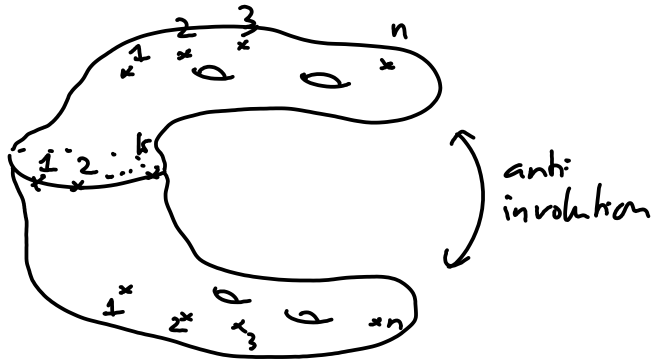
E. Getzler - 27/5/08 - Open-closed topological field theories



Σ closed Riem surface with anti-involution σ

+ marked points, invt. under σ

+ orientation of Σ/σ
(singles out one half of Σ that we're really interested in)

The half of Σ we consider has genus g & h holes (boundaries)

$$\mathbb{T}^n \longrightarrow \mathcal{M}^-_{g,n|k_1\dots k_h}$$
$$\downarrow$$
$$\mathcal{M}_{g,n|k_1\dots k_h}$$

(g = genus, h = # holes
n = # interior marked pts
k_i = # ∂ marked pts on i^{th} boundary)

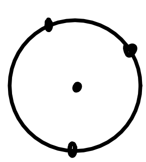
principal fiber bundle associated to the frames of tangent space at the interior marked points (⇐ dual to cotangent lines)

real dimensions: $\dim \mathcal{M} = 6g+3h-6+2n+|k|$
$\dim \mathcal{M}^- = 3(2g+h-2+n)+|k|$

Ex: • $\mathcal{M}^-_{0,3} \cong T^3$ (vs. $\mathcal{M}_{0,3} = pt$)

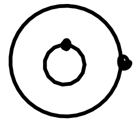
• $\mathcal{M}^-_{0,0|k} = \mathcal{M}_{0,0|k} = \coprod (k-1)!$ interiors of associahedra
$\uparrow$
orderings of mkd pts along boundary

• $\mathcal{M}^-_{0,1|k}$ = $(k-1)!$ copies of $\mathbb{T} \times$ (interior of $(k-1)$-dim! cyclohedra)



$k=1$: $\mathbb{T}$
$k=2$: $\mathbb{T}^2 \times (0,1)$
$k=3$: $2 \times \mathbb{T}^3 \times$ hexagon

• $\mathcal{M}_{0,0|1,1} \cong$ punctured disk

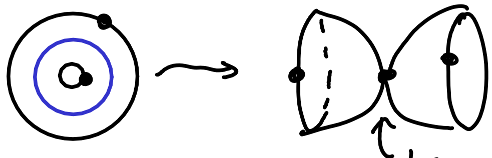


(modulus + rotation difference)

∃ <u>compactification</u> $\mathcal{M}^- \hookrightarrow \mathcal{M}^+$ (stable compactification)

<u>Ex.</u>: for $\mathcal{M}^+_{0,0|1,1}$ this includes:

(1) • collapsing of a loop:

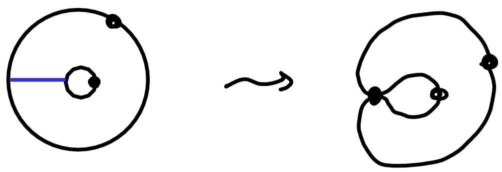


keep track of tangent space identification at node

⇝ this stratum $\approx \left(\mathcal{M}^-_{0,1|1} \times \mathcal{M}^-_{0,1|1}\right)/\mathbb{T}^1 \cong \mathbb{T}^1$

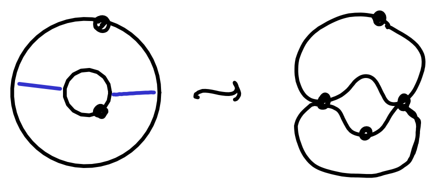
gluing must respect triv^{ns} of tgt space at node

(2) • collapsing of an arc:



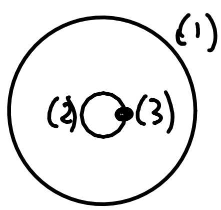
this stratum = interval (a component of $\mathcal{M}_{0,0|4}$)

(3) • at end pts of interval, get



stratum = point

$\Rightarrow \mathcal{M}^+_{0,0|1,1} =$

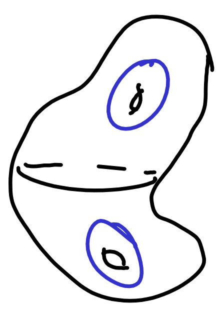


closed annulus, top. retracts onto $\mathcal{M}^-$.

[NOTHING BEYOND these because of <u>stability</u> requirement!]

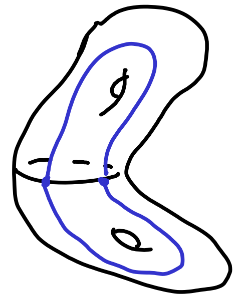
<u>Rmk.</u>: can view these collapsings on the doubled surface:

• collapse 2 symmetric loops which don't intersect Fix(σ)



⟷ collapse closed loop (case (1))

- or collapse a symm. loop which hits $\mathrm{Fix}(\sigma)$



$\longleftrightarrow$ collapse arc.

<u>Ex.:</u> $\mathcal{M}^-_{0,0|5} \hookrightarrow \mathcal{M}^+_{0,0|5} \xrightarrow[2:1]{\text{generally}} \bar{\mathcal{M}}_{0,5}(\mathbb{R}) \hookrightarrow \bar{\mathcal{M}}_{0,5}(\mathbb{C})$

24 copies of interior of a pentagon

24 pentagons

loss of choice of orient^n of Σ/σ

$\cong \mathbb{RP}^2$ blown up at 4 pts

$\cong \mathbb{CP}^2$ blown up at 4 pts

(Morava: unorientable dodecahedron 12 pentagons, 30 edges, 15 vertices)

on 24 pentagons = orientation double cover of faces

<u>Open-closed TFT</u>:

- <u>open states</u>: (V, δ) — V carries a non-deg. inner product; δ BRST differential (skew-adjoint) — e.g. $CF^*(L,L)$, L Lagrangian

- <u>closed states</u>: (W, δ) — e.g. $C^*(M)$, M symplectic

  carries a non-deg. inner product st. δ skew adjoint <u>and</u> a <u>circle action</u> generated by an operator

  $$L = [\delta, G], \quad G^2 = 0.$$

- <u>Correlators of the theory</u>: $\omega \in \Omega^*\left(\mathcal{M}^+_{g,n|k_1 \dots k_h}, V^{\otimes \sum k_i} \otimes W^{\otimes n}\right)$

s.t. 0) ω's are equivariant wrt symmetric group action
(permutation of marked point labels)
and basic wrt $\mathbb{T}^n$-action (rotation at interior marked pts)

1) $(d+\delta)\,\omega = 0$

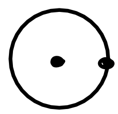
2) formulas for ω restricted to $\partial \mathcal{M}^+$

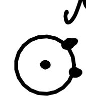
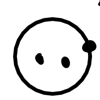
<u>Consequently:</u> • $\mathcal{M}^+_{0,3}$ induces a <u>commutative product</u> <u>on $H^*(W,\delta)$</u>
(in fact a BV-algebra because of S^1-actions)

$\mathcal{M}^+_{0,4} \Rightarrow$ product is <u>associative</u> (look at arc connecting 2 pts on ∂ of $\overline{\mathcal{M}}_{0,4}$).

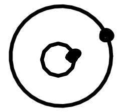
• $\mathcal{M}^+_{0,0|3}$ induces a <u>product</u> <u>on $H^*(V,\delta)$</u>

$\mathcal{M}^+_{0,0|4} \Rightarrow$ <u>associativity</u>

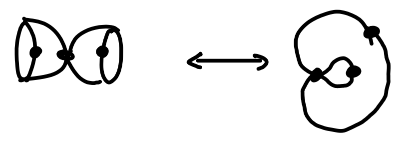
• $\mathcal{M}^+_{0,1|1}$  gives a pairing between $H^*(W,\delta)$ and $H^*(V,\delta)$:

- $\tau_*: H^*(V) \to H^*(W)$ (open → closed) $\mathcal{M}^+_{0,1|2}$  says this is a <u>trace</u>
- $\tau^*: H^*(W) \to H^*(V)$ $\mathcal{M}^+_{0,2|1}$  says this is an algebra <u>homomorphism</u>

• One more axiom: <u>Cardy condition</u>

$\mathcal{M}^+_{0,0|1,1}$  $\rightsquigarrow \langle \tau_* x, \tau_* y\rangle_W = \mathrm{Tr}_V\big(\lambda(x)\cdot\rho(y)\big)$ $\forall x,y \in V$

left mult by x, right mult by y



<u>Ex:</u> holds if $V = \mathbb{C}[G]$ G finite group, and $W =$ center of $\mathbb{C}[G]$.

- We can filter $\mathcal{M}^+$ by "skeleta".

  (generalizes usual filtration by skeleton of associahedra cyclohedra)

Thm: $(\mathcal{M}^+, \partial\mathcal{M}^+)$ is α-connected where

	g	n	h	α	
	0	n	0	$n-3$	$\mathcal{M}^+_{0,n}$
	0	0	1	$k-3$	$\mathcal{M}^+_{0,0\vert k}$
	0	1	1	$k-1$	$\mathcal{M}^+_{0,1\vert k}$
$(g>0)$	g	n	0	$2g-2+n$	(use Harer stability)
$(h>0,\ g>0$ or $h>1)$	g	n	h	$2g+n+\vert k\vert-1$	

Corollary: Let $sk_p(\mathcal{M}^+)$ be the union of strata where $\sum_{\text{compact } \Sigma_i} \alpha_i \leq p$ (α as in theorem). Then $(\mathcal{M}^+, sk_p\mathcal{M}^+)$ is p-connected.

Rmk: topological statement of Costello's thm:

Thm (Costello): $\mathcal{M}^0 \subset \mathcal{M}^+$ is a retraction

$\uparrow$ union of strata which are products of associahedra & cyclohedra

($\rightsquigarrow$ consequences for TFT's)