Developments in the noncommutative Batalin-Vilkovisky formalism

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- $S = \sum_{g \geq 0} \hbar^{2g-1+i} S_{g,i}\,, \quad S_{g,i} \in Symm^{i},$

$$\{S_{0,1}, S_{0,1}\} = 0,$$

$S_{0,1}$- $A_{\infty}-$algebra with (even/odd) scalar product, so $S-$*multiloop, higher genus* generalization of $A_{\infty}-$algebra.

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- $$\Longleftrightarrow \Delta(\exp\frac{1}{\hbar}S) = 0$$

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- gives framework in order to find nice higher genus analogue for the theory of variations of (nc-)Hodge structures (of CY-type), (recall (S.B., 2000), $A_\infty-$periods:

$$nc-VHS \quad (\ HC^-_t \subset HP) \to \left(H_*(\overline{\mathcal{M}}_{0,n}) - action\right) \ on \ HH$$

also with exponential (nc-)Hodge

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- Related on fundamental level with *supersymmetric* simple associative superalgebras: *odd general linear algebra* $q(N)$ of Bernstein-Leites, and with $gl(N|N)$, via invariant calculus on $q(N) \otimes \Pi V$, $gl(N|N) \otimes \Pi V$.

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- (S.B.,2006b) "SUPER-SYMMETRIC MATRIX INTEGRALS" integration theory (*super*-invariant w.r.t $q(N)$ and $gl(N|N)$) in the non-commutative setting with finite-dimensional integrals, $\Delta \leftrightarrow d_{DR}$

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Noncommutative Batalin-Vilkovisky differential (even inner product)

- Let $V = V_0 \oplus V_1$, $\beta : V^{\otimes 2} \to k$ be an even symmetric inner product on V:

$$\beta(x, y) = (-1)^{\overline{xy}} \beta(y, x)$$

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- Define the noncommutative BV differential on F via

$$\Delta(a_{\rho_1}\dots a_{\rho_r})^{\lambda}(a_{\tau_1}\dots a_{\tau_t})^{\lambda} =$$
$$= \sum_{p,q}(-1)^{\varepsilon}\beta_{\rho_p\tau_q}(a_{\rho_1}\dots a_{\rho_{p-1}}a_{\tau_{q+1}}\dots a_{\tau_{q-1}}a_{\rho_{p+1}}\dots a_{\rho_r})^{\lambda}+$$
$$\sum_{p\pm 1\neq q}(-1)^{\widetilde{\varepsilon}}\beta_{\rho_p\rho_q}(a_{\rho_1}\dots a_{\rho_{p-1}}a_{\rho_{q+1}}\dots a_{\rho_r})^{\lambda}(a_{\rho_{p+1}}\dots a_{\rho_{q-1}})^{\lambda}(a_{\tau_1}\dots a_{\tau_t})^{\lambda}$$
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- Theorem (S.B.,2009b) $Ker \Delta_1 + \Delta_2 = \mathrm{Im}\, \Delta_1 + \Delta_2$ (∼?related with Madsen-Weiss)...

Solutions (A-model).

- Conjecture (S.B,2006a). Counting of holomorphic curves $(\Sigma, \partial\Sigma, p_i) \longrightarrow (M, \coprod L_i, \oplus H_*(L_i \bigcap L_j))$, with $\mathbb{Z}/2\mathbb{Z}$-graded local systems, gives solution to the nc-BV equations.

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- subtleties:

$$\frac{1}{2}\dim_R M - even \Longrightarrow F = \Lambda(C^\lambda)$$

$$\frac{1}{2}\dim_R M - odd \Rightarrow F = S(C^\lambda),$$

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- Assume: I - an *odd derivation* acting on V, preserving the scalar product: , in general $I^2 \neq 0$ (!), $\exists \widetilde{I}$, $[I, \widetilde{I}] = 1$, $str([a, \cdot]) = 0$ for any $a \in A$.

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- Example $q(N)$, $q(N) = \{[X, \pi] = 0 | X \in gl(N|N)\}$,where $\pi-$odd involution, $q(N)$ has *odd trace otr*, $I = [\Xi, \cdot]$, Ξ- odd element $\Xi = \begin{pmatrix} 0 & |diag(\lambda_1, \ldots, \lambda_n) \end{pmatrix}$, ($I^2 \neq 0$ (!))

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- Similarly, with even scalar product and an *odd derivation,* with, in general $I^2 \neq 0$.

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