

18.966 – Homework 3 – Solutions.

1. Given a point $p \in C'$ (a two-dimensional oriented submanifold), let (e, f) be an oriented basis of $T_p C'$, orthonormal with respect to the metric g induced by ω and J . Then $\omega(e, f) = g(Je, f) \leq |Je| |f| = |e| |f| = 1$. Meanwhile, the area form $dvol_{g|_{C'}}$ induced by g on C' is given by $dvol_{g|_{C'}}(e, f) = 1$. Hence $\omega|_{C'} \leq dvol_{g|_{C'}}$ at every point of C' ; integrating, we deduce that $[\omega] \cdot [C'] = \int_{C'} \omega \leq vol_g(C')$.

In the case of C (an almost-complex submanifold, equipped with the orientation induced by J), an oriented orthonormal basis of $T_p C$ is given by (e, Je) where e is any unit length vector in $T_p C$. (Note that $|Je| = |e| = 1$ and $g(Je, e) = \omega(e, e) = 0$). Then $\omega(e, Je) = g(Je, Je) = 1 = dvol_{g|_C}(e, Je)$, so $\omega|_C = dvol_{g|_C}$, and $[\omega] \cdot [C] = \int_C \omega = vol_g(C)$.

In conclusion, $vol_g(C) = [\omega] \cdot [C] = [\omega] \cdot [C'] \leq vol_g(C')$.

2. a) The homogeneous coordinate x_n is a linear form on \mathbb{C}^{n+1} (namely, $(x_0, \dots, x_n) \mapsto x_n$) and hence, by restriction to the tautological line, a linear form on L . This section of L^* vanishes precisely at those points $[x_0 : \dots : x_n]$ for which the last coordinate is zero, so its zero set is $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$. Moreover, it vanishes transversely, and the orientation induced on its zero set is the natural one (because all orientations agree with those induced by the complex structure). So $c_1(L^*) = e(L^*)$ is Poincaré dual to $[\mathbb{C}P^{n-1}] \in H_{2n-2}(\mathbb{C}P^n)$, i.e. $c_1(L^*) = h$. Therefore $c_1(L) = -c_1(L^*) = -h$.

Given a line $\ell \subset \mathbb{C}^{n+1}$ (defining a point $p = [\ell] \in \mathbb{C}P^n$), any nearby line can be parametrized by a map $\ell \rightarrow \mathbb{C}^{n+1}$, $x \mapsto x + u(x)$, where $u \in \text{Hom}(\ell, \mathbb{C}^{n+1})$. This gives a map (in fact a local submersion) $\psi : \text{Hom}(\ell, \mathbb{C}^{n+1}) \rightarrow \mathbb{C}P^n$ defined by $\psi(u) = [\text{Im}(\text{Id} + u)]$. Its differential at the origin is $\Psi = d_0 \psi : \text{Hom}(\ell, \mathbb{C}^{n+1}) \rightarrow T_p \mathbb{C}P^n$. We claim that Ψ is surjective, with kernel $\text{Hom}(\ell, \ell) \simeq \mathbb{C}$ (those linear maps whose image is contained in ℓ). Indeed, this can be checked easily in the case where ℓ is the first coordinate axis, and $\psi((u_0, \dots, u_n)) = [1 + u_0 : u_1 : \dots : u_n]$. Therefore, we have a short exact sequence of vector bundles $0 \rightarrow \underline{\mathbb{C}} = \text{Hom}(L, L) \rightarrow \text{Hom}(L, \underline{\mathbb{C}}^{n+1}) = (L^*)^{n+1} \rightarrow T\mathbb{C}P^n \rightarrow 0$ (where $\underline{\mathbb{C}}$ denotes the trivial line bundle over $\mathbb{C}P^n$).

Taking a complement F to the subbundle $\text{Hom}(L, L) \subset \text{Hom}(L, \underline{\mathbb{C}}^{n+1})$ (e.g. its orthogonal complement for some Hermitian metric), the restriction of Ψ to F is an isomorphism, so we conclude that $\text{Hom}(L, \underline{\mathbb{C}}^{n+1}) = (L^*)^{n+1}$ is isomorphic to $T\mathbb{C}P^n \oplus \underline{\mathbb{C}}$.

Since Chern classes behave multiplicatively under direct sums, and $c(L^*) = 1 + c_1(L^*) = 1 + h$, we have $c(T\mathbb{C}P^n) = c(T\mathbb{C}P^n \oplus \underline{\mathbb{C}}) = c((L^*)^{n+1}) = (1 + h)^{n+1}$. Expanding into powers of h , we deduce that $c_k(T\mathbb{C}P^n) = \binom{n+1}{k} h^k$ for all $1 \leq k \leq n$.

b) Consider $X = P^{-1}(0)$, where P is a homogeneous polynomial of degree d in the homogeneous coordinates, i.e. a section of $(L^*)^{\otimes d}$. Fix any connection on $(L^*)^{\otimes d}$. If we assume that P is transverse to the zero section, then at any point $x \in X$ the linear map $(\nabla P)_x : T_x \mathbb{C}P^n \rightarrow (L^*)_x^{\otimes d}$ (which does not depend on the chosen connection since $P(x) = 0$) is surjective and its kernel is $T_x X$ (see Homework 2). Therefore we get a short exact sequence of vector bundles $0 \rightarrow TX \rightarrow T\mathbb{C}P^n|_X \rightarrow (L^*)^{\otimes d}|_X \rightarrow 0$, and considering again a complement to TX in $T\mathbb{C}P^n|_X$ we conclude that $T\mathbb{C}P^n|_X \simeq TX \oplus (L^*)^{\otimes d}|_X$.

Using additivity of the first Chern class of a line bundle under tensor product, we have $c_1((L^*)^{\otimes d}) = 1 + dh$. Let $\alpha = h|_X \in H^2(X, \mathbb{Z})$ (the pullback of h by the inclusion $i : X \hookrightarrow \mathbb{C}\mathbb{P}^n$). Using the multiplicativity of Chern classes under direct sums and their functoriality under pullback, we deduce that $(1 + \alpha)^{n+1} = c(TX) \cdot (1 + d\alpha)$.

Since $\alpha^n = 0$ in the cohomology of X (for dimension reasons), $1 + d\alpha$ is invertible, with inverse $(1 + d\alpha)^{-1} = \sum_{k=0}^{n-1} (-1)^k d^k \alpha^k$. The total Chern class of TX is then $1 + c_1(TX) + \dots + c_{n-1}(TX) = (1 + d\alpha)^{-1}(1 + \alpha)^{n+1}$.

3. a) The Hodge $*$ operator on $\Omega^2(M^4)$ satisfies $*^2 = 1$, and every 2-form α decomposes into the sum of a selfdual part $\alpha^+ = \frac{1}{2}(\alpha + *\alpha)$ and an antiselfdual part $\alpha^- = \frac{1}{2}(\alpha - *\alpha)$. On an even-dimensional manifold, $d^* = -*d*$ in all degrees, so $\Delta = dd^* + d^*d = -d*d* - *d*d$ commutes with $*$. Therefore, if α is harmonic then so is $*\alpha$, and hence so are α^+ and α^- .

So every harmonic form α decomposes into the sum of a harmonic selfdual form (α^+) and a harmonic antiselfdual form (α^-). Moreover, selfdual and antiselfdual forms are obviously in direct sum; so $\mathcal{H}^2 = \mathcal{H}_+^2 \oplus \mathcal{H}_-^2$ (this decomposition corresponds to the ± 1 eigenspaces of $*$: $\mathcal{H}^2 \rightarrow \mathcal{H}^2$).

If α is a nontrivial selfdual form then $\int_M \alpha \wedge \alpha = \int_M \alpha \wedge *\alpha = \int_M \langle \alpha, \alpha \rangle \text{dvol}_g = \|\alpha\|_{L^2}^2 > 0$; and if $\beta \neq 0$ is antiselfdual then $\int_M \beta \wedge \beta = -\int_M \beta \wedge *\beta = -\|\beta\|_{L^2}^2 < 0$. Moreover $\langle \alpha, \beta \rangle = \alpha \wedge *\beta = -\alpha \wedge \beta = -\beta \wedge \alpha = -\beta \wedge *\alpha = -\langle \beta, \alpha \rangle$, so $\alpha \wedge \beta$ is pointwise 0, and $\int_M \alpha \wedge \beta = 0$. Thus \mathcal{H}_\pm^2 are orthogonal and definite positive (resp. definite negative) for the intersection pairing.

b) At any point of M , the tangent space and the compatible triple (ω, J, g) can be identified with $(\mathbb{R}^4, \omega_0, J_0, g_0)$, with standard basis (e_1, e_2, e_3, e_4) , and $J_0(e_1) = e_2, J_0(e_3) = e_4$. In terms of the dual basis,

$$\begin{aligned} \Lambda_+^2 &= \text{span}(e^1 \wedge e^2 + e^3 \wedge e^4, e^1 \wedge e^3 - e^2 \wedge e^4, e^1 \wedge e^4 + e^2 \wedge e^3), \\ \Lambda_-^2 &= \text{span}(e^1 \wedge e^2 - e^3 \wedge e^4, e^1 \wedge e^3 + e^2 \wedge e^4, e^1 \wedge e^4 - e^2 \wedge e^3). \end{aligned}$$

Meanwhile, $\omega = e^1 \wedge e^2 + e^3 \wedge e^4$, and $\Lambda^{2,0}$ is spanned by

$$(e^1 + ie^2) \wedge (e^3 + ie^4) = (e^1 \wedge e^3 - e^2 \wedge e^4) + i(e^1 \wedge e^4 + e^2 \wedge e^3),$$

while $\Lambda^{0,2}$ is the complex conjugate; it follows that $\Lambda_+^2 \otimes \mathbb{C} = \Lambda^{2,0} \oplus \Lambda^{0,2} \oplus \mathbb{C}\omega$. Moreover, the summands in this decomposition are clearly orthogonal (both for the standard Hermitian product $\langle \alpha, \beta \rangle = \alpha \wedge \overline{*}\beta$ and for the complexified intersection pairing $(\alpha, \beta) \mapsto \alpha \wedge \overline{\beta}$; in fact the two coincide in the selfdual case), as follows from considering the types.

Next, we observe that Λ_-^2 is the orthogonal to Λ_+^2 (for either one of the two above-mentioned inner products on Λ^2); so $\Lambda_-^2 \otimes \mathbb{C} = (\Lambda^{2,0} \oplus \Lambda^{0,2})^\perp \cap \omega^\perp = \Lambda^{1,1} \cap \omega^\perp$.

Let $\alpha \in \mathcal{H}_\mathbb{R}^{1,1}$ be a real harmonic $(1, 1)$ -form. Then $*\alpha$ is also a harmonic $(1, 1)$ -form, and hence so are α^+ and α^- . At every point of M we have $\Lambda_+^2 \cap \Lambda^{1,1} = \text{span}(\omega)$, so $\alpha^+ = f\omega$ for some function $f : M \rightarrow \mathbb{R}$. Moreover, $d\alpha^+ = df \wedge \omega = 0$. However, exterior product with ω induces an isomorphism from Λ^1 to Λ^3 , so $df \wedge \omega = 0$ if and only if $df = 0$. Therefore f is constant, and α^+ is a constant multiple of ω . We conclude that $\mathcal{H}_\mathbb{R}^{1,1} \subset \mathcal{H}_-^2 \oplus \mathbb{R}\omega$. Conversely, ω is a real $(1,1)$ -form, and so is any antiselfdual form since $\Lambda_-^2 \subset \Lambda^{1,1}$, so $\mathcal{H}_\mathbb{R}^{1,1} = \mathcal{H}_-^2 \oplus \mathbb{R}\omega$.