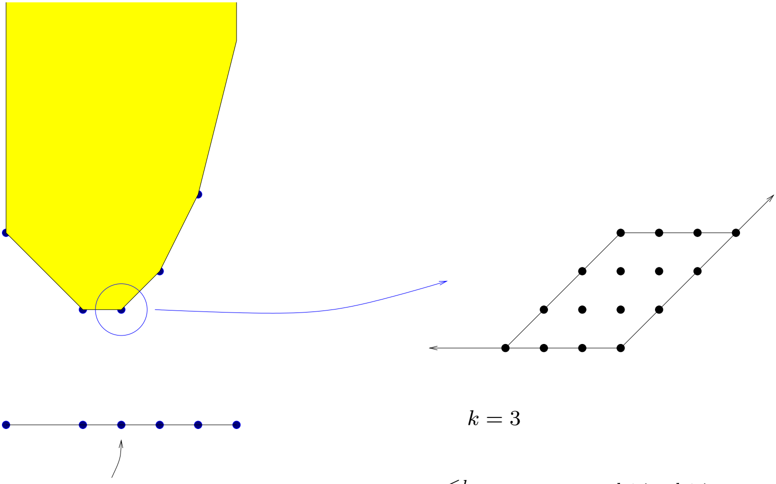
This is a *thickening* of the affine subset of the projective toric variety $\mathbb{P}_{\tau_2}$ determined by τ_1.



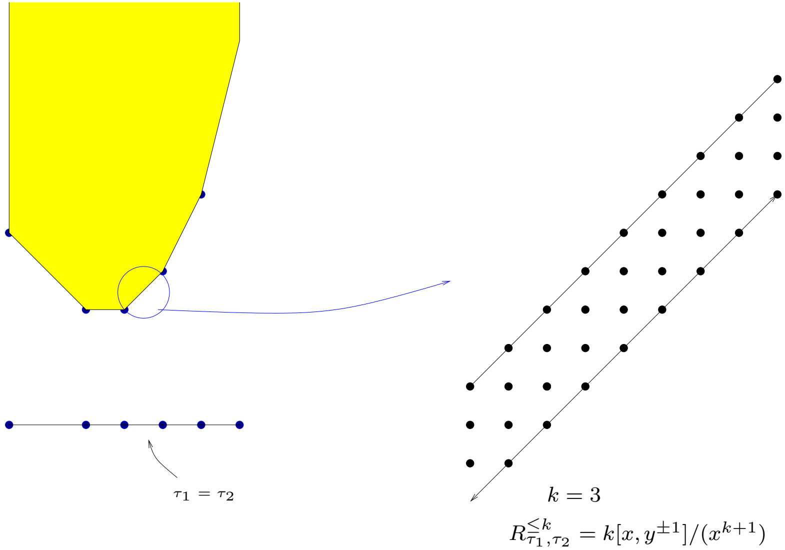


$k = 3$

$$R^{\leq k}_{\tau_1,\tau_2} = k[x,y]/(x^{k+1})$$



$$R^{\leq k}_{\tau_1,\tau_2} = k[x,y]/(x^{k+1}, y^{k+1})$$


$\tau_1 = \tau_2$
$k = 3$
$R^{\leq k}_{\tau_1,\tau_2} = k[x, y^{\pm 1}]/(x^{k+1})$

For a vertex $v \in \mathscr{P}$, there are natural surjections

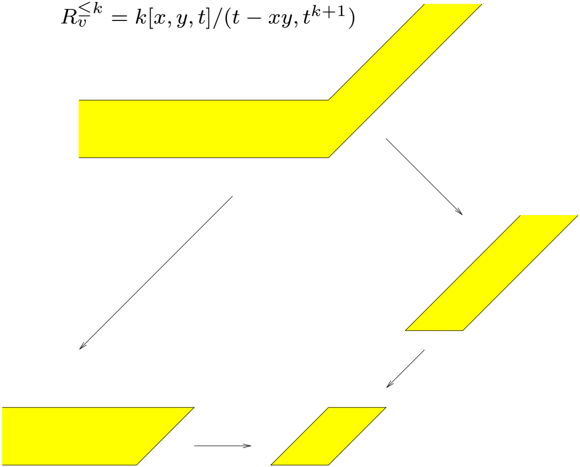
$$R^k_{v,\tau_2} \to R^k_{v,\tau_1}$$

whenever $\tau_1 \subseteq \tau_2$.

We can then construct

$$R^k_v = \varinjlim R^k_{v,\tau}.$$

$$R_{\overline{v}}^{\leq k} = k[x,y,t]/(t-xy, t^{k+1})$$



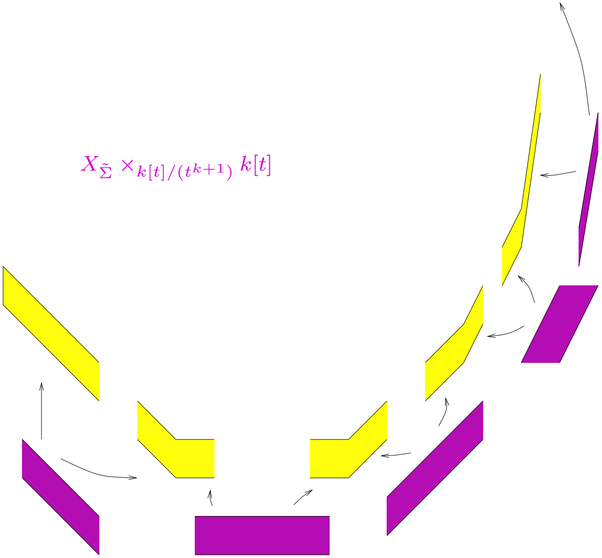
Next, the schemes

$$\mathrm{Spec} R_v^k$$

can be glued together using common open sets.

This produces the desired scheme

$$\mathcal{X}_k \to \mathrm{Spec}\mathbb{k}[t]/(t^{k+1}).$$


$X_{\tilde\Sigma} \times_{k[t]/(t^{k+1})} k[t]$

3. Onwards to affine manifolds.

We now replace $B, \mathscr{P}, \varphi$ with the following data:

- B is an integral affine manifold.
- $\mathscr{P}$ is a decomposition of B into integral lattice polytopes.
- φ is a *multi-valued integral strictly convex piecewise linear function*, represented on an open cover $\{(U_i, \varphi_i)\}$ with φ_i strictly convex piecewise linear function with integral slopes on U_i, with $\varphi_i - \varphi_j$ affine linear on $U_i \cap U_j$.

Where do the monoids P_y live?

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On B have an extension

$$0 \to \mathbb{Z} \to \tilde{\Lambda} \to \Lambda \to 0$$

determined by the extension class $c_1 \in \mathrm{Ext}^1(\Lambda, \mathbb{Z}) = H^1(B, \check{\Lambda})$ represented by the Čech 1-cocycle

$$\big(U_i \cap U_j, d(\varphi_i - \varphi_j)\big)$$

For any point $y \in B$, $y \in \tau_1 \subseteq \tau_2$ as before, can choose a representative for φ in a neighbourhood of y, yielding a splitting

$$\tilde{\Lambda}_y \cong \Lambda_y \oplus \mathbb{Z}$$

and then defining

$$P_{y,\tau_1} \subseteq \Lambda_y \oplus \mathbb{Z}$$

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Different choices of representatives yield isomorphic choices of P_{y,τ_1}.

We then similarly obtain

$$\begin{aligned} I^{>k}_{y,\tau_1,\tau_2} &\subseteq \Bbbk[P_{y,\tau_1}] \\ R^k_{y,\tau_1,\tau_2} &= \Bbbk[P_{y,\tau_1}]/I^{>k}_{y,\tau_1,\tau_2}. \end{aligned}$$

Parallel transport of monomials:

We can use parallel transport in the sheaf to compare monomials defined at different points.

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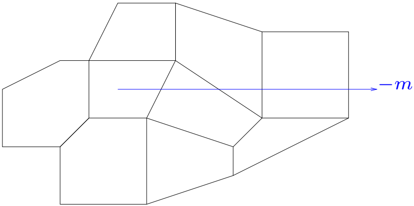
In particular, this may depend on a choice of path connecting these two points!

However, if B has no singularities, $\tilde{\Lambda}$ is has no monodromy in a neighbourhood of any given cell.

Basic fact. If we parallel transport a monomial (m, r) along a straight line in the direction $-m$, the order of (m, r) on maximal cells σ traversed by this line *increases*!



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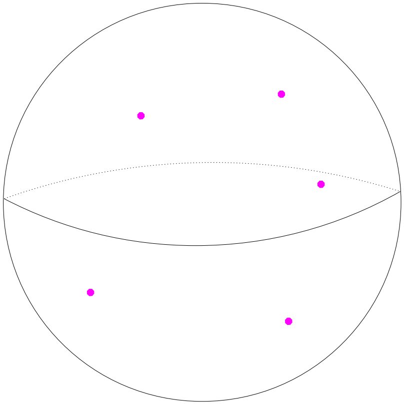
Example. If we take

$$B = \mathbb{R}^n/\Gamma$$

for $\Gamma \subseteq \mathbb{Z}^n$ an integral lattice, this construction yields Mumford's degenerations of abelian varieties, (see also [Alexeev], [Alexeev,Nakamura]).

4. Introducing singularities.

Now let B be an integral affine manifold with singularities, still with a polyhedral decomposition (slightly subtle).



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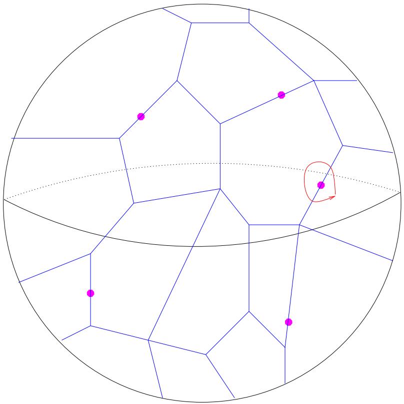

In 2-dim. case, assume monodromy of singularities is

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

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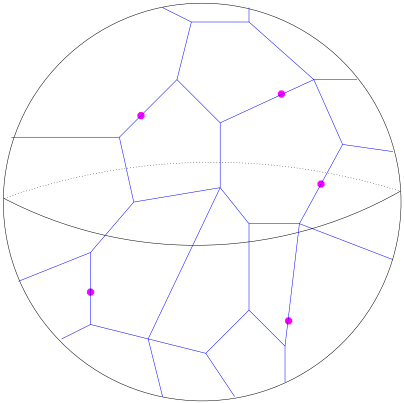
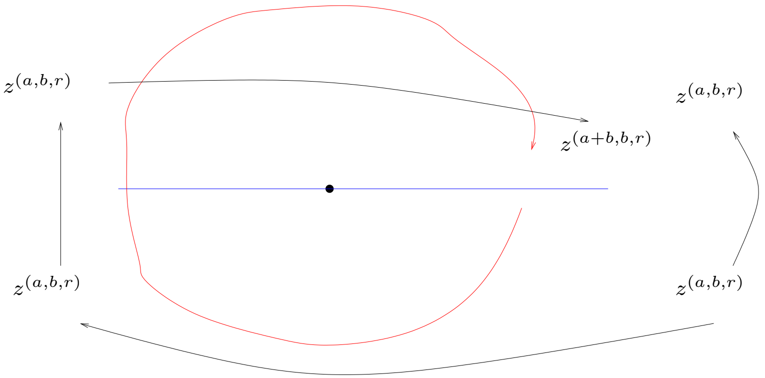
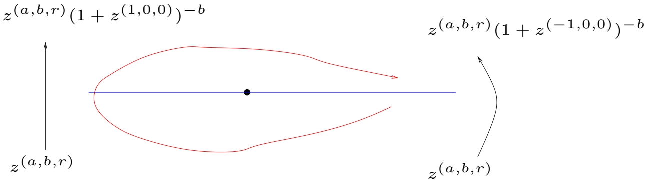


We insist there is an edge through each singular point in the invariant direction.

Problem. Gluing is not well-defined now!



Solution. Modify gluing!



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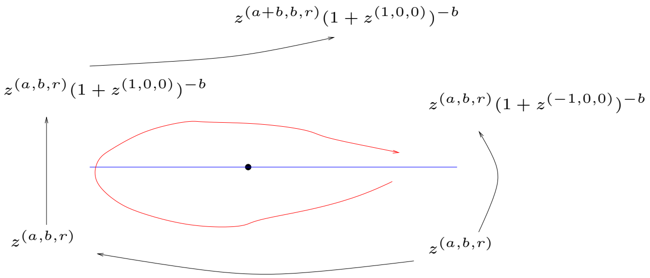

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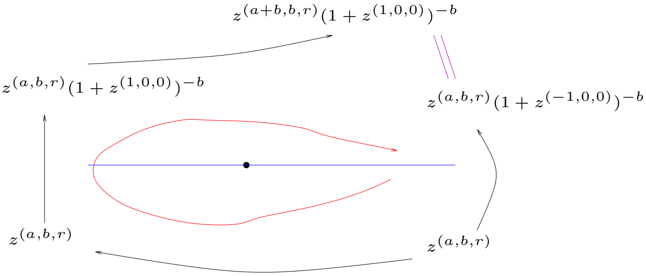

How does this help us?

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Let's examine the gluing closely, taking φ to be given by

$$\varphi(a,b) = \begin{cases} 0 & \text{if } b \leq 0 \\ b & \text{if } b \geq 0 \end{cases}$$

and set

$$\begin{aligned} x &= z^{(0,1,1)} \\ y &= z^{(0,-1,0)} \\ w &= z^{(-1,0,0)} \end{aligned}$$

The glued ring is the fibred product

$$\Bbbk[x,y,w^{\pm 1}]/(y^k) \times_{(\Bbbk[x,y,w^{\pm 1}]/(x^k,y^k))_{1+w}} \Bbbk[x,y,w^{\pm 1}]/(x^k)$$

with the two maps given by

$$x, y, w \mapsto x, y, w$$

and

$$\begin{array}{rcl} x & \mapsto & x/(1+w) \\ y & \mapsto & y(1+w) \\ w & \mapsto & w \end{array}$$

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and

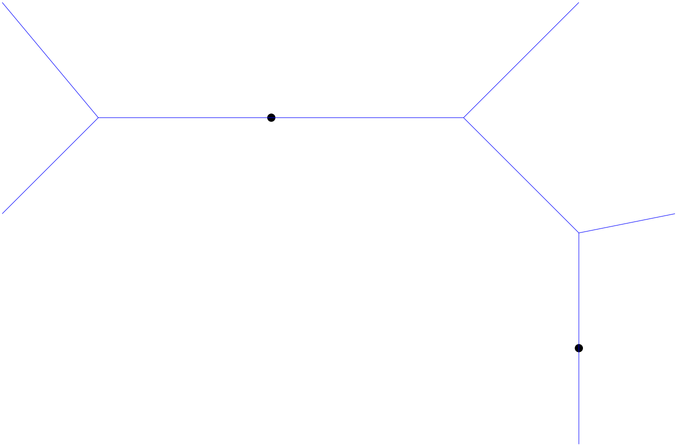
$$\begin{aligned} x &\mapsto x/(1+w) \\ y &\mapsto y(1+w) \\ w &\mapsto w \end{aligned}$$

This fibred product is

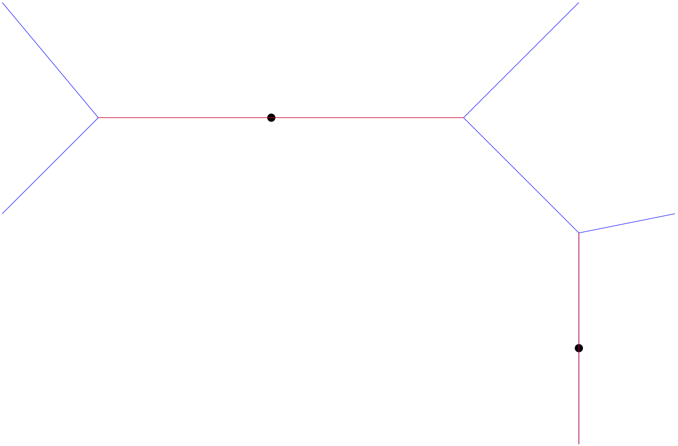
$$\mathbb{k}[X,Y,W,T]/(XY-(1+W)T),$$

$$\begin{aligned} X &= (x, x(1+w)) \\ Y &= (y(1+w), y)) \\ W &= (w, w) \\ T &= (xy, xy) \end{aligned}$$

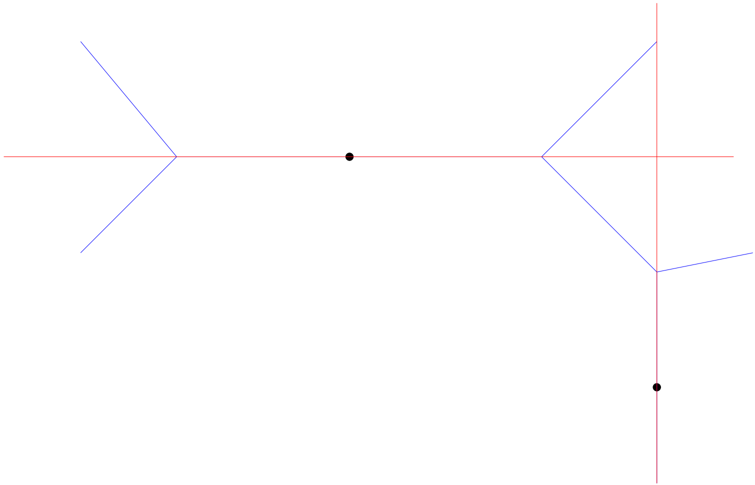
Idea. Attach these gluing automorphisms to straight lines emanating from singularities, along the invariant direction:



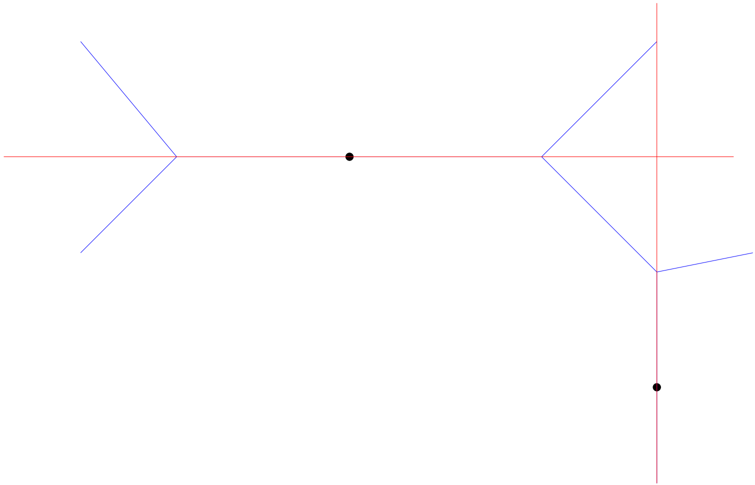
Idea. Attach these gluing automorphisms to straight lines emanating from singularities, along the invariant direction:



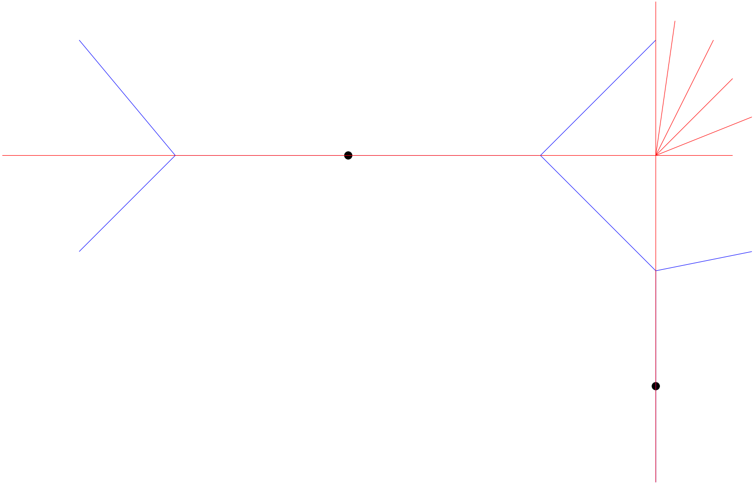
Problem. We then have compatibility problems for gluing at the vertices, so we have to extend the lines and continue to glue affine pieces using this automorphism.

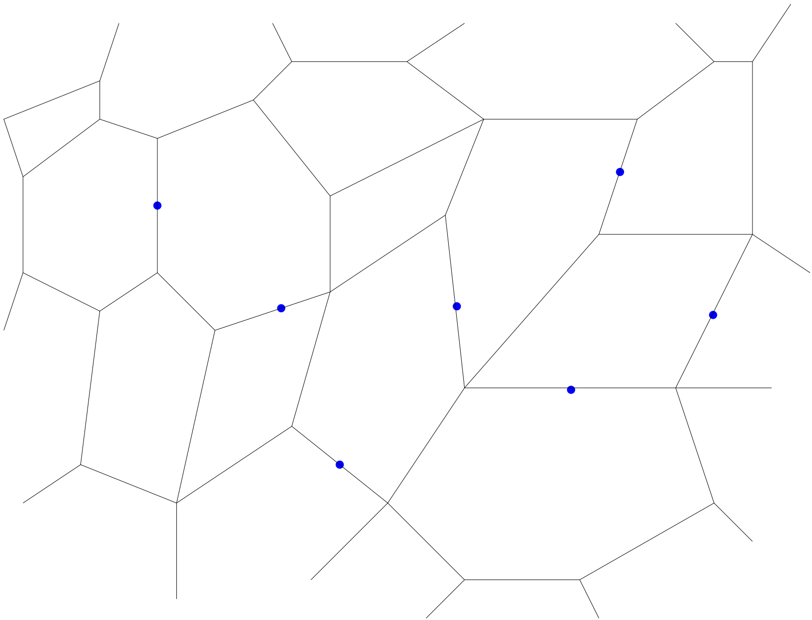


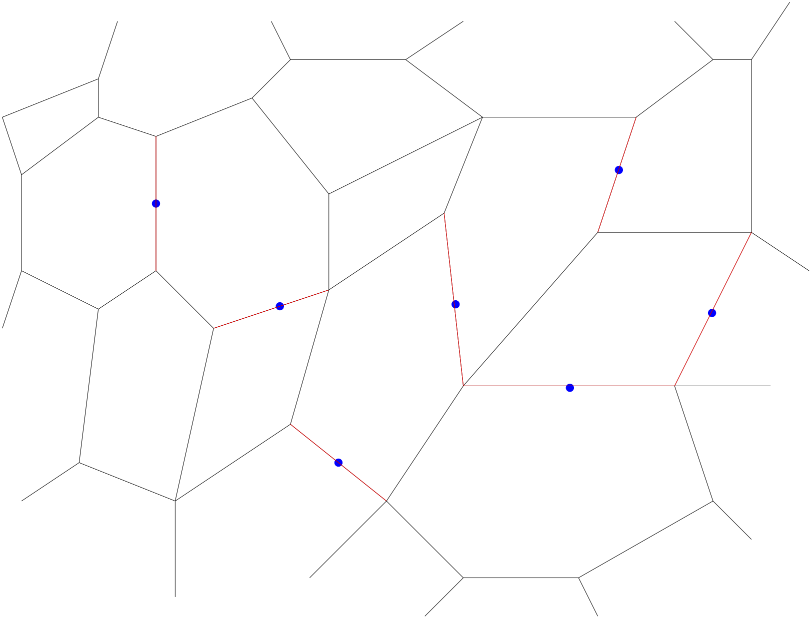
Next problem. When two of these lines collide, we need the composition of automorphisms around the collision point to be the identity, to ensure compatibility of gluing.

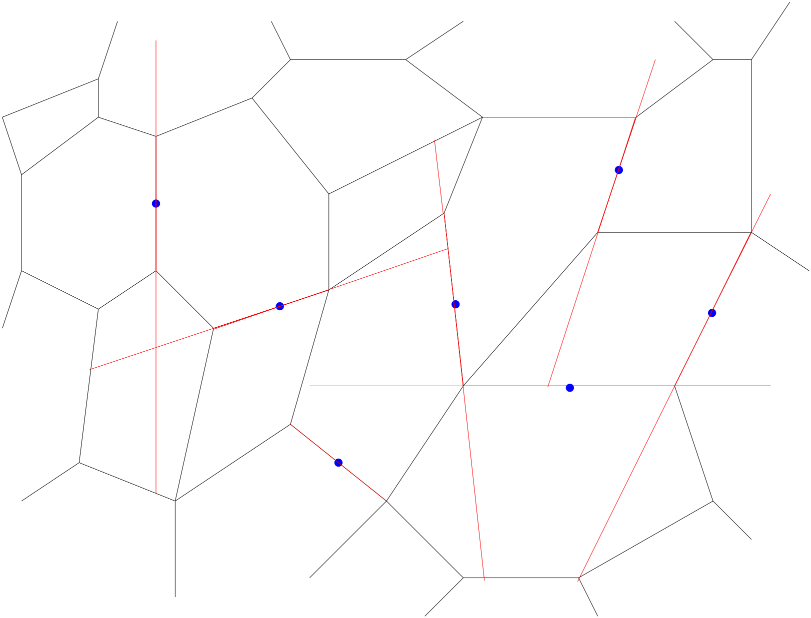


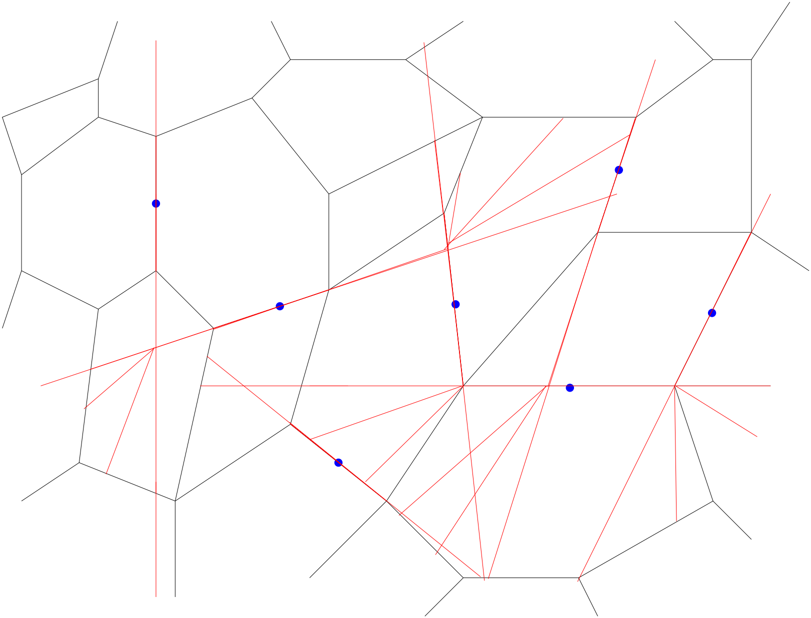
Solution. (This is the main idea of [Kontsevich and Soibelman, 2004]). Add new lines with automorphisms emanating from collision point.

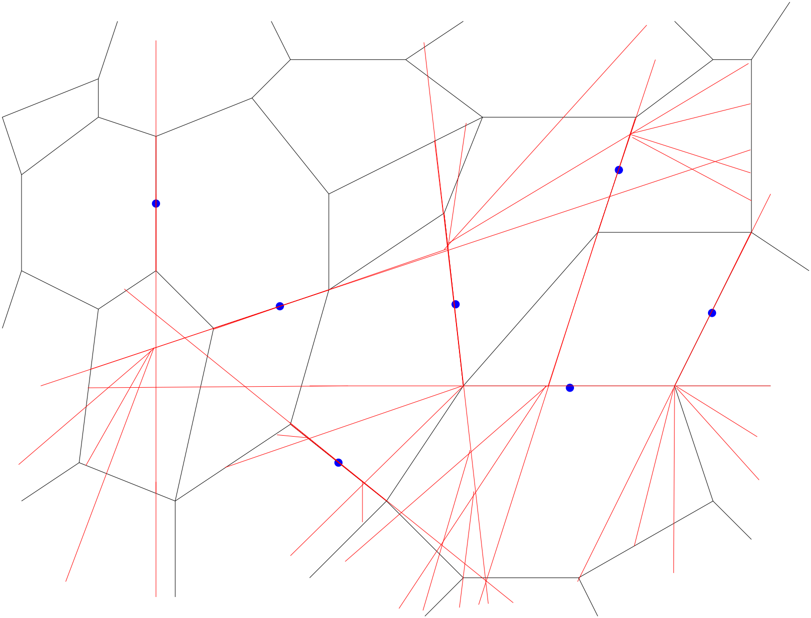












Main Theorem. Given an integral affine manifold B with singularities, polyhedral decomposition $\mathscr{P}$, and strictly convex multi-valued piecewise linear function φ, and given some local conditions on the singularities of B (*local rigidity*) then there exists a degeneration of Calabi-Yau varieties $\mathcal{X} \rightarrow \mathrm{Spec}\,\Bbbk[[t]]$ controlled by this data. The degeneration is uniquely determined by some initial data (*the log structure on the singular fibre*), and is described by a union of affine hyperplanes with attached automorphisms, containing all tropical trees.

Remarks. The existence of such a deformation implies the existence of a flat deformation of the central fibre in the analytic category, i.e. a flat deformation $\mathcal{X} \to D$, where D is a disk. The general fibre will be a compactification of $X(\check{B}_0)$, where $(\check{B}, \check{\varphi})$ is the *discrete Legendre transform* of (B, φ). In particular $\check{X}(B_0) \cong X(\check{B}_0)$ (as topological spaces).

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Finally, given that the deformation $\mathcal{X} \to \mathrm{Spec}\Bbbk[[t]]$ is described in terms of tropical trees, tropical rational curves emerge naturally in the calculation of periods, thus making a direct connection between the B side of mirror symmetry and tropical geometry.