

MIRROR SYMMETRY: LECTURE 1

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Goal of the class: this is not always going to be the most rigorous class, but the goal is to tell the story of mirror symmetry.

1. PHYSICAL ORIGINS

Mirror symmetry comes from statements in supersymmetric string theory. Basic idea of string theory: replace particles with vibrating strings, which propagate through space and form surfaces. We thus get 2-d quantum field theories on these surfaces, called *worldsheets*, with the fields taking values in some manifolds. Supersymmetric means that there are lots of symmetries acting here, and physicists have lots of buzzwords here that won't make sense to mathematicians. For instance "superconformal field theory" means that the theory depends only on the conformal structure on Σ rather than the Riemannian structure. There are various flavors of these theories, but the ones more relevant to mirror symmetry are the *nonlinear sigma-models*

1.1. Non-linear Sigma Models. Here, we look at maps $\phi : \Sigma \rightarrow X$, where X (the "target space") is a *Calabi-Yau manifold*.

Definition 1. A Calabi-Yau manifold is a complex manifold (X, J) (ideally, compact, 3-dimensional, maybe $b_1 = 0$) s.t. $K_X = \bigwedge^n T^*X \cong \mathcal{O}_X$, so \exists a section $\Omega \in H^0(X, K_X)$, i.e. a holomorphic volume form.

On this manifold, we have a *complexified Kähler form* $\omega^{\mathbb{C}} = B + i\omega$ which is a closed $(1, 1)$ -form and $\text{Im } \omega^{\mathbb{C}} = \omega$ is nondegenerate (B is supposed to make the space of symplectic forms a subspace of complex cohomology). The "moduli" of the theory, i.e. how one can deform the complex structure J and the complexified symplectic structure $\omega^{\mathbb{C}}$, is governed by $H^1(X, T_X)$ and $H^{1,1}(X) = H^1(X, \Omega_X^1)$ respectively.

More physics: the field theory is governed by a really big lie algebra, but two of the interesting generators (Q, \overline{Q}) are called "supersymmetric charges": each particle is assigned some eigenvalue of these indicating its charge. We have simultaneous eigenspaces $H^q(X, \Omega^p T_X), H^q(X, \Omega_X^p)$. But many of our choices here are arbitrary: for instance, we should be able to replace Q with $-Q$, which exchanges these two eigenspaces. Thus, we have an idea that for any target

space on which superstring theory acts, these two spaces (a priori of different dimension) are the opposite spaces for a different target space (the “mirror”).

Conjecture 1. *Given a Calabi-Yau manifold $(X, J, \omega^{\mathbb{C}})$, one can find another Calabi-Yau manifold $(X^{\vee}, J^{\vee}, \omega^{\mathbb{C}\vee})$ s.t. $H^q(X, \Omega^p TX) \cong H^q(X^{\vee}, \Omega^p_{X^{\vee}})$ naturally and vice versa, i.e. we get a local isomorphism on the tangent spaces to their corresponding moduli spaces. Additionally, we’d like to say that the superconformal field theories are equivalent (mostly physics-related statement).*

This statement is definitely not true in the generality above, and one manifold can have various mirrors.

1.2. A-model and B-model. There exist two “topologically twisted” variants of nonlinear sigma models called the A -model and the B -model which have the following nice feature: although the model above depends on both the complex and symplectic (i.e. complexified Kähler) structure, the A -model depends only on $\omega^{\mathbb{C}}$ while the B -model only depends on J . Mirror symmetry thus says that the A -model on $(X, \omega^{\mathbb{C}}) \leftrightarrow B$ -model on (X^{\vee}, J^{\vee}) and the B -model on $(X, J) \leftrightarrow A$ -model on $(X^{\vee}, \omega^{\mathbb{C}\vee})$.

To make mathematical sense of this, we would like numerical quantities that define what these models do (note that the A and B -models don’t actually correspond to the physical world). We specifically look at “correlation functions”, for instance Gromov-Witten invariants.

2. HODGE THEORY AND QUANTUM COHOMOLOGY

The first feature that we expect is that $H^q(X, \Omega^p TX) \cong H^q(X^{\vee}, \Omega^p_{X^{\vee}})$. In the Calabi-Yau case, the exterior powers $\bigwedge^p TX$ are naturally isomorphic to Ω^{n-p} , so $v_1 \wedge \cdots \wedge v_p \mapsto i_{v_1} \cdots i_{v_p} \Omega$. In terms of Dolbeault cohomology, $H^{n-p,q}(X) \cong H^{p,q}(X^{\vee})$. This has some odd consequences: for instance, $H^3(X, \mathbb{C}) = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3} \cong H^{0,0} \oplus H^{1,1} \oplus H^{2,2} \oplus H^{3,3}$. One can cook up examples that match these numerically (i.e. by dimension), but we want a stronger duality.

A stronger notion relies on “Yukawa couplings” (which are actually triplings) on $H^{1,1}$ and $H^1(X, TX)$. On $H^{1,1}(X)$ (i.e. the A -model), we have

(1)

$$\langle \omega_1, \omega_2, \omega_3 \rangle = \int_X \omega_1 \wedge \omega_2 \wedge \omega_3 + \sum_{\beta \in H_2(X, \mathbb{Z}), \beta \neq 0} n_{\beta} \int_{\beta} \omega_1 \int_{\beta} \omega_2 \int_{\beta} \omega_3 \frac{e^{2\pi i \int_{\beta} \omega^{\mathbb{C}}}}{1 - e^{2\pi i \int_{\beta} \omega^{\mathbb{C}}}}$$

The latter term will have infinitely many terms, and should converge, but one can think of this as a power series in the final quantity. n_{β} is the “number of genus 0 (rational) complex curves in X representing the class β ”. It is not clear why this should be a number, and what specific kinds of curves we actually want to count: the values here emerge from *Gromov-Witten theory*.

On the other hand, on $H^1(X, TX) \cong H^{2,1}(X)$, we define

$$(2) \quad \langle \theta_1, \theta_2, \theta_3 \rangle = \int_X \Omega \wedge (\theta_1 \cdot \theta_2 \cdot \theta_3 \cdot \Omega)$$

where the interior product on the right is

$$(3) \quad H^1(X, TX)^{\otimes 3} \otimes H^0(X, \Omega_X^3) \rightarrow H^3(X, \bigwedge^3 TX \otimes \Omega_X^3) = H^{0,3}(X)$$

We can also interpret this as derivatives of Ω as we change the complex structure (via the Gauss-Manin connection). The claim of mirror symmetric is that, given two mirror manifolds, we have the above isomorphisms and these two couplings are identified with each other.

Remark. As stated, this depends on choice of holomorphic volume form, and one could choose any multiple: one thus needs to normalize the volume form.

How do we identify these relations? We hope that we have a relation between $\omega^{\mathbb{C}}$ on X and J^{\vee} on X^{\vee} that exists on the entire moduli space of Calabi-Yau manifolds. That is, this should be induced by a *mirror map* $\mathcal{M}_{\text{sympl}}(X) \rightarrow \mathcal{M}_{\text{cx}}(X^{\vee})$ between the moduli spaces of symplectic and complex forms s.t., on tangent spaces, we have the correspondence $H^{1,1}(X) \cong H^1(X^{\vee}, TX^{\vee})$ under which the above couplings should agree.

The first mathematical prediction was due to Candelas-de la Ossa-Green-Parkes in 1991. For X a quintic 3-fold, i.e. a degree 5 hypersurface in $\mathbb{C}\mathbb{P}^4$ (we do not need to identify a particular one, as the symplectic structure only depends on the volume of $\mathbb{C}\mathbb{P}^4$). Since $H_2(X, \mathbb{Z}) \cong \mathbb{Z}$, we identify the curves by degree d . For instance, $n_1 = 2875$ lines on X , $n_2 = 609250$ conics, and a few higher values were known. CdGP predicted the values n_d in general (more precisely, they calculated the associated Gromov-Witten invariants, and conjectured that they actually do count rational curves). They did this by computing on $X^{\vee} = \widetilde{\text{quintic}/(\mathbb{Z}/5)^3}$ which is equipped with a one-parameter complex deformation: note that $h^{2,1}(X^{\vee}) = 1 = h^{1,1}(X)$, $h^{1,1}(X^{\vee}) = 101 = h^{2,1}(X)$. CdGP defined the mirror map using equations defined by Hodge theory. On the symplectic side, the couplings are fairly easy to compute. Expanding the equation on the complex side and extracting the coefficients of the power series, one obtains the n_d . We will go over all of this in more detail over the next month.

3. HOMOLOGICAL MIRROR SYMMETRY (KONTSEVICH '94)

All the above lies in the world of “closed string theory”, i.e. particles are loops and the world sheet is compact. We can instead look at “open string theory”, where the worldsheet is a surface with bound and we have constraints on the values of fields at the boundary (called D -branes after Dirichlet). Moreover, the axioms of field theory on gluing boundaries should give us a category of

D -branes. On the A -model, the branes are Lagrangian submanifolds with a flat bundle over L , and in the B -model branes are complex analytic submanifolds with a holomorphic vector bundle. Kontsevich came up with a precise mathematical conjecture of what these categories should be.

Conjecture 2 (Kontsevich's HMS). *If $(X, J, \omega^{\mathbb{C}})$ and $(X^{\vee}, J^{\vee}, \omega^{\mathbb{C}\vee})$ are mirrors, then*

$$(4) \quad \begin{aligned} D^b\text{Fuk}(X, \omega^{\mathbb{C}}) &\cong D^b\text{Coh}(X^{\vee}, J^{\vee}) \\ D^b\text{Coh}(X, J) &\cong D^b\text{Fuk}(X^{\vee}, \omega^{\mathbb{C}\vee}) \end{aligned}$$

as an equivalence of triangulated categories.

Here, the Fukaya category $\text{Fuk}(X, \omega^{\mathbb{C}})$ is the category whose objects are Lagrangian submanifolds with flat bundle, and the morphisms are given by intersection theory with their compositions given by Floer homology and operations on it. On the other side, we have the category of coherent sheaves $\text{Coh}(X, J)$, examples of which are vector bundles on complex submanifolds (which correspond to skyscraper sheaves). The symbol D^b corresponds to enlarging to the triangulated *derived category*, i.e. complexes of such objects up to homotopy.

Example. The case $X = T^2/\text{elliptic curve}$ was understood by Polishchuk-Zazlow.

4. STROMINGER-YAU-ZASLOW CONJECTURE (1996)

All the above stuff is properties of pairs of manifold that one knows to be mirrors. The SYZ conjecture helps us construct mirror pairs.

Conjecture 3 (SYZ). *For X, X^{\vee} mirrors, they carry mutually dual fibrations by special Lagrangian tori, i.e.*

$$(5) \quad \begin{array}{ccc} T^n & \longrightarrow & X, \quad (T^{\vee})^n & \longrightarrow & X^{\vee} \\ & & \downarrow & & \downarrow \\ & & B & & B \end{array}$$

Here, $L^n \subset X$ is special Lagrangian if $\omega|_L = 0$ and $\text{Im } \Omega|_L = 0$, and $T^{\vee} = \text{Hom}(\pi_1(T), U(1))$.