

18.966 – Homework 1 – Solutions.

1. Let E be a Lagrangian subspace of a symplectic vector space (V, Ω) , and let e_1, \dots, e_n be a basis of E . We proceed by induction, assuming we have constructed $f_1, \dots, f_{k-1} \in V$ such that the family $(e_1, \dots, e_n, f_1, \dots, f_{k-1})$ is free and $\Omega(e_i, f_i) = 1$, $\Omega(e_i, f_j) = 0$ for $i \neq j$, and $\Omega(f_i, f_j) = 0$.

Because $(e_1, \dots, e_n, f_1, \dots, f_{k-1})$ is free, there exists a (non-unique) linear form $\tau \in V^*$ such that $\tau(e_i) = 0$ for $i \neq k$, $\tau(f_i) = 0$ for $i < k$, and $\tau(e_k) = 1$. Using the fact that Ω is non-degenerate (induces an isomorphism between V and V^*), there exists $f_k \in V$ such that $\Omega(\cdot, f_k) = \tau$.

Let us check that the family $(e_1, \dots, e_n, f_1, \dots, f_k)$ is free. Indeed, if $v = \sum_{i=1}^n \lambda_i e_i + \sum_{i=1}^k \mu_i f_i = 0$, then $\Omega(e_i, v) = \mu_i = 0$ for all $1 \leq i \leq k$, and $v = \sum \lambda_i e_i = 0$; since the (e_i) form a basis of E , we also have $\lambda_i = 0$ for all i . Moreover, $\Omega(e_i, f_k)$ and $\Omega(f_i, f_k)$ are as prescribed.

Therefore, by induction we can construct f_1, \dots, f_n such that $(e_1, \dots, e_n, f_1, \dots, f_n)$ is a basis of V (it's a free family and $\dim V = 2n$) and the expression of Ω in this basis is the standard one.

2. a)

$$\begin{aligned} \iint_{[0,1] \times S^1} \Gamma^* \omega &= \int_0^1 \int_{S^1} \omega_{\gamma_t(s)} \left(\frac{\partial}{\partial t} \gamma_t(s), \frac{\partial}{\partial s} \gamma_t(s) \right) ds dt \\ &= \int_0^1 \int_{S^1} \omega_{\gamma_t(s)} (X_t(\gamma_t(s)), \dot{\gamma}_t(s)) ds dt \\ &= \int_0^1 \left(\int_{\gamma_t} i_{X_t} \omega \right) dt = \int_0^1 \langle [i_{X_t} \omega], [\gamma_t] \rangle dt. \end{aligned}$$

Observing that γ_t and γ are mutually homologous (the restriction of Γ to $[0, t] \times S^1$ provides a bounding 2-chain), the r.h.s. is equal to $\int_0^1 \langle [i_{X_t} \omega], [\gamma] \rangle dt = \langle \text{Flux}(\rho_t), [\gamma] \rangle$.

b) Assume $\phi : (x, \xi) \mapsto (x, \xi + 1)$ is generated by a time-dependent Hamiltonian vector field X_t (i.e., $\phi = \rho_1$, and $i_{X_t} \omega = dH_t$ for some Hamiltonian $H_t : M \rightarrow \mathbb{R}$). Then $\text{Flux}(\rho_t) = 0$ by definition ($[i_{X_t} \omega] = 0$ for all t).

Recall that $\omega = -d\alpha$, where $\alpha = \xi dx$, and consider the loop $\gamma : S^1 \rightarrow T^*S^1$ defined by $\gamma(x) = (x, 0)$, and its image $\gamma_1 = \phi(\gamma)$ given by $\gamma_1(x) = (x, 1)$. recall that by (1) and Stokes' theorem we have

$$\langle \text{Flux}(\rho_t), [\gamma] \rangle = \iint_{[0,1] \times S^1} \Gamma^*(-d\alpha) = - \int_{S^1} \gamma_1^* \alpha + \int_{S^1} \gamma_0^* \alpha,$$

which implies that $\int_{\gamma_1} \alpha = \int_{\gamma_0} \alpha$, in contradiction with the direct calculation ($\int_{\gamma_0} \xi dx = 0$ and $\int_{\gamma_1} \xi dx = 2\pi$). Therefore ϕ is not Hamiltonian.

3. First observe that N^*S is a subbundle of $T^*X|_S$, whose fiber at $x \in S$ is the annihilator of $T_x S$ and hence has rank $n - k$. So N^*S is an n -dimensional submanifold of T^*X . We claim that the Liouville form α restricts to zero on N^*S : indeed, denote by $\pi : T^*X \rightarrow X$

the bundle projection, and recall that $\alpha_{(x,\xi)}(v) = \xi(\pi_*(v))$. Assume that $(x, \xi) \in N^*S$, and let $v \in T_{(x,\xi)}(N^*S)$. Then $\pi_*(v) \in T_xS$, and ξ annihilates T_xS , so $\alpha_{(x,\xi)}(v) = \xi(\pi_*(v)) = 0$. So $\alpha|_{N^*S} = 0$. Since $\omega = -d\alpha$, it follows that $\omega|_{N^*S} = 0$: so N^*S is Lagrangian.

Alternate proof in coordinates: near a point of S , choose local coordinates (x_i) on X such that S is locally defined by the equations $x_{k+1} = \dots = x_n = 0$. Let (ξ_i) be the dual coordinates on the fibers of T^*X : then TS is locally spanned by $\partial/\partial x_1, \dots, \partial/\partial x_k$, so its annihilator is the span of dx_{k+1}, \dots, dx_n . Thus N^*S is locally defined by the equations $x_{k+1} = \dots = x_n = 0$, $\xi_1 = \dots = \xi_k = 0$. It is then easy to check that $T(N^*S) = \text{span}(\partial/\partial x_1, \dots, \partial/\partial x_k, \partial/\partial \xi_{k+1}, \dots, \partial/\partial \xi_n)$ is Lagrangian with respect to $\omega = \sum dx_i \wedge d\xi_i$.

4. a) Let X, Y, Z be vector fields on N such that $X, Y \in (TN)^\omega$ everywhere. By the coordinate-independent formula for the exterior differential, $d\omega(X, Y, Z)$ can be written as

$$L_X(\omega(Y, Z)) + L_Y(\omega(Z, X)) + L_Z(\omega(X, Y)) + \omega(X, [Y, Z]) + \omega(Y, [Z, X]) + \omega(Z, [X, Y]).$$

Since $X, Y \in (TN)^\omega \subseteq TN$, $\omega(Y, Z) = \omega(Z, X) = \omega(X, Y) = 0$, so the first three terms vanish. The brackets $[Y, Z]$ and $[Z, X]$ are vector fields on N , so the next two terms also vanish. Therefore, $\omega(Z, [X, Y]) = d\omega(X, Y, Z) = 0$ (since ω is closed). This implies that $[X, Y] \in (TN)^\omega$.

b) Let (V, Ω) be a symplectic vector space, and W a coisotropic subspace. Let $u, v \in W$, and let $u', v' \in W^\Omega$. Then $\Omega(u + u', v + v') = \Omega(u, v)$ (by expanding the left-hand side). Thus, given two equivalence classes $[u], [v] \in W/W^\Omega$, the quantity $\tilde{\Omega}([u], [v]) = \Omega(u, v)$ does not depend on the chosen representatives $u \in [u]$ and $v \in [v]$. We check that the skew-symmetric form $\tilde{\Omega}$ on W/W^Ω is non-degenerate: assume $[u] \in W/W^\Omega$ is such that $\tilde{\Omega}([u], [v]) = 0$ for all $[v] \in W/W^\Omega$, and pick any representative $u \in [u]$. Then, for any $v \in W$, $\Omega(u, v) = \tilde{\Omega}([u], [v]) = 0$, so $u \in W^\Omega$ and $[u] = 0$. Thus $(W/W^\Omega, \tilde{\Omega})$ is symplectic.

Now consider a coisotropic submanifold $N \subset (M, \omega)$ with regular isotropic foliation, and let $\pi : N \rightarrow Q$ be the fibration whose fibers are the isotropic leaves: by definition, at any point $p \in N$, we have $(T_pN)^\omega = \ker(d\pi_p)$, $T_pN/(T_pN)^\omega = T_pN/\ker(d\pi_p) \simeq \text{im}(d\pi_p) = T_{\pi(p)}Q$, and the projection map $T_pN \rightarrow T_pN/(T_pN)^\omega \simeq T_{\pi(p)}Q$ is simply $d\pi_p$. By the above argument, $T_{\pi(p)}Q$ carries a symplectic structure $\tilde{\omega}_p$ such that $\omega_p(u, v) = \tilde{\omega}_p(d\pi_p(u), d\pi_p(v)) \forall u, v \in T_pN$.

We must show that $\tilde{\omega}_p$ does not depend on the choice of p in the fiber $F = \pi^{-1}(q)$. Consider a tangent vector field X on F , and extend it to a vector field on a neighborhood of F in N , everywhere tangent to the fibers of π (such an extension is easy to construct, e.g. by taking a local trivialization of π); we again denote this extension by X . Then by the Cartan formula, $L_X(\omega|_N) = di_X(\omega|_N) + i_Xd(\omega|_N)$, where the first term vanishes because $X \in \ker(d\pi) = (TN)^\omega$ and the second term vanishes because $\omega|_N$ is closed. Thus the flow ϕ^t generated by X preserves $\omega|_N$ (here ϕ^t are local diffeomorphisms of a neighborhood of F in N , defined for small enough t). Let $p \in F$ and $u, v \in T_pN$, and consider $p' = \phi^t(p) \in F$, $u' = d\phi^t(u)$, $v' = d\phi^t(v)$. Observe that $d\pi(u') = d(\pi \circ \phi^t)(u) = d\pi(u)$ (since the flow ϕ^t is tangent to the fibers of π and hence satisfies $\pi \circ \phi^t = \pi$), and similarly $d\pi(v) = d\pi(v')$. Then

$$\tilde{\omega}_p(d\pi(u), d\pi(v)) = \omega_p(u, v) = \omega_{p'}(u', v') = \tilde{\omega}_{p'}(d\pi(u'), d\pi(v')),$$

which shows that $\tilde{\omega}_p = \tilde{\omega}_{p'}$. This argument applies for any pair of points $p, p' \in F$ which are sufficiently close to each other (we can always find a vector field X whose flow takes p to p'); thus by connectedness $\tilde{\omega}_p$ is independent of the choice of $p \in F$; from now on we can call it $\tilde{\omega}_q$ (where $q = \pi(p)$).

The above construction gives us a 2-form $\tilde{\omega}$ on Q , nondegenerate (since $\tilde{\omega}_q$ was shown to be a symplectic form on $T_q Q$), and with the property that $\pi^*(\tilde{\omega}) = \omega|_N$, since given any $p \in N$ and $u, v \in T_p N$, $\omega_p(u, v) = \tilde{\omega}_{\pi(p)}(d\pi(u), d\pi(v)) = (\pi^*\tilde{\omega})_p(u, v)$. Finally, $\pi^*(d\tilde{\omega}) = d\pi^*(\tilde{\omega}) = d\omega|_N = 0$, and one easily checks that $\pi^* : \Omega^*(Q) \rightarrow \Omega^*(N)$ is injective, so $d\tilde{\omega} = 0$. Therefore $\tilde{\omega}$ is a symplectic form.

5. a) (i) By definition, $\omega(X_f, \cdot) = df$; therefore $\{f, g\} = \omega(X_f, X_g) = df(X_g) = L_{X_g}f$.

(ii) by the given formula, $i_{[X_g, X_f]}\omega = di_{X_g}i_{X_f}\omega + i_{X_g}di_{X_f}\omega - i_{X_f}di_{X_g}\omega - i_{X_f}i_{X_g}d\omega$. Since ω is closed, and $i_{X_f}\omega = df$ and $i_{X_g}\omega = dg$ are closed, the last three terms vanish, and we get $i_{[X_g, X_f]}\omega = di_{X_g}i_{X_f}\omega = d(\omega(X_f, X_g)) = d(\{f, g\})$, which by definition is $i_{X_{\{f, g\}}}\omega$. Using the non-degeneracy of ω we deduce that $[X_g, X_f] = X_{\{f, g\}}$.

b) It is obvious that $\{\cdot, \cdot\}$ is skew-symmetric (it follows from the fact ω is). Next, we compute, using the formulas proved in (a):

$$\begin{aligned} \{f, \{g, h\}\} &= L_{X_{\{g, h\}}}f = L_{[X_h, X_g]}f = L_{X_h}L_{X_g}f - L_{X_g}L_{X_h}f \\ &= L_{X_h}(\{f, g\}) - L_{X_g}(\{f, h\}) = \{\{f, g\}, h\} - \{\{f, h\}, g\} \end{aligned}$$

which using skew-symmetry yields the Jacobi identity.

c) Let $N = F^{-1}(c)$ be a regular level set of F . At every point of N , the tangent space is $TN = \ker(dF) = \ker(df_1) \cap \dots \cap \ker(df_k)$, and so $(TN)^\omega = \ker(df_1)^\omega + \dots + \ker(df_k)^\omega$. Observe that $\ker(df_i) = \{v \mid df_i(v) = \omega(X_{f_i}, v) = 0\} = \text{span}(X_{f_i})^\omega$, so $\ker(df_i)^\omega = \text{span}(X_{f_i})$. Hence $(TN)^\omega = \text{span}(X_{f_1}, \dots, X_{f_k})$.

Next, using (a) we have $df_i(X_{f_j}) = \{f_j, f_i\} = 0$, so each X_{f_j} belongs to $TN = \ker(dF) = \bigcap \ker(df_i)$. Hence $(TN)^\omega = \text{span}(X_{f_1}, \dots, X_{f_k}) \subseteq TN$, i.e. N is coisotropic, and the tangent space to the isotropic foliation $(TN)^\omega$ is spanned by the X_{f_i} .