18.966 – Homework 2 – due Wednesday March 19, 2008.

1. a) Let M be a contact manifold equipped with a contact form α, and recall that the symplectization of (M, α) is $(\mathbb{R} \times M, d(e^r\alpha))$. Show that, for any $f \in C^\infty(M, \mathbb{R}_+)$, the symplectizations of (M, α) and $(M, f\alpha)$ are symplectomorphic. (In other terms, the symplectization only depends on the contact structure, not on the contact form).

b) Let (M, α) be a contact manifold, and let R be the Reeb vector field on M (defined by $\alpha(R) = 1$, $d\alpha(R, \cdot) = 0$). Consider the Hamiltonian function $H = -e^r$ on the symplectization of (M, α). Show that the associated Hamiltonian vector field is R (or more precisely, $X_H(r, x) = (0, R(x)) \in T_{(r,x)}(\mathbb{R} \times M) = T_r\mathbb{R} \oplus T_xM$).

2. a) Let (M, ω) be an exact symplectic manifold, and let X be a *Liouville vector field*, i.e. a vector field such that $L_X\omega = \omega$. Let N be a hypersurface in M, such that X is transverse to TN at every point of N. Show that $\alpha = i_X\omega$ is a contact form on N.

b) Assume that the flow ϕ_t of X is defined for all $t \in \mathbb{R}$. Consider the map $\Phi : \mathbb{R} \times N \to M$ defined by $\Phi(r, x) = \phi_r(x)$. Show that $\Phi^*\omega = d(e^r\alpha)$.

(In particular, Φ induces a symplectomorphism from a neighborhood of $\{0\} \times N$ in $(\mathbb{R} \times N, d(e^r\alpha))$ to a neighborhood of N in M.)

c) Let $N \subset \mathbb{R}^{2n}$ be a star-shaped hypersurface, i.e. the image of a map $i : S^{2n-1} \to \mathbb{R}^{2n}$ of the form $x \mapsto f(x)\,x$, where $f \in C^\infty(S^{2n-1}, \mathbb{R}_+)$. Let $\alpha = \frac{1}{2}\sum x_i\,dy_i - y_i\,dx_i$. Show that α is a contact form on N, and that the symplectization of (N, α) is symplectomorphic to $\mathbb{R}^{2n} \setminus \{0\}$ equipped with the standard symplectic form ω_0.

Hint: consider a suitable Liouville vector field on $(\mathbb{R}^{2n}, \omega_0)$.

3. Show that the sphere S^6 carries a natural almost-complex structure, induced by a vector cross-product on $\mathbb{R}^7$.

Hint: view $\mathbb{R}^7$ as the space of imaginary octonions. Octonions are the non-commutative, non-associative normed division algebra structure on $\mathbb{R}^8 = \mathbb{H} \oplus e\mathbb{H}$ with product given by the formula

$$(a + be)(a' + b'e) = (aa' - \overline{b'}b) + (b'a + b\overline{a'})e, \qquad \forall a, b, a', b' \in \mathbb{H}$$

($\overline{a'}$ is the conjugate of a', i.e. $\overline{x + yi + zj + tk} = x - yi - zj - tk$). (You may use the fact that $\|(a+be)(a'+b'e)\| = \|a+be\|\,\|a'+b'e\|$, where $\|\cdot\|$ is the usual Euclidean norm on $\mathbb{R}^8$.)