18.966 – Homework 1 – due Wednesday February 27, 2008.

1. Show that, if E is a Lagrangian subspace of a symplectic vector space (V, Ω), then any basis $e_1, \dots, e_n$ of E can be extended to a standard basis $e_1, \dots, e_n, f_1, \dots, f_n$ of (V, Ω).

2. Let $\{\rho_t\}_{t\in[0,1]}$ be the isotopy generated by a time-dependent *symplectic* vector field X_t on a symplectic manifold (M, ω), i.e. $\rho_0 = \mathrm{Id}$, $\dfrac{d\rho_t}{dt} = X_t \circ \rho_t$, and $i_{X_t}\omega$ is closed. Then the *flux* of $\{\rho_t\}$ is defined to be

$$\mathrm{Flux}(\rho_t) = \int_0^1 [i_{X_t}\omega]\, dt \in H^1(M, \mathbb{R}).$$

a) Let $\gamma : S^1 \to M$ be an arbitrary closed loop, and define $\Gamma : [0,1] \times S^1 \to M$ by the formula $\Gamma(t,s) = \rho_t(\gamma(s))$, so $\gamma_t(\cdot) = \Gamma(t, \cdot)$ is the image of the loop γ by ρ_t. Prove that

$$\langle \mathrm{Flux}(\rho_t), [\gamma] \rangle = \iint_{[0,1]\times S^1} \Gamma^*\omega. \tag{1}$$

(Remark: the right-hand side is simply the symplectic area swept by the family of loops $\{\gamma_t\}_{t\in[0,1]}$. In particular, equation (1) implies that this area depends only on the homology class represented by γ!)

b) Does the symplectomorphism $\phi : (x, \xi) \mapsto (x, \xi + 1)$ of $T^*S^1 \simeq S^1 \times \mathbb{R}$ belong to the group of Hamiltonian diffeomorphisms?

Hint: assume ϕ is generated by a Hamiltonian isotopy, and use the exactness property ($\omega = d\alpha$) to rewrite the right-hand side of equation (1) in terms of the 1-form α.

3. Let S be a k-dimensional submanifold of a smooth n-dimensional manifold X. Equip T^*X with its standard symplectic form. The *conormal bundle* of S is

$$N^*S = \{(x, \xi) \in T^*X \mid x \in S \text{ and } \xi(v) = 0\ \forall v \in T_xS\}.$$

Show that N^*S is a Lagrangian submanifold of T^*X.

4. Let N be a coisotropic submanifold in a symplectic manifold (M, ω). (i.e., at every point $p \in N$, $(T_pN)^\omega \subseteq T_pN$).

a) Show that, if X and Y are two vector fields on N such that $X, Y \in (TN)^\omega$ everywhere, then their Lie bracket $[X, Y]$ also lies in $(TN)^\omega$.

(Hint: show that, for any vector field Z on N, $\omega(Z, [X, Y]) = 0$, by expressing $d\omega(X, Y, Z)$ as a sum of terms including this one).

By the Frobenius integrability theorem, this implies that $(TN)^\omega$ defines an *integrable foliation* on N: given any $p \in N$, there exists a neighborhood U and a submanifold $F \subset U$, with $p \in F$, such that $TF = (TN)^\omega$ at every point of F. It is easy to check that F is an isotropic submanifold ($TF = (TN)^\omega \subset TN = (TF)^\omega$), called the *isotropic leaf* through p.

b) Assume that the isotropic foliation of N is regular, i.e. there exists a locally trivial fibration $\pi : N \to Q$ whose fibers are the isotropic leaves (connected). Show that Q carries a natural symplectic form $\tilde{\omega}$ such that $\omega_{|N} = \pi^*\tilde{\omega}$.

Hint: first show that, if W is a coisotropic subspace in a symplectic vector space (V, Ω), then Ω induces a natural symplectic structure on the quotient W/W^Ω.

5. Let (M, ω) be a symplectic manifold. The *Poisson bracket* of two smooth functions $f, g \in C^\infty(M, \mathbb{R})$ is defined by $\{f, g\} = \omega(X_f, X_g)$, where X_f and X_g are the Hamiltonian vector fields defined by f and g.

a) Show that $L_{X_g} f = \{f, g\}$, and that $[X_g, X_f] = X_{\{f,g\}}$.

Hint: to prove the second identity, you may use without proof the identity $i_{[X,Y]}\alpha = d i_X i_Y \alpha + i_X d i_Y \alpha - i_Y d i_X \alpha - i_Y i_X d\alpha$.

b) Deduce that $(C^\infty(M, \mathbb{R}), \{\cdot, \cdot\})$ is a Lie algebra, i.e. $\{f, g\} = -\{g, f\}$ (skew-symmetry) and $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ (Jacobi identity).

(Note: by the first identity proved in (a) and skew-symmetry of the bracket, $\{f, g\} = 0 \Leftrightarrow$ the flow of X_f preserves the level sets of $g \Leftrightarrow$ the flow of X_g preserves the level sets of f)

c) Assume that $f_1, \dots, f_k$ satisfy $\{f_i, f_j\} = 0\ \forall i, j$, and let $F = (f_1, \dots, f_k) : M \to \mathbb{R}^k$. Show that any regular level set of F is a coisotropic submanifold of M, and that the vector fields X_{f_i} are all tangent to this submanifold and span the tangent space to its isotropic foliation.

(For example, if $k = \frac{1}{2}\dim M$ then the regular levels of F are Lagrangian; this situation is called an *integrable system*).