

ALCOVED POLYTOPES I

THOMAS LAM AND ALEXANDER POSTNIKOV

ABSTRACT. The aim of this paper is to initiate the study of alcoved polytopes, which are polytopes arising from affine Coxeter arrangements. This class of convex polytopes includes many classical polytopes, for example, the hypersimplices. We compare two constructions of triangulations of hypersimplices due to Stanley and Sturmfels and explain them in terms of alcoved polytopes. We study triangulations of alcoved polytopes, the adjacency graphs of these triangulations, and give a combinatorial formula for volumes of these polytopes. In particular, we study a class of matroid polytopes, which we call the multi-hypersimplices.

1. INTRODUCTION

The affine Coxeter arrangement of a crystallographic root system $\Phi \subset V \simeq \mathbb{R}^r$ is obtained by taking all integer affine translations $H_{\alpha,k} = \{x \in V \mid (\alpha, x) = k\}$, $\alpha \in \Phi$, $k \in \mathbb{Z}$, of the hyperplanes perpendicular to the roots. The regions of the affine Coxeter arrangements are called alcoves. They are in a one-to-one correspondence with elements of the associated affine Weyl group. If Φ is irreducible, then the alcoves are simplices congruent to each other. We define an *alcoved polytope* \mathcal{P} as a convex polytope that is the union of several alcoves. In other words, an alcoved polytope is the intersection of some half-spaces bounded by the hyperplanes $H_{\alpha,k}$:

$$\mathcal{P} = \{x \in V \mid b_\alpha \leq (\alpha, x) \leq c_\alpha, \alpha \in \Phi\},$$

where b_α and c_α are some integer parameters. These polytopes come naturally equipped with coherent triangulations into alcoves. Alcoved polytopes include many interesting classes of polytopes: hypersimplices, order polytopes, some special matroid polytopes, Fomin-Zelevinsky's generalized associahedra, and many others. This is the first of two papers about alcoved polytopes. In this paper, we concentrate on alcoved polytopes of the Lie type A case and on related combinatorial objects. In [AP2], we will treat the general case of an arbitrary root system.

Hypersimplices are integer polytopes which appear in algebraic and geometric contexts. For example, they are moment polytopes for torus actions on Grassmannians. They are also weight polytopes of the fundamental representations of the general linear group GL_n . The (k, n) -th hypersimplex can be defined as the slice of the hypercube $[0, 1]^{n-1}$ located between the two hyperplanes $\sum x_i = k - 1$ and $\sum x_i = k$. It is well-known that the normalized volume of this hypersimplex equals the Eulerian number $A_{k, n-1}$, i.e., the number of permutations of size $n - 1$ with $k - 1$ descents. Stanley [Sta1] explained this fact by constructing a triangulation

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of the hypersimplex into $A_{k,n-1}$ unit simplices. Another construction of a triangulation of the hypersimplex was given by Sturmfels [Stu]. It naturally appears in the context of Gröbner bases. These two constructions of triangulations are quite different.

In this paper, we compare these triangulations. In Sections 2-6, we show that these triangulations are actually identical to each other and that they can be naturally described in terms of alcoved polytopes. We also give an additional description of this triangulation in terms of minimal circuits in some graph. In Section 7, we describe the adjacency graph for simplices in this triangulation. In Section 8, we give a formula for the normalized volume of alcoved subpolytopes of the hypersimplex in terms of permutations with descents at certain specified positions.

We can view a collection \mathcal{M} of k -element subsets of $[n] := \{1, \dots, n\}$ as a subset of the vertices of the hypersimplex. Define the polytope $\mathcal{P}_{\mathcal{M}}$ as the convex hull of those vertices. In Section 9, we classify those collections \mathcal{M} for which $\mathcal{P}_{\mathcal{M}}$ is alcoved. We call such collections \mathcal{M} *sort-closed* collections.

In Section 10, we specialize to the case where \mathcal{M} is the collection of bases of a matroid. The associated polytope $\mathcal{P}_{\mathcal{M}}$, called a *matroid polytope*, is the moment map polytope for some torus orbit closure X in the grassmannian $Gr_{k,n}$. The normalized volume of $\mathcal{P}_{\mathcal{M}}$ equals the degree of X . We describe two classes of sort-closed matroids. One of them leads to an interesting set of polytopes, which we call *multi-hypersimplices*. Whereas a hypersimplex is a slice of a cube, a multi-hypersimplex is a slice of a direct product of simplices.

In Sections 11-13, we study certain graphs, following Sturmfels [Stu], corresponding to the case of rank two matroids and give a formula for the normalized volume of their associated polytopes. We also calculate the generating function for the f -vectors of the alcoved triangulations of the second hypersimplices.

In Section 14, we define the multi-Eulerian polynomials $A_{\Pi}(t)$ as generating functions for volumes of multi-hypersimplices. These polynomials generalize the usual Eulerian polynomials and have many interesting combinatorial properties. In the first non-trivial case, we give a formula for the multi-Eulerian polynomial in terms of the usual Eulerian polynomial.

In Sections 15 and 16, we study triangulations of general alcoved polytopes (of type A) and the graphs associated to adjacent simplices in these triangulations. We prove a theorem expressing the volume of an alcoved polytope in terms of the number of lattice points contained in other alcoved polytopes. In Section 17, we give a description of the alcoves of an alcoved polytope in terms of homotopy-classes of maps from a circle with n marked points. In Section 18, we give some final remarks. We discuss relations with order and weight polytopes and also discuss degrees on torus orbits on the Grassmannian. In Appendix 19, we give a short introduction to the relationship between Gröbner bases and coherent triangulations, following [Stu].

In the second part [AP2] of this paper, we will extend the hypersimplices to all Lie types and calculate their volumes. We will prove a general theorem on volumes of alcoved polytopes. We will give uniform generalizations of the descent and major index statistics, appropriate for our geometric approach.

2. HYPERSIMPLICES AND EULERIAN NUMBERS

Let us fix integers $0 < k < n$. Let $[n] := \{1, \dots, n\}$ and $\binom{[n]}{k}$ denote the collection of k -element subsets of $[n]$. To each k -subset $I \in \binom{[n]}{k}$ we associate the 01-vector $\epsilon_I = (\epsilon_1, \dots, \epsilon_n)$ such that $\epsilon_i = 1$, for $i \in I$; and $\epsilon_i = 0$, for $i \notin I$.

The *hypersimplex* $\Delta_{k,n} \subset \mathbb{R}^n$ is the convex polytope defined as the convex hull of the points ϵ_I , for $I \in \binom{[n]}{k}$. All these $\binom{[n]}{k}$ points are actually vertices of the hypersimplex because they are obtained from each other by permutations of the coordinates. This $(n-1)$ -dimensional polytope can also be defined as

$$\Delta_{k,n} = \{(x_1, \dots, x_n) \mid 0 \leq x_1, \dots, x_n \leq 1; x_1 + \dots + x_n = k\}.$$

The hypersimplex is linearly equivalent to the polytope $\tilde{\Delta}_{k,n} \subset \mathbb{R}^{n-1}$ given by

$$\tilde{\Delta}_{k,n} = \{(x_1, \dots, x_{n-1}) \mid 0 \leq x_1, \dots, x_{n-1} \leq 1; k-1 \leq x_1 + \dots + x_{n-1} \leq k\}.$$

Indeed, the projection $p : (x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1})$ sends $\Delta_{k,n}$ to $\tilde{\Delta}_{k,n}$. The hypersimplex $\tilde{\Delta}_{k,n}$ can be thought of as the region (slice) of the unit hypercube $[0, 1]^{n-1}$ contained between the two hyperplanes $\sum x_i = k-1$ and $\sum x_i = k$.

Recall that a *descent* in a permutation $w \in S_n$ is an index $i \in \{1, \dots, n-1\}$ such that $w(i) > w(i+1)$. Let $\text{des}(w)$ denotes the number of descents in w . The *Eulerian number* $A_{k,n}$ is the number of permutations in S_n with $\text{des}(w) = k-1$ descents.

Let us normalize the volume form in \mathbb{R}^{n-1} so that the volume of a unit simplex is 1 and, thus, the volume of a unit hypercube is $(n-1)!$. It is a classical result, implicit in the work of Laplace, that the normalized volume of the hypersimplex $\Delta_{k,n}$ equals the Eulerian number $A_{k,n-1}$. One would like to present a triangulation of $\Delta_{k,n}$ into $A_{k,n-1}$ unit simplices.

In the following sections we give four different constructions of such triangulations of the hypersimplex $\Delta_{k,n}$. One of these constructions is due to Stanley [Sta1] and another is due to Sturmfels [Stu].

3. STANLEY'S TRIANGULATION

The hypercube $[0, 1]^{n-1} \subset \mathbb{R}^{n-1}$ can be naturally triangulated into $(n-1)$ -dimensional unit simplices ∇_w labelled by permutations $w \in S_{n-1}$ given by

$$\nabla_w = \{(y_1, \dots, y_{n-1}) \in [0, 1]^{n-1} \mid 0 < y_{w(1)} < y_{w(2)} < \dots < y_{w(n-1)} < 1\}.$$

Stanley [Sta1] defined a transformation of the hypercube $\psi : [0, 1]^{n-1} \rightarrow [0, 1]^{n-1}$ by $\psi(x_1, \dots, x_{n-1}) = (y_1, \dots, y_{n-1})$, where

$$y_i = (x_1 + x_2 + \dots + x_i) - \lfloor x_1 + x_2 + \dots + x_i \rfloor.$$

The notation $\lfloor x \rfloor$ denotes the integer part of x . The map ψ is piecewise-linear, bijective on the hypercube (except for a subset of measure zero), and volume preserving.

The inverse map ψ^{-1} is linear on the open simplices ∇_w . Thus it transforms the triangulation of the hypercube given by ∇_w 's into another triangulation.

Theorem 3.1 (Stanley [Sta1]). *The collection of simplices $\psi^{-1}(\nabla_w)$, $w \in S_{n-1}$, gives a triangulation of the hypercube $[0, 1]^{n-1}$ compatible with the subdivision of the hypercube into hypersimplices. The collection of the simplices $\psi^{-1}(\nabla_w)$, where w^{-1} varies over permutations in S_{n-1} with $k-1$ descents, gives a triangulation of the*

k -th hypersimplex $\tilde{\Delta}_{k,n}$. Thus the normalized volume of $\tilde{\Delta}_{k,n}$ equals the Eulerian number $A_{k,n-1}$.

Proof. Let $\psi(x_1, \dots, x_{n-1}) = (y_1, \dots, y_{n-1}) \in \nabla_w$. For $i = 1, \dots, n-2$, we have,

$$\lfloor x_1 + \dots + x_{i+1} \rfloor = \begin{cases} \lfloor x_1 + \dots + x_i \rfloor & \text{if } y_i < y_{i+1}; \\ \lfloor x_1 + \dots + x_i \rfloor + 1 & \text{if } y_i > y_{i+1}. \end{cases}$$

Thus $\lfloor x_1 + \dots + x_{n-1} \rfloor = \text{des}(w^{-1})$. In other words, if $\text{des}(w^{-1}) = k-1$, then $k-1 \leq x_1 + \dots + x_{n-1} \leq k$, i.e., $(x_1, \dots, x_{n-1}) \in \tilde{\Delta}_{k,n}$. \square

4. STURMFELS' TRIANGULATION

Let S be a multiset of elements from $[n]$. We define $\text{sort}(S)$ to be the unique non-decreasing sequence obtained by ordering the elements of S . Let I and J be two k -element subsets of $[n]$, and let $\text{sort}(I \cup J) = (a_1, a_2, \dots, a_{2k})$. Then we set $U(I, J) = \{a_1, a_3, \dots, a_{2k-1}\}$ and $V(I, J) = \{a_2, a_4, \dots, a_{2k}\}$. For example, for $I = \{1, 2, 3, 5\}$, $J = \{2, 4, 5, 6\}$, we have $\text{sort}(I \cup J) = (1, 2, 2, 3, 4, 5, 5, 6)$, $U(I, J) = \{1, 2, 4, 5\}$, and $V(I, J) = \{2, 3, 5, 6\}$.

We say that an ordered pair (I, J) is *sorted* if $I = U(I, J)$ and $J = V(I, J)$. We call an ordered collection $\mathcal{I} = (I_1, \dots, I_r)$ of k -subsets of $[n]$ *sorted* if (I_i, I_j) is sorted for every $1 \leq i < j \leq r$. Equivalently, if $I_l = \{I_{l1} < \dots < I_{lk}\}$, for $l = 1, \dots, r$, then \mathcal{I} is sorted if and only if $I_{11} \leq I_{21} \leq \dots \leq I_{r1} \leq I_{12} \leq I_{22} \leq \dots \leq I_{rk}$. For such a collection \mathcal{I} , let $\nabla_{\mathcal{I}}$ denote the $(r-1)$ -dimensional simplex with the vertices $\epsilon_{I_1}, \dots, \epsilon_{I_r}$.

Theorem 4.1 (Sturmfels [Stu]). *The collection of simplices $\nabla_{\mathcal{I}}$, where \mathcal{I} varies over all sorted collections of k -element subsets in $[n]$, is a simplicial complex that forms a triangulation of the hypersimplex $\Delta_{k,n}$.*

It follows that the maximal by inclusion sorted collections, which correspond to the maximal simplices in the triangulation, all have the same size $r = n$.

Corollary 4.2. *The normalized volume of the hypersimplex $\Delta_{k,n}$ is equal to the number of maximal sorted collections of k -subsets in $[n]$.*

This triangulation naturally appears in the context of Gröbner bases. Let $k[x_I]$ be the polynomial ring in the $\binom{n}{k}$ variables x_I labelled by k -subsets $I \in \binom{[n]}{k}$. Define the map $\phi : k[x_I] \rightarrow k[t_1, t_2, \dots, t_n]$ by $x_I \mapsto t_{i_1} t_{i_2} \dots t_{i_k}$, for $I = \{i_1, \dots, i_k\}$. The kernel of this map is an ideal in $k[x_I]$ that we denote by $\mathcal{J}_{k,n}$. Recall that a sufficiently generic height function on the vertices ϵ_I of the hypersimplex $\Delta_{k,n}$ induces a term order on monomials in $k[x_I]$ and defines a Gröbner basis for the ideal $\mathcal{J}_{k,n}$. On the other hand, such a height function gives a coherent triangulation of $\Delta_{k,n}$. This gives a correspondences between Gröbner bases and coherent triangulations. The initial ideal associated with a Gröbner is square-free if and only if the corresponding triangulation is unimodular. For more detail on Gröbner bases, see Appendix 19.

Theorem 4.3 (Sturmfels [Stu]). *The marked set of quadratic binomials*

$$\mathcal{G}_{k,n} = \left\{ \underline{x_I x_J} - x_{U(I,J)} x_{V(I,J)} \mid I, J \in \binom{[n]}{k} \right\},$$

is a Gröbner basis for $\mathcal{J}_{k,n}$ under some term order on $k[x_I]$ such that the underlined term is the initial monomial. The simplices of the corresponding triangulation are $\nabla_{\mathcal{I}}$, where \mathcal{I} varies over sorted collections of k -subsets of $[n]$. Moreover, this triangulation is unimodular.

In Section 9, we state and prove a more general statement.

5. ALCOVE TRIANGULATION

The *affine Coxeter arrangement* of type A_{n-1} is the arrangement of hyperplanes in \mathbb{R}^{n-1} given by

$$H_{ij}^l = \{(z_1, \dots, z_{n-1}) \in \mathbb{R}^{n-1} \mid z_i - z_j = l\}, \quad \text{for } 0 \leq i < j \leq n-1, l \in \mathbb{Z},$$

where we assume that $z_0 = 0$. It follows from the general theory of affine Weyl groups, see [Hum], that the hyperplanes H_{ij}^l subdivide \mathbb{R}^{n-1} into unit simplices, called *alcoves*.

We say that a polytope \mathcal{P} in \mathbb{R}^{n-1} is *alcoved* if \mathcal{P} is an intersection of some half-spaces bounded by the hyperplanes H_{ij}^l . In other words, an alcoved polytope is a polytope given by inequalities of the form $b_{ij} \leq z_j - z_i \leq c_{ij}$, for some collection of integer parameters b_{ij} and c_{ij} . We will denote this alcoved polytope by $\mathcal{P}(b_{ij}, c_{ij})$. If the parameters satisfy $b_{ij} = c_{ij} - 1$, for all i, j , then the corresponding polytope consists of a single alcove (or is empty). Each alcoved polytope comes naturally equipped with the triangulation into alcoves. Conversely, if \mathcal{P} is a convex polytope which is a union of alcoves, then \mathcal{P} is an alcoved polytope.

Assume that $z_i = x_1 + \dots + x_i$, for $i = 1, \dots, n-1$. The hypersimplex $\tilde{\Delta}_{k,n}$ is given by the following inequalities in the z -coordinates:

$$(1) \quad 0 \leq z_1 - z_0, \dots, z_{n-1} - z_{n-2} \leq 1; \quad k-1 \leq z_{n-1} - z_0 \leq k.$$

Thus the hypersimplex is an alcoved polytope. Let us call its triangulation into alcoves the *alcove triangulation*.

6. CIRCUIT TRIANGULATION

Let $G_{k,n}$ be the directed graph on the vertices ϵ_I , $I \in \binom{[n]}{k}$, of the hypersimplex $\Delta_{k,n}$ defined as follows. Let us regard the indices i of a vector $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ as elements of $\mathbb{Z}/n\mathbb{Z}$. Thus we assume that $\epsilon_{n+1} = \epsilon_1$. We connect a vertex $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ with a vertex ϵ' by an edge $\epsilon \xrightarrow{i} \epsilon'$ labelled by $i \in [n]$ whenever $(\epsilon_i, \epsilon_{i+1}) = (1, 0)$ and the vector ϵ' is obtained from ϵ by switching ϵ_i and ϵ_{i+1} . In other words, each edge in the graph $G_{k,n}$ is given by cyclically shifting a “1” in vector ϵ one step to the right to the next adjacent place. It is possible to perform such a shift if the next place is not occupied by another “1”.

A circuit in the graph $G_{k,n}$ of minimal possible length is given by a sequence of shifts of “1”s so that the first “1” in ϵ moves to the position of the second “1”, the second “1” moves to the position of the third “1”, and so on, finally, the last “1” cyclically moves to the position of the first “1”. The length of such a circuit is n . We will call such circuits in $G_{k,n}$ *minimal*. Here is an example of a minimal circuit in G_{26} :

$$\begin{array}{ccccc} (1, 0, 1, 0, 0, 0) & \xrightarrow{3} & (1, 0, 0, 1, 0, 0) & \xrightarrow{1} & (0, 1, 0, 1, 0, 0) \\ & \uparrow_6 & & & \downarrow_2 \\ (0, 0, 1, 0, 0, 1) & \xleftarrow{5} & (0, 0, 1, 0, 1, 0) & \xleftarrow{4} & (0, 0, 1, 1, 0, 0) \end{array}$$

The sequence of labels of edges in a minimal circuit forms a permutation $w = w_1 \cdots w_n \in S_n$. For example, the permutation corresponding to the above minimal circuit is $w = 312456$.

If we do not specify the initial vertex in a minimal circuit, then the permutation w is defined modulo cyclic shifts $w_1 \cdots w_n \sim w_n w_1 \cdots w_{n-1}$. By convention, we will pick the representative w of the class of permutations modulo cyclic shifts such that $w_n = n$. This corresponds to picking the initial point in a minimal circuit with lexicographically maximal 01-vector ϵ . Indeed, if $\epsilon \xrightarrow{i} \epsilon'$ is an edge in $G_{k,n}$, then $\epsilon > \epsilon'$ in the lexicographic order, for $i = 1, \dots, n-1$; and $\epsilon < \epsilon'$, for $i = n$.

Lemma 6.1. *A minimal circuit in the graph $G_{k,n}$ is uniquely determined by the permutation w modulo cyclic shifts. A permutation $w \in S_n$ such that $w_n = n$ corresponds to a minimal circuit in the graph $G_{k,n}$ if and only if the inverse permutation w^{-1} has exactly $k-1$ descents.*

Proof. The inverse permutation w^{-1} has a descent for each pair $(i < j)$ such that $b = w_i = w_j + 1$. Thus a “1” was moved from the b -th position to the $(b+1)$ -th position in ϵ before a “1” was moved from the $(b-1)$ -th position to the b -th position. This happens if and only if $\epsilon_b = 1$. Since ϵ has k “1”s, this happens exactly k times. But the occurrence corresponding to $w^{-1}(n) = n > w^{-1}(1)$ is not counted as a descent, so w^{-1} has exactly $k-1$ descents. Conversely, if w^{-1} has $k-1$ descents, then we obtain a vector ϵ with k “1”s, so that w corresponds to a minimal circuit containing ϵ . \square

For a permutation, $w = w_1 \cdots w_n \in S_n$, let (w) denote the long cycle in S_n given by $(w) = (w_1, \dots, w_n)$ in cycle notation. Two permutations $u, w \in S_n$ are equivalent modulo cyclic shifts if and only if $(u) = (w)$. The reader should not confuse *circuits* in the graph $G_{k,n}$ with *cycles* in the symmetric group S_n .

Let $C_{k,n}$ denote the set of long cycles $(w) = (w_1, \dots, w_{n-1}, n) \in S_n$ such that w^{-1} has exactly $k-1$ descents. For $(w) \in C_{k,n}$, let $c_{(w)}$ be the corresponding minimal circuit in the graph $G_{k,n}$, whose edges are labelled by w_1, \dots, w_n . Lemma 6.1 shows that the map $(w) \mapsto c_{(w)}$ is one-to-one correspondence between the set long cycles $C_{k,n}$ and the set of minimal circuits in $G_{k,n}$.

Each minimal circuit $c_{(w)}$ in $G_{k,n}$ determines the simplex $\Delta_{(w)}$ inside the hypersimplex $\Delta_{k,n}$ with the vertex set $c_{(w)}$.

Theorem 6.2. *The collection of simplices $\Delta_{(w)}$ corresponding to all minimal circuits in $G_{k,n}$ forms a triangulation of the hypersimplex $\Delta_{k,n}$.*

Let us call this triangulation of the hypersimplex the *circuit triangulation*.

Theorem 6.3. *The following four triangulations of the hypersimplex are identical: Stanley’s triangulation, Sturmfels’ triangulation, the alcove triangulation, and the circuit triangulation.*

Let us prove these two claims together. Let $\Gamma_{k,n}$ be the collection of (maximal) simplices of the triangulation of Theorem 6.3.

Proof. The fact that Stanley’s triangulation coincides with the alcove triangulation follows directly from the definitions. We leave this as an exercise for the reader.

Let us show that the simplices $\Delta_{(w)}$ are exactly those in Sturmfels’ triangulation. An ordered pair of subsets $I = \{i_1 < \cdots < i_k\}$ and $J = \{j_1 < \cdots < j_k\}$ is sorted

if and only if the interleaving condition $i_1 \leq j_1 \leq i_2 \leq j_2 \leq \dots \leq j_k$ is satisfied. When two vertices ϵ_I and ϵ_J belong to the same minimal circuit, a “1” from ϵ_I is moved towards the right in ϵ_J but never past the original position of another “1” in ϵ_J . Thus the interleaving $i_a \leq j_a \leq i_{a+1}$ condition is satisfied, and similarly we obtain the other interleaving inequalities. Conversely, the interleaving condition implies that each sorted collection belongs to a minimal circuit in $G_{k,n}$.

Let us now show that the circuit triangulation coincides with Stanley’s triangulation. Recall that the latter triangulation occurs in the space \mathbb{R}^{n-1} . To be more precise, in order to obtain Stanley’s triangulation we need to apply the projection $p : (x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1})$ to the circuit triangulation. Let us identify a permutation $w = w_1 \cdots w_{n-1} \in S_{n-1}$ with $k-1$ descents with the permutation $w_1 \cdots w_{n-1}n \in S_n$.

We claim that the projected simplex $p(\Delta_{(w)})$ is exactly the simplex $\psi^{-1}(\nabla_w)$ in Stanley’s triangulation. Indeed, the map $\psi^{-1} : (y_1, \dots, y_{n-1}) \mapsto (x_1, \dots, x_{n-1})$ restricted to the simplex $\nabla_w = \{0 < y_{w(1)} < \dots < y_{w(n-1)} < 1\}$, is given by $x_1 = y_1$ and

$$x_{i+1} = \begin{cases} y_{i+1} - y_i & \text{if } w^{-1}(i+1) > w^{-1}(i), \\ y_{i+1} - y_i + 1 & \text{if } w^{-1}(i+1) < w^{-1}(i) \end{cases}$$

for $i = 1, \dots, n-2$. The vertices of the simplex ∇_w are the points $v_0, \dots, v_{n-1} \in \mathbb{R}^{n-1}$ such that $v_r = (y_1, \dots, y_{n-1})$ is given by $y_{w(1)} = \dots = y_{w(r)} = 0$ and $y_{w(r+1)} = \dots = y_{w(n-1)} = 1$. The map ψ^{-1} sends the vertex $v_0 = (0, \dots, 0)$ to the point (x_1, \dots, x_{n-1}) such that $x_1 = 0$ and $x_{i+1} = 1$ if $w^{-1}(i+1) < w^{-1}(i)$ and $x_{i+1} = 0$ if $w^{-1}(i+1) > w^{-1}(i)$, for $i = 1, \dots, n-2$. The vertex v_r is obtained from v_{r-1} by changing $y_{w(n-r)}$ from 0 to 1. Thus $\psi^{-1}(v_r)$ differs from $\psi^{-1}(v_{r-1})$ exactly in the coordinates $x_{w(n-r)}$ and $x_{w(n-r)+1}$. Here $x_n = k - (x_1 + \dots + x_{n-1})$. In fact, going from $\psi^{-1}(v_{r-1})$ to $\psi^{-1}(v_r)$ we move a “1” from $x_{w(n-r)+1}$ to $x_{w(n-r)}$. Finally, moving from v_{n-1} to v_0 we are changing x_1 from 1 to 0. Thus as we go from $\psi^{-1}(v_r)$ to $\psi^{-1}(v_{r+1})$ we are traveling along the edges of the graph $G_{k,n}$ in the reverse direction. So the vertices $\psi^{-1}(v_r)$ of the simplex $\psi^{-1}(\nabla_w)$ are exactly the vertices of $p(\Delta_{(w)})$. This completes the proof of the theorem. \square

An explicit bijection θ between maximal sorted collections of k -subsets of $[n]$ and permutations $w \in S_n$ with $k-1$ descents satisfying $w_n = n$ can be constructed as follows. Let $\mathcal{I} = (I_1, \dots, I_n)$ be such a collection. Every number in $[n]$ must occur in $\bigcup_i I_i$. Set $(a_1, \dots, a_{kn}) = \text{sort}(\bigcup_i I_i)$. Let b_k be such that $a_{b_k} = k$ and $a_{b_k+1} = k+1$. Then $\theta(\mathcal{I}) = w_1 w_2 \cdots w_{n-1} n$, where $w_i \equiv b_i \pmod{n-1}$ with representatives taken from $[n-1]$. This bijection is compatible with the correspondences in Theorem 6.3. For example, $b_i \pmod{n-1}$ tells us when a “1” is moved from ϵ_i to ϵ_{i+1} where by convention a “1” is moved from ϵ_n to ϵ_1 in the last edge of a cycle.

7. ADJACENCY OF MAXIMAL SIMPLICES IN THE HYPERSIMPLEX

Let us say that two simplices in a triangulation are *adjacent* if they share a common facet. Let us describe the adjacent simplices in the triangulation $\Gamma_{k,n}$, using first the construction of the circuit triangulation.

Theorem 7.1. *Two simplices $\Delta_{(u)}$ and $\Delta_{(w)}$ of $\Gamma_{k,n}$ are adjacent if and only if there exists $i = 1, \dots, n$ such that $u_i - u_{i+1} \not\equiv \pm 1 \pmod{n}$ and the cycle (w) is obtained from (u) by switching u_i with u_{i+1} , i.e., $(w) = (u_i, u_{i+1})(u)(u_i, u_{i+1})$. Here again we assume that $u_{n+1} = u_1$.*

Proof. The two simplices $\Delta_{(u)}$ and $\Delta_{(w)}$ are adjacent if and only if exactly one pair of their vertices differ. This means that the corresponding minimal circuits $c_{(u)}$ and $c_{(w)}$ differ in exactly one place. Let $\epsilon' \xrightarrow{u_i} \epsilon \xrightarrow{u_{i+1}} \epsilon''$ be three vertices in order along the minimal cycle $c_{(u)}$. Then we can obtain another cycle $c_{(w)}$ from $c_{(u)}$ by changing only ϵ if and only if $u_i - u_{i+1} \neq \pm 1 \pmod{n}$ so that $\epsilon' \xrightarrow{u_{i+1}} \epsilon^* \xrightarrow{u_i} \epsilon''$ are valid edges. When $u_i - u_{i+1} = \pm 1 \pmod{n}$ we are either moving the same “1” twice or moving two adjacent “1”s one after another. In both cases, the order of the shifts cannot be reversed, and so ϵ cannot be replaced by another vertex. \square

Alternatively, let $\mathcal{I} = (I_1, \dots, I_n)$ be a sorted subset corresponding to the maximal simplex $\nabla_{\mathcal{I}}$ of $\Gamma_{k,n}$. Let $t \in [n]$ and $I_t = \{i_1, i_2, \dots, i_k\}$. Then we can replace I_t in \mathcal{I} by another $I'_t \in \binom{[n]}{k}$ to obtain an adjacent maximal simplex $\nabla_{\mathcal{I}'}$ if and only if the following holds. We must have $I'_t = \{i_1, \dots, i'_a, \dots, i'_b, \dots, i_k\}$ for some $a \neq b \in [n]$ and $i'_a \neq i'_b$, satisfying $i_a - i'_a = i'_b - i_b = \pm 1 \pmod{n}$ and also both k -subsets $\{i_1, \dots, i'_a, \dots, i_b, \dots, i_k\}$ and $\{i_1, \dots, i_a, \dots, i'_b, \dots, i_k\}$ must lie in \mathcal{I} . For example, we may replace $\{1, 3, 5, 8\}$ by $\{1, 2, 6, 8\}$ if and only if both $\{1, 2, 5, 8\}$ and $\{1, 3, 6, 8\}$ lie in \mathcal{I} .

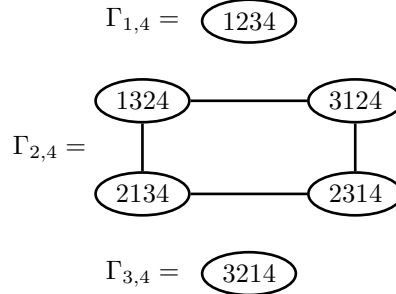


FIGURE 1. The graphs of the triangulations of $\Delta_{1,4}$, $\Delta_{2,4}$ and $\Delta_{3,4}$.

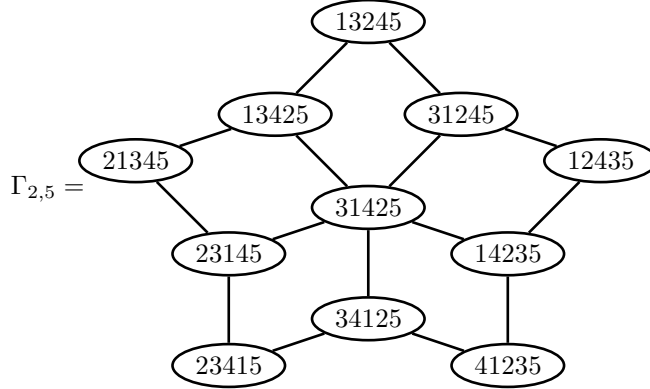


FIGURE 2. Graph of the triangulation of the hypersimplex $\Delta_{2,5}$.

We can give $\Gamma_{k,n}$ the structure of a graph by letting the simplices be the vertices and letting an edge join two simplices if the two simplices are adjacent. Figures 1 and 2 give examples of these graphs which we will also denote as $\Gamma_{k,n}$.

We will investigate degrees of vertices in these graphs in Section 12. In Section 15, we will give another construction for these graphs that works for general alcoved polytopes.

8. DESCENT-RESTRICTED PERMUTATIONS AND ALCOVED POLYTOPES

Let us consider an alcoved polytope \mathcal{P} (as in Section 5) which lies within a hypersimplex $\Delta_{k,n}$. In the x -coordinates, $\mathcal{P} \subset \mathbb{R}^n$ is defined by the hyperplane $x_1 + x_2 + \cdots + x_n = k$, the inequalities $0 \leq x_i \leq 1$ together with inequalities of the form

$$b_{ij} \leq x_{i+1} + \cdots + x_j \leq c_{ij}$$

for integer parameters b_{ij} and c_{ij} for each pair (i, j) satisfying $0 \leq i < j \leq n - 1$.

Let $W_{\mathcal{P}} \subset S_{n-1}$ be the set of permutations $w = w_1 w_2 \cdots w_{n-1} \in S_{n-1}$ satisfying the following conditions:

- (1) w has $k - 1$ descents.
- (2) The sequence $w_i \cdots w_j$ has at least b_{ij} descents. Furthermore, if $w_i \cdots w_j$ has exactly b_{ij} descents, then $w_i < w_j$.
- (3) The sequence $w_i \cdots w_j$ has at most c_{ij} descents. Furthermore, if $w_i \cdots w_j$ has exactly c_{ij} descents, then we must have that $w_i > w_j$.

In the above conditions we assume that $w_0 = 0$.

Let $p : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ denote the projection as before. We may apply Stanley's piecewise-linear map ψ to $p(\mathcal{P})$.

Theorem 8.1. *A unimodular triangulation of $p(\mathcal{P})$ is given by the set of simplices $\psi^{-1}(\nabla_{w^{-1}})$ as $w = w_1 w_2 \cdots w_{n-1}$ varies over permutations in $W_{\mathcal{P}}$.*

In particular, $\Gamma_{k,n}$ induces a triangulation of \mathcal{P} .

Proof. By Theorem 6.3, the polytope $p(\mathcal{P})$ is the union of a set of simplices of the form $\psi^{-1}(\nabla_{w^{-1}})$ for $w \in S_{n-1}$. We check that the three conditions defining $W_{\mathcal{P}}$ hold. The first condition follows as in the proof of Theorem 3.1. As before, write $z_i = x_1 + \cdots + x_i$ and $y_i = z_i - \lfloor z_i \rfloor$. We assume $z_0 = 0$ in addition. The inequality $0 \leq x_i \leq 1$ gives $0 \leq z_{i+1} - z_i \leq 1$. For a generic point, the number of descents in the sequence $(y_i, y_{i+1}, \dots, y_j)$, for $0 \leq i < j \leq n - 1$, is equal to the number of indices $l \in [i, j - 1]$ such that $z_l < a < z_{l+1}$ for a whole number a . If $y \in \nabla_{w^{-1}}$ then this is also the number of descents in the sequence $w_i w_{i+1} \cdots w_j$. Since we have $b_{ij} \leq z_j - z_i \leq c_{ij}$ the latter two conditions defining $W_{\mathcal{P}}$ follow. Conversely, that $w \in W_{\mathcal{P}}$ implies $\psi^{-1}(\nabla_{w^{-1}}) \subset p(\mathcal{P})$ follows in the same manner. \square

9. SORT-CLOSED SETS

Let \mathcal{M} be a collection of k -subsets of $[n]$. The polytope $\mathcal{P}_{\mathcal{M}}$ is the convex hull in \mathbb{R}^n of the points $\{\epsilon_I \mid I \in \mathcal{M}\}$ and is a subpolytope of the hypersimplex $\Delta_{k,n}$. Let (I_1, \dots, I_r) be a sorted collection of k -subsets of $[n]$. If in addition the sets I_l are distinct elements of \mathcal{M} then we call (I_1, \dots, I_r) a \mathcal{M} -sorted subset of size r .

Definition 9.1. *A collection \mathcal{M} of k -subsets of $[n]$ is sort-closed if for every two elements I and J in \mathcal{M} , the subsets $U(I, J)$ and $V(I, J)$ are both in \mathcal{M} .*

Theorem 9.2. *The triangulation $\Gamma_{k,n}$ of the hypersimplex induces a triangulation of the polytope $\mathcal{P}_{\mathcal{M}}$ if and only if \mathcal{M} is sorted. The normalized volume of $\mathcal{P}_{\mathcal{M}}$ is equal to the number of \mathcal{M} -sorted subsets of size $\dim(\mathcal{P}_{\mathcal{M}}) + 1$.*

The proof is analogous to that of Theorem 4.3. We work in the polynomial ring $k[x_I \mid I \in \mathcal{M}]$. The ideal $\mathcal{J}_{\mathcal{M}}$ is the kernel of the ring homomorphism $\phi : k[x_I \mid I \in \mathcal{M}] \rightarrow k[t_1, t_2, \dots, t_n]$ given by $x_I \mapsto t_{i_1} t_{i_2} \cdots t_{i_k}$, for $I = \{i_1, \dots, i_k\}$.

Proposition 9.3. *Suppose that \mathcal{M} is sorted. Then there is a term order, $\prec_{\mathcal{M}}$, on $k[x_I \mid I \in \mathcal{M}]$ so that the reduced Gröbner basis of $\mathcal{J}_{\mathcal{M}}$ is given by the nonzero marked binomials of the form*

$$(2) \quad \{x_I x_J - x_{U(I,J)} x_{V(I,J)}\}$$

with the first monomial being the leading term.

Proof. This proof is essentially the same as that of Sturmfels in [Stu, Chapter 14], see Appendix 19 for background.

We say a monomial $x_A x_B \cdots x_V = x_{a_1 \dots a_k} x_{b_1 \dots b_k} \cdots x_{v_1 \dots v_k}$ is *sorted* if the ordered collection of sets (A, B, \dots, V) is sorted. As remarked above this means that

$$(3) \quad a_1 \leq b_1 \leq \dots \leq v_1 \leq a_2 \leq b_2 \leq \dots \leq v_k.$$

Note that the variables in a monomial are considered to be ordered for the purposes of this argument. The *inversion number* of a string of integers $i_1 i_2 \dots i_r$ is the number of pairs $\{(e, f) \mid e < f, i_e > i_f\}$. The inversion number of a monomial $x_{a_1 \dots a_k} x_{b_1 \dots b_k} \cdots x_{v_1 \dots v_k}$ is given by the inversion number of $a_1 b_1 \cdots v_k$. Thus a monomial is *sorted* if and only if its inversion number is 0.

If a monomial is not sorted then one of the inequalities in (3) is reversed. This implies that there is a pair of adjacent variables $x_I x_J$ which is unsorted. Using the binomial $x_I x_J - x_{U(I,J)} x_{V(I,J)}$ we can sort this pair. The inversion number is strictly reduced in this process. Thus we can sort a monomial modulo the ideal generated by the marked binomials of (2) in a finite number of steps. Using general results in [Stu], we conclude that there is a term order $\prec_{\mathcal{M}}$ which selects the marked term for each binomial of (2).

We now prove that sorted monomials are exactly the $\prec_{\mathcal{M}}$ -standard monomials (it is clear that only sorted monomials may be $\prec_{\mathcal{M}}$ -standard). This will imply that the quadratic binomials of (2) form a (reduced) Gröbner basis of the ideal $\mathcal{J}_{\mathcal{M}}$. So suppose m_1 is a sorted monomial lying in $\text{in}_{\prec_{\mathcal{M}}}(\mathcal{J}_{\mathcal{M}})$; then there exists a binomial $m_1 - m_2$ which lies in $\mathcal{J}_{\mathcal{M}}$ with m_2 not lying in $\text{in}_{\prec_{\mathcal{M}}}(\mathcal{J}_{\mathcal{M}})$ (this is a consequence of the fact that we are working in a toric ideal which is generated by binomials). Thus m_2 must be sorted, which is impossible as two sorted monomials with the same image under $\phi : k[x_I \mid I \in \mathcal{M}] \rightarrow k[t_1, \dots, t_n]$ must be the same. \square

Proof of Theorem 9.2. The ‘if’ direction follows from Proposition 9.3 and Theorem 19.1. We may assume that $\prec_{\mathcal{M}}$ arises from a weight vector since only finitely many binomials are involved in (2). For the ‘only if’ direction, suppose $\mathcal{P} = \mathcal{P}_{\mathcal{M}}$ is a convex polytope which is a union of simplices in $\Gamma_{k,n}$. Since the triangulation $\Gamma_{k,n}$ is coherent, this triangulation $\Gamma_{\mathcal{P}}$ is also coherent. We know already that all the faces of $\Gamma_{\mathcal{P}}$ are sorted collections of k -subsets of $[n]$ (if we identify a simplex with its set of vertices). By the correspondence of Theorem 19.1 and Proposition 19.2, $\Gamma_{\mathcal{P}}$ arises from some term-order $\prec_{\mathcal{M}}$ which gives rise to an initial ideal which is the Stanley-Reisner ideal of the triangulation $\Gamma_{\mathcal{P}}$. Let $(I, J) \in \mathcal{M} \times \mathcal{M}$ not be sorted, then $m_1 = x_I x_J \in \text{in}_{\prec_{\mathcal{M}}}(\mathcal{J}_{\mathcal{M}})$. As in the proof of Proposition 9.3, this means there is another monomial m_2 so that $m_1 - m_2 \in \mathcal{J}_{\mathcal{M}}$. After a finite iteration of this argument, we see that $x_I x_J = x_A x_B \pmod{\mathcal{J}_{\mathcal{M}}}$ for some $\prec_{\mathcal{M}}$ -standard monomial

$x_A x_B$. But this means that $x_A x_B$ must be a sorted monomial since it is an edge of an alcove. Thus $A = U(I, J)$ and $B = V(I, J)$ satisfy $A, B \in \mathcal{M}$ so that \mathcal{M} is sorted. \square

By Theorem 6.3, the set of vertices (viewed as a collection of k -subsets of $[n]$) of an alcoved subpolytope of the hypersimplex is sort-closed. The converse follows from the description of an alcoved polytope as a convex union of alcoves; see Section 5.

10. SORT-CLOSED MATROIDS

Let k and n be positive integers satisfying $k \leq n$. A (non-empty) collection \mathcal{M} of k -subsets (called bases) of $[n]$ (the ground set) is a *matroid* if it satisfies the following axiom (Exchange Axiom):

Let I and J be two bases of \mathcal{M} . Then for any $i \in I$ there exists $j \in J$ so that $I - \{i\} \cup \{j\}$ is a base of \mathcal{M} .

The matroid \mathcal{M} is then said to be a rank k matroid on n elements. If I is a base of \mathcal{M} we write $I \in \mathcal{M}$. To a matroid \mathcal{M} (of rank k on n elements) we associate the *matroid polytope* $\mathcal{P}_{\mathcal{M}}$ as in Section 9. We say that \mathcal{M} is *sort-closed* if it is sort-closed as a collection of k -subsets. Thus by Theorem 9.2, the triangulation $\Gamma_{k,n}$ of the hypersimplex induces a triangulation of the polytopes $\mathcal{P}_{\mathcal{M}}$ for sort-closed matroids \mathcal{M} .

We now describe two classes of matroids which are sort-closed. Let Π be a set partition of $[n]$ with parts $\{\pi_i\}_{i=1}^r$ of sizes $|\pi_i| = a_i$, and $\underline{b} = (b_1, \dots, b_r)$, $\underline{c} = (c_1, \dots, c_r)$ be two sequences of non-negative integers. We will call the data $(\Pi, \underline{b}, \underline{c})$ a *weighted set partition*. Define $\mathcal{M}_{\Pi, \underline{b}, \underline{c}, k}$ to be the collection of k -subsets I of $[n]$ such that

$$(4) \quad b_j \leq |I \cap \pi_j| \leq c_j$$

for all j .

Lemma 10.1. *The collection of k -subsets $\mathcal{M}_{\Pi, \underline{b}, \underline{c}, k}$ defined above is a matroid.*

Proof. Let I and J be two such subsets and $i \in I$, say $i \in \pi_s$ for some s . If $|I \cap \pi_k| = |J \cap \pi_k|$ for all k or if $|I \cap \pi_s| \leq |J \cap \pi_s|$, one can again find some $j \in J \cap \pi_s - (I - \{i\})$ to add to $I - \{i\}$ form a base. Otherwise there is some t such that $|I \cap \pi_t| < |J \cap \pi_t| \leq c_t$ in which case one can find some $j \in (J \cap \pi_t)$ to add to $I - \{i\}$ without violating any of the inequalities in (4). This verifies the exchange axiom. \square

This class of matroids is closed under duality. We call the polytope $\Delta_{\Pi, \underline{b}, \underline{c}, k}$ associated to the matroid $\mathcal{M}_{\Pi, \underline{b}, \underline{c}, k}$ a *weighted multi-hypersimplex*. When $b_j = 0$ and $c_j = 1$ for all j we will denote the matroid and polytope by $\mathcal{M}_{\Pi, k}$ and $\Delta_{\Pi, k}$ respectively, and we call the polytope $\Delta_{\Pi, k}$ a *multi-hypersimplex*. Up to affine equivalence the polytope $\Delta_{\Pi, k}$ depends only on the multiset $\{a_i\}_{i=1}^r$. The polytope $\Delta_{\Pi, k}$ is the intersection of the hyperplane $x_1 + \dots + x_n = k$ with a product of simplices $\Delta_{\Pi} \simeq \Delta_{a_1} \times \dots \times \Delta_{a_r}$ just as the hypersimplices are slices of cubes. In the z -coordinates, this polytope is determined by intersecting the hypersimplex $\hat{\Delta}_{k,n}$ with the inequalities

$$0 \leq z_{a_1} - z_0 \leq 1; \quad 0 \leq z_{a_1+a_2} - z_{a_1} \leq 1; \dots; \quad 0 \leq z_n - z_{a_1+a_2+\dots+a_{r-1}} \leq 1$$

where we assume $z_n = k$. A weighted multi-hypersimplex can be viewed as a slice of a product of unions of hypersimplices. In particular, when $b_j = c_j - 1$ for all j , the polytope $\Delta_{\Pi, \underline{b}, \underline{c}, k}$ is the slice $x_1 + \dots + x_n = k$ of

$$\Delta_{\Pi, \underline{b}, \underline{c}} \simeq \tilde{\Delta}_{c_1, a_1+1} \times \tilde{\Delta}_{c_2, a_2+1} \times \dots \times \tilde{\Delta}_{c_r, a_r+1}$$

where the hypersimplex $\tilde{\Delta}_{c_i, a_i}$ (in the notation of Section 2) lives in the coordinates $(x_{a_1+\dots+a_{i-1}+1}, \dots, x_{a_1+\dots+a_i})$. Let $\Pi(a_1, \dots, a_r)$ denote the set partition

$$\{\pi_1 = \{1, \dots, a_1\}, \dots, \pi_r = \{a_1 + \dots + a_{r-1} + 1, \dots, a_1 + \dots + a_r = n\}\}.$$

Proposition 10.2. *The matroid $\mathcal{M}_{\Pi, \underline{b}, \underline{c}, k}$ for any $\underline{b}, \underline{c} \in \mathbb{N}^r$ is sort-closed.*

Proof. Let $I, J \in \mathcal{M}_{\Pi, \underline{b}, \underline{c}, k}$. Suppose to the contrary that one of $U(I, J)$ or $V(I, J)$ were not a base. Let $(q_1, q_2, \dots, q_{2k}) = \text{sort}(I \cup J)$. If $|U(I, J) \cap \pi_s| > c_s$ or $|V(I, J) \cap \pi_s| > c_s$ then it must be the case that for some i the entries $q_i, q_{i+2}, \dots, q_{i+2c_s}$ belonged to the same part $\pi_s \in \Pi$. Then $q_{i+1}, q_{i+3}, \dots, q_{i+2c_s-1}$ belong to π_s as well since $q_{i+2k} \leq q_{i+2k+1} \leq q_{i+2k+2}$. This is impossible as I and J were legitimate bases to begin with and contain at most c_s elements from π_s each. A similar argument guarantees that $|U(I, J) \cap \pi_s| \geq b_s$ and $|V(I, J) \cap \pi_s| \geq b_s$ for all s . \square

A matroid \mathcal{M} is *cyclically transversal* if it is a transversal matroid specified by a set of cyclic intervals $\{S_1, \dots, S_k\}$ of $[n]$. Recall that the bases of a transversal matroid are the k -element subsets $I = (i_1, \dots, i_k)$ of $[n]$ such that $i_s \in S_s$.

Proposition 10.3. *Let \mathcal{M} be a cyclically transversal matroid defined by the subsets $\{S_1, \dots, S_k\}$. Then \mathcal{M} is sort-closed.*

Proof. Let I be a k -element subset of $[n]$. By the Hall marriage theorem, I is a base of \mathcal{M} if and only if

$$(5) \quad |I \cap \bigcup_{r \in R} S_r| \geq |R|$$

for every subset R of $[k]$. Now let I and J be bases of \mathcal{M} and we now check (5) for $U(I, J)$ and $V(I, J)$. Since each S_i is a cyclic interval of $[n]$ it suffices to consider the case where $\bigcup_{r \in R} S_r$ is itself a cyclic interval $[a, b]$. By hypothesis, the multiset $I \cup J$ intersects $[a, b]$ in at least $2|R|$ elements. Thus each of $U(I, J)$ and $V(I, J)$ will intersect $[a, b]$ in at least $|R|$ elements. \square

Let us describe $\mathcal{P}_{\mathcal{M}} \subset \mathbb{R}^n$ for a cyclically transversal matroid explicitly in terms of inequalities. It is given by the hyperplane $x_1 + x_2 + \dots + x_n = n$, the inequalities $0 \leq x_i \leq 1$ together with the inequalities

$$\sum_{s \in S_R} x_s \geq |R|$$

with $S_R = \bigcup_{r \in R} S_r$ for every subset R of $[k]$.

We end this section with the question: what other matroids are sort-closed?

11. GRAPHS AND RANK TWO MATROIDS

When the rank k is equal to two (which we will assume throughout this section), every matroid \mathcal{M} arises as $\mathcal{M}_{\Pi, 2}$ for some set partition Π . Following [Stu, Chapter 9] and [LST], we associate a graph on $[n]$ to each maximal simplex of the matroid polytope $\mathcal{M}_{\Pi, 2}$. The vertices are drawn on a circle so that they are labelled clockwise in increasing order. Throughout this section, a “graph on $[n]$ ” will refer

to such a configuration of the vertices in the plane. Since the bases are two element subsets of $[n]$, we may identify them with the edges.

Lemma 11.1. *Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$ be two bases. Then the pair A, B are sorted if and only if the edges (a_1, a_2) and (b_1, b_2) intersect (not necessarily in their interior) when drawn on the circle.*

Proof. Sorted implies that $a_1 \leq b_1 \leq a_2 \leq b_2$ which immediately gives the lemma. \square

Thus $\binom{[n]}{2}$ -sorted subsets correspond to a graph on $[n]$ drawn on a circle, so that every pair of edges cross. Note that two edges sharing a vertex are considered to cross.

Proposition 11.2. *Let Π be a set partition and $\Delta_{\Pi,2}$ have dimension d . The maximal simplices in the alcoved triangulation of $\Delta_{\Pi,2}$ are in one-to-one correspondence with graphs on $[n]$ with $d + 1$ edges such that all edges are bases and every pair of edges cross.*

Proof. Follows immediately from Theorem 9.2 and Lemma 11.1. \square

Without the condition on the number of edges in the graphs of Proposition 11.2, one would obtain graphs corresponding to all simplices (not just the maximal ones) of the triangulation.

When the dimension of $\Delta_{\Pi,2}$ is n , each such graph G is determined by picking an odd-cycle C such that all the edges cross pairwise. The remaining edges of G joins a vertex not on C to the unique ‘opposite’ vertex lying on C (so that the edge crosses every edge of C); see Figure 3. We will call the resulting graph $G(C)$. Let C be a cycle, with pairwise crossing edges, of length $2k + 1$ with vertices $V(C) = \{v_1, v_2, \dots, v_{2k+1}\} \subset [n]$ labelled so that $v_1 < v_2 < \dots < v_{2k+1}$. Then the edges of C are of the form (v_i, v_{k+i+1}) or (v_1, v_{k+i}) , where the indices are taken modulo $2k + 1$. Thus the condition that all the edges of C are bases is equivalent to $|V(C) \cap \pi_i| \leq k$ for all i . In fact this is enough to guarantee that $G(C)$ corresponds to a valid maximal simplex of $\Delta_{\Pi,2}$ – that the remaining edges not on the cycle are bases is implied.

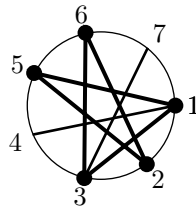


FIGURE 3. A graph $G(C)$. The cycle C has been drawn in bold.

Suppose G arises from a sorted subset (I_1, \dots, I_r) . Let $w = \theta(I_1, \dots, I_r)$ where θ is the bijection of Section 6. The vertices i not on the odd cycle of G are exactly the positions such that $w_i = w_{i-1} + 1$.

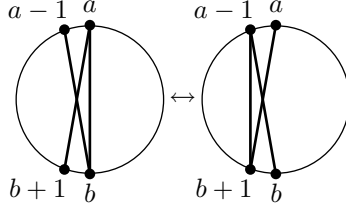


FIGURE 4. The two simplices in $\Gamma_{2,n}$ corresponding to two graphs G and G' are adjacent if G and G' are related by the above move.

Proposition 11.3. *Let a_1, \dots, a_r be positive integers and $n = a_1 + \dots + a_r$. Then the volume of the second multi-hypersimplex $\Delta_{\Pi(a_1, \dots, a_r), 2}$ is given by*

$$\begin{aligned} \text{Vol}(\mathcal{P}_{\Pi(a_1, \dots, a_r), 2}) &= \sum_{k=1}^{\infty} \left(\sum_{c_1, \dots, c_r \leq k; c_1 + \dots + c_r = 2k+1} \binom{a_1}{c_1} \dots \binom{a_r}{c_r} \right) \\ &= 2^{n-1} - \sum_{i=1}^r \sum_{b, d \geq 0} \binom{a_i}{2b+d+1} \binom{n-a_i}{d} \end{aligned}$$

Proof. The first formula follows from enumerating odd subsets $S \subset \{1, 2, \dots, n\}$ with size $2k+1$ satisfying $c_i := |S \cap \pi_i| \leq k$ for all i . The second formula comes from counting the odd subsets $S' \subset \{1, 2, \dots, n\}$, where $|S' \cap \pi_i| > |S'|/2$ for some $i \in \{1, \dots, r\}$, and subtracting them from all odd subsets of $\{1, \dots, n\}$. \square

One can also describe the simplices of the polytopes $\mathcal{P}_{\mathcal{M}}$ for higher rank matroids as hypergraphs G satisfying the following conditions:

- (1) Every hyperedge $A \subset [n]$ of G is a base of \mathcal{M} .
- (2) Let $A = \{a_1 < \dots < a_k\}$, $B = \{b_1 < \dots < b_k\}$ be a pair of hyperedges belonging to G . Let C_A be the cycle on $[n]$ drawn with usual edges $(a_1, a_2), \dots, (a_k, a_1)$ and similarly for C_B . Then each edge of C_A must touch C_B and vice versa.

12. ADJACENCY OF ALCOVES IN THE SECOND HYPERSIMPLEX

Let $\Delta \in \Gamma_{k,n}$ be a (maximal) simplex. We say that Δ has *degree* d if it is adjacent to d other simplices. We call Δ an *internal simplex* if none of its facets lies on the boundary of $\Delta_{k,n}$. In this case Δ has maximal degree, namely n .

Proposition 12.1. *The two simplices of $\Gamma_{2,n}$ corresponding to two graphs G and G' (via the correspondence of Proposition 11.2) are adjacent if and only if there are four distinct vertices labelled $a, a+1, b, b+1 \pmod n$ such that G contains the edges $(a, b), (a-1, b), (b+1, a)$ and G' is obtained from G by changing the edge (a, b) to $(a-1, b+1)$; see Figure 4.*

Proof. The proposition follows immediately from Proposition 11.2 and Theorem 7.1 applied to the case $k=2$ (more precisely, the comments after the proof of the theorem). \square

In fact the proposition is true also for smaller dimensional faces of the simplices in $\Gamma_{k,n}$. Let C be an odd cycle such that all edges cross and let $|C|$ denote its length.

Theorem 12.2. *Let Δ be the simplex of $\Gamma_{2,n}$ corresponding to the graph $G(C)$. If $G(C)$ is a triangle then $\Delta = \Delta_{2,3}$ is the unique simplex in $\Gamma_{2,3}$ (and has degree 0). If $|C| = 3$ and $G(C)$ has two vertices of degree two then Δ has degree two. Otherwise, Δ has degree $|C|$.*

Proof. The Theorem follows from Proposition 12.1. Indeed we can perform the move shown in Figure 4 to an edge (a, b) if it joins two vertices a, b each of degree at least two, with the exception of the case where a and b both have degree exactly two and are joined to the same vertex c . When the edge (a, b) can be replaced, the change is necessarily unique. The case where a and b both have degree two and are joined to the same vertex c occurs only when C is a three-cycle. In all other cases, every edge of C can be replaced, and so Δ has degree $|C|$. \square

The following corollary is immediate from Theorem 12.2.

Corollary 12.3. *For $d > 1$, the second hypersimplex $\Gamma_{2,n}$ has $\binom{n}{2d+1}$ simplices with degree $2d+1$. No simplex has even degree greater than or equal to 4. In particular, if n is odd then $\Gamma_{2,n}$ contains a unique internal simplex. If n is even, then $\Gamma_{2,n}$ has no internal simplices.*

13. F -VECTOR OF THE TRIANGULATION OF THE SECOND HYPERSIMPLEX

Let $\tilde{\Gamma}_{k,n}$ be the simplicial complex associated with the triangulation $\Gamma_{k,n}$. That is, $\tilde{\Gamma}_{k,n}$ includes all simplices of $\Gamma_{k,n}$ and all their faces. In this section we discuss the f -vector of $\tilde{\Gamma}_{k,n}$.

Let T be a tree drawn on a circle such that every pair of edges cross. Every such tree is obtained from a unique path P_T by inserting edges as shown in Figure 5. Let $T_{a,b}$ denote the number of such unlabeled trees which as a bipartite graph has a vertices in one part and b vertices in the other part. A direct bijective proof of the following lemma would be interesting.

Lemma 13.1. *Set $T(x, y) = \sum_{a,b \geq 1} T_{a,b} x^a y^b$. We have*

$$T(x, y) = \frac{xy}{1-x-y}.$$

Proof. Figure 6 gives the four kinds of paths P_T which can occur, where the paths are drawn on a circle, and treated as a bipartite graph where the two parts are distinguished. The trees arising from these four kinds of paths correspond to the four terms $\frac{x}{1-x}$, $\frac{y}{1-y}$, $\frac{xy}{(1-x)(1-y)}$ and $\frac{xy}{(1-x)(1-y)}$ inside the brackets of the expression $T(x, y) = xy \left(1 + \left(\frac{x}{1-x} + \frac{y}{1-y} + \frac{2xy}{(1-x)(1-y)} \right) \frac{1}{1 - \frac{xy}{(1-x)(1-y)}} \right)$. The right-hand side of this expression simplifies to $\frac{xy}{1-x-y}$. \square

Now let F be a forest drawn on a circle such that every pair of edges cross and no vertex is isolated. Thus $F = T_1 \cup T_2 \cup \dots \cup T_l$ is a union of trees, each of which contains at least one edge. If, for each tree T_i , we remove all but one edge and all non-incident edges, then we obtain a forest F' where all components are just single edges, and all edges cross. Conversely, every forest F can be obtained from some F' by replacing each edge (s, t) with a tree T in the following manner. Treating T as a bipartite graph, all vertices of one part are placed very close to s and all vertices of the other part are placed very close to t , and in addition all edges must cross; see Figure 7.

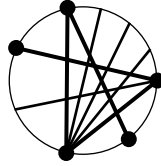


FIGURE 5. A tree drawn on a circle with all edges crossing. The corresponding path is drawn in bold with solid dots.

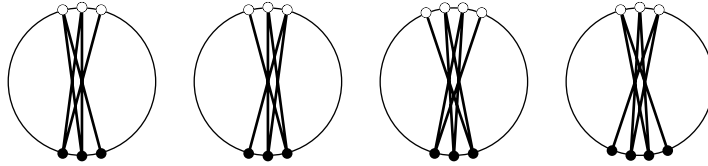


FIGURE 6. Four kinds of paths where every pair of edges cross. The two parts of the path are to be considered distinguished.

Let $\mathcal{B}_{i,n}$ be the number of such forests which are labelled along the circle in the natural manner. Thus

$$\mathcal{B}_{i,n} = \sum_{a_1+b_1+\dots+a_l+b_l=n} a_1 T_{a_1,b_1} T_{a_2,b_2} \cdots T_{a_l,b_l}$$

where $l = n - i$ is the number of connected components of the forest and the vertex labelled ‘1’ is placed in the part of size a_1 of the first tree. All the other labels are determined by the writing $\{1, 2, 3, \dots, n\}$ in clockwise order. Let $B(t, q) = \sum_{i,n} \mathcal{B}_{i,n} t^n q^i$. By Lemma 13.1, we compute that $B(t, q)$ equals

$$\left[\sum_{j=0}^{\infty} x \frac{\partial T(x, y)}{\partial x} (T(x, y))^j z^{j+1} \right]_{y \mapsto tq, x \mapsto tq, z \mapsto t^{-1}} = \frac{tq(q-tq)}{(1-2tq)(1-2tq-tq^2)}.$$

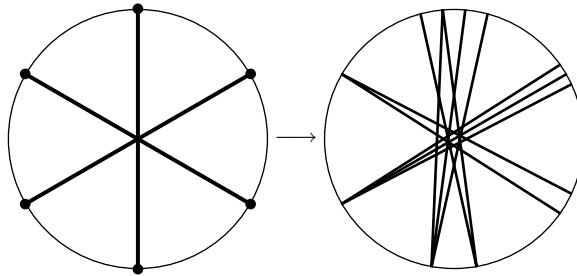


FIGURE 7. Replacing edges by trees, to obtain a forest where all edges cross.

Recall that we define the f -vector $f(\Delta) = (f_{-1}, f_0, \dots, f_{d-1})$ of a simplicial complex Δ of dimension $d-1$ by setting $f_i(\Delta)$ to be the number of faces of dimension

i. The f -polynomial is defined as $f(\Delta, t) = \sum_{i=0}^d f_{i-1} t^i$. Define the generating function $F(t, x) = \sum_{n \geq 1} f(\tilde{\Gamma}_{2,n}, t) x^n$, where we define $f(\tilde{\Gamma}_{2,1}, t) = 1$.

Theorem 13.2. *The generating function $F(t, x)$ is given by*

$$F(t, x) = \frac{1}{1-x} \left[\frac{tq^2(t^2q^2 + t^2q - tq + 1)}{(1-tq)^2(1-2tq-tq^2)} \right]_{q \rightarrow \frac{x}{1-x}} + \frac{x}{1-x}.$$

Proof. According to Proposition 11.2, $f_i(\tilde{G}_{2,n})$ is given by the number of graphs on $[n]$, with $i+1$ edges such that every pair of edges cross. Let $\mathcal{C}_{i,n}$ be the number of such graphs with i edges and n vertices containing no isolated vertex. Thus $f_{i-1}(\tilde{\Gamma}_{2,n}) = \sum_{r=0}^n \binom{n}{r} \mathcal{C}_{i,r}$ for $i \geq 1$ and $f_{-1}(\tilde{\Gamma}_{2,n}) = 1$. Set $C(t, q) = \sum_{1 \leq i \leq n} \mathcal{C}_{i,n} t^i q^n$. Then one can check that the generating functions are related by $F(t, x) = C(t, x/(1-x))/(1-x) + x/(1-x)$.

Every graph on $[n]$ such that every pair of edges cross and has no isolated vertices either contains a unique odd-cycle or is a forest of the type considered above. The number of graphs which contain a unique odd-cycle is exactly the Eulerian number $A_{2,n-1} = 2^{n-1} - n$. All these graphs have as many edges as vertices, so adding them to the forests we get

$$C(t, q) = B(t, q) + \sum_n (2^{n-1} - n) t^n q^n = \frac{tq^2(t^2q^2 + t^2q - tq + 1)}{(1-tq)^2(1-2tq-tq^2)}.$$

The theorem follows immediately. \square

One should compare the results of this section to known formulae for the h -polynomials of the triangulation of the hypersimplex derived from a calculation of the Ehrhart polynomial of $\Delta_{k,n}$; see, e.g., [Kat]. The h -polynomial of a simplicial complex Δ of dimension $d-1$ is given by $t^d h(\Delta, t^{-1}) = [t^d f(\Delta, t^{-1})]_{t \rightarrow t^{-1}}$. For more details see for example [Hib, EC1, Zie]. Write $\binom{n}{r}_l$ to denote the coefficient of t^r in $((1+t+t^2+\dots+t^{l-1})^n)$.

Proposition 13.3. *The h -polynomial of the simplicial complex $\tilde{G}_{k,n}$ is given by*

$$h(\tilde{\Gamma}_{k,n}, t) = \sum_{d=0}^{n-1} \left(\sum_{i=0}^{k-1} (-1)^i \binom{n}{i} \binom{n}{(k-i)d-i}_{k-i} \right) t^d.$$

Note that it is not clear from the above formula that the h -vector is non-negative, though this follows from general results.

14. MULTI-EULERIAN POLYNOMIALS

Let Π be a fixed set partition with parts of sizes $\{a_i\}_{i=1}^r$ of total size $n = \sum_i a_i$ and let $\underline{b} = (b_1, \dots, b_r), \underline{c} = (c_1, \dots, c_r) \in \mathbb{N}^r$. Define the *weighted multi-Eulerian number* $A_{\Pi, \underline{b}, \underline{c}, k} = \text{Vol}(\Delta_{\Pi, \underline{b}, \underline{c}, k})$ as the normalized $(n-1)$ -dimensional volume of the corresponding weighted multi-hypersimplex. We will consider polytopes of smaller dimension to have volume 0 for what follows. Now define the *weighted multi-Eulerian polynomial* $A_{\Pi, \underline{b}, \underline{c}}(t)$ as

$$A_{\Pi, \underline{b}, \underline{c}}(t) = \sum_{k=1}^r A_{\Pi, \underline{b}, \underline{c}, k} t^k.$$

Note that when Π is the set partition with all $a_i = 1$ and $b_i = 0$ and $c_i = 1$ for all i , then $A_{\Pi, \underline{b}, \underline{c}}(t)$ reduces to the usual Eulerian polynomial $A_{n-1}(t) = \sum_k A_{k, n-1} t^k$.

If Π is a set partition of $[n]$, we denote by Π^* the set partition of $[n+1]$ with an additional part of size one containing $n+1$. If $\underline{d} = (d_1, \dots, d_r) \in \mathbb{Z}^r$, then we denote by $\underline{d}^* \in \mathbb{Z}^{r+1}$ to be the integer vector with an additional coordinate $d_{r+1} = 0$ and $\underline{d}' \in \mathbb{Z}^{r+1}$ similarly with $d_{r+1} = 1$. Then $\Delta_{\Pi^*, \underline{b}^*, \underline{c}', k}$ is affinely equivalent to (and has the same normalized volume as) the intersection of $k-1 \leq x_1 + \dots + x_n \leq k$ with the product of unions of hypersimplices

$$\left(\bigcup_{i=b_1+1}^{c_1} \Delta_{i, a_1+1} \right) \times \dots \times \left(\bigcup_{i=b_r+1}^{c_r} \Delta_{i, a_r+1} \right).$$

Thus

$$A_{\Pi^*, \underline{b}^*, \underline{c}'}(1) = \binom{n}{a_1, a_2, \dots, a_r} \left(\sum_{i=b_1+1}^{c_1} A_{i, a_1} \right) \dots \left(\sum_{i=b_r+1}^{c_r} A_{i, a_r} \right).$$

In particular if $b_j = 0$ and $c_j = 1$ then this value is simply a multinomial coefficient. For this special case, we will omit \underline{b} and \underline{c} in the notation and omit the prefix *weighted* from the names. We write out the combinatorial interpretation for $A_{\Pi, k}$ explicitly.

Proposition 14.1. *Let k be a positive integer and Π be a set partition of $[n]$ with parts of sizes $\{a_i\}_{i=1}^r$ as before. The following quantities are equal:*

- (1) *the multi-Eulerian number $A_{\Pi, k}$,*
- (2) *the number of permutations of $w \in S_{n-1}$ with $k-1$ descents such that the substring $w_{a_1} w_{a_1+a_2} \dots w_{a_1+a_2+\dots+a_{r-1}}$ has $k-1$ descents,*
- (3) *the number \mathcal{M}_{Π} -sorted subsets (I_1, \dots, I_n) of size n .*

Proof. As described in Section 10 we may consider multi-hypersimplices subpolytopes of the hypersimplex so the proposition follows from Theorems 6.3, 9.2 and 8.1. \square

Note that with $\Pi = \Pi(a_1, a_2, \dots, a_r)$, then $A_{\Pi, k}$ is a function symmetric in the inputs $\{a_i\}$. Let $\Pi = \Pi(a, 1^{n-a})$. By Proposition 14.1(2), $A_{\Pi}(t)$ is the generating function by descents for permutations satisfying $w_1 < w_2 < \dots < w_a$. Thus

$$A_{\Pi(a, 1^{n-a})}(t) = \sum_{w \in S_{n-a}} \binom{a+w_1-2}{a-1} t^{\text{des}(w)+1}.$$

We give an explicit formula for $a = 2$.

Proposition 14.2. *For $n \geq 3$, $A_{\Pi(2, 1^{n-2})}(t) = t \frac{d}{dt} A_{n-1}(t)$, where $A_{n-1}(t)$ is the usual Eulerian polynomial. In other words,*

$$(6) \quad \sum_{w \in S_{n-1}} w_1 t^{\text{des}(w)+1} = \sum_{w \in S_{n-1}} (\text{des}(w) + 1) t^{\text{des}(w)+1}.$$

We will prove this statement bijectively. Let $w \in S_n$. A *circular descent* of w is either a usual descent or the index n if $w_n > w_1$. Define the *circular descent number* $\text{cdes}(w)$ as the number of circular descents. Let C_n denote the subgroup of S_n generated by the long cycle $c = (12 \dots n)$, written in cycle notation. We have the following easy lemma.

Lemma 14.3. *The statistic cdes is constant on double cosets $C_n \backslash S_n / C_n$.*

Proof. Left multiplication by the long cycle c maps $w_1w_2 \cdots w_n$ to $(w_1 + 1)(w_2 + 1) \cdots (w_n + 1)$ where “ $n + 1$ ” is identified with “1”. Right multiplication by c maps $w_1w_2 \cdots w_n$ to $w_2w_3 \cdots w_nw_1$. The lemma is immediate from the definition of circular descent number. \square

Actually in the following we will use this Lemma for S_{n+1} .

Proof of Proposition 14.2. If $w = w_1w_2 \cdots w_n \in S_n$ and $w' = w_1w_2 \cdots w_n(n + 1) \in S_{n+1}$ then $\text{cdes}(w') = \text{des}(w) + 1$. By this and our earlier comments, the left hand side of (6) is the generating function for permutations in S_{n+1} satisfying $w_1 < w_2$ and $w_{n+1} = n + 1$, according to their circular descent number. Alternatively, we may view this as the cdes -generating function of right cosets $\bar{w} \in S_{n+1}/C_{n+1}$ satisfying the property that the two numbers w_i, w_{i+1} cyclically located after $w_{i-1} = n + 1$ satisfy $w_i < w_{i+1}$ for any representative $w \in \bar{w}$. Here the indices are taken modulo $n + 1$ and by Lemma 14.3, $\text{cdes}(\bar{w}) := \text{cdes}(w)$ does not depend on the representative w of \bar{w} and so is well defined.

The right hand side of (6) is the generating function for permutations $u = u_1u_2 \cdots u_n \in S_n$ satisfying $u_n = n$ where one of the circular descents has been marked, again according to circular descent number. If $u_i > u_{i+1}$ is the marked circular descent (where $i + 1$ is to be taken modulo n), then we can insert the number $n + 1$ between u_i and u_{i+1} to obtain a coset $\bar{v} \in S_{n+1}/C_{n+1}$ with the same number of circular descents. If $v \in \bar{v}$ and $v_i = n + 1$ then we automatically have $v_{i-1} > v_{i+1}$ (in fact this is a cdes -preserving bijection between cosets \bar{v} satisfying this property and permutations of $u \in S_n$ satisfying $u_n = n$ with a marked circular descent). Let $c = (123 \cdots (n + 1))$ be the generator of C_{n+1} and consider $v' = c^{n+1-v_{i-1}}v$ for any $v \in \bar{v}$ where i is determined by $v_i = n + 1$. Let $\bar{v}' = v'C$. By Lemma 14.3, $\text{cdes}(\bar{v}') = \text{cdes}(\bar{v})$. However, it is easy to see that \bar{v}' is exactly one of the cosets which are enumerated by the left hand side of (6). Thus we obtain a cdes -preserving bijection $\bar{v} \mapsto \bar{v}'$ between two classes of cosets in S_{n+1}/C_{n+1} , enumerated by the two sides of (6).

We illustrate the bijection with an example, where we will pick representatives of appropriate cosets at our convenience. Let $u = 53162748 \in S_8$ satisfying $u_8 = 8$ with marked circular descent index 4 corresponding to $u_4 = 6 > 2 = u_5$. This is an object enumerated by the right hand side of (6), with $n = 8$. Inserting “9” between “6” and “2” we obtain $v = 531692748 \in S_9$. Multiplying on the left by c^3 (where $c = (123456789)$) adds 3 to every value, changing “6” to “9”, giving the permutation $v' = 864935172$. Multiplying on the right by c^4 we move the 9 to the last position to get $w = 351728649 \in \bar{v}' = v'C$, which satisfies $w_9 = 9$ and $w_1 < w_2$. This is exactly a permutation enumerated by the left hand side of (6). Note that $\text{cdes}(u) = 5 = \text{cdes}(w)$ and that all the steps can be reversed to give a bijection. \square

It would be interesting if algebro-geometric proofs of some of our results concerning (weighted) multi-Eulerian numbers could be given; see Section 18.4.

15. TRIANGULATIONS OF ALCOVED POLYTOPES

In the next few sections we study triangulations of general alcoved polytopes of type A . Most of the constructions here are generalized to other crystallographic root systems in [AP2].

Let $\mathcal{P} = \mathcal{P}(b_{ij}, c_{ij}) \subset \mathbb{R}^n$ be an alcoved polytope which we realize in the x -coordinates. In other words, \mathcal{P} is a $(n-1)$ -dimensional polytope lying in a hyperplane $x_1 + x_2 + \cdots + x_n = k$ for some $k \in \mathbb{Z}$; and given by the inequalities $b_{ij} \leq x_{i+1} + \cdots + x_j \leq c_{ij}$ for each pair (i, j) satisfying $0 \leq i < j \leq n-1$. By translating \mathcal{P} by (m, \dots, m) for some $m \in \mathbb{Z}$ to obtain an affinely equivalent polytope, we can assume that all the coordinates of the points of \mathcal{P} are non-negative. Let $Z_{\mathcal{P}} = \mathcal{P} \cap \mathbb{Z}^n \subset \mathbb{N}^n$ denote the set of integer points lying inside \mathcal{P} .

Let $G_{\mathcal{P}}$ be the directed graph defined as follows in analogy with $G_{k,n}$ in Section 6. The graph $G_{\mathcal{P}}$ has vertices labelled by points $a \in Z_{\mathcal{P}}$. Two vertices $a, b \in Z_{\mathcal{P}}$ are connected by an edge $a \rightarrow b$ labelled i if there exists an index $i \in [1, n]$ such that $a + e_{i+1} - e_i = b$, where e_i, e_{i+1} are the coordinate vectors and $e_{n+1} := e_1$. Let $C_{\mathcal{P}}$ denote the set of minimal circuits of $G_{\mathcal{P}}$, which have length n .

For an integer vector $a = (a_1, a_2, \dots, a_n) \in \mathbb{N}^n$ with non-negative coordinates lying on $x_1 + x_2 + \cdots + x_n = k$, we let I_a denote the multiset of size k of $\{1, 2, \dots, n\}$ with a_1 1's, a_2 2's and so on. If I, J are multisets of size k with elements from $\{1, 2, \dots, n\}$ then we can define $U(I, J)$ and $V(I, J)$ by sorting $I \cup J$ as in Section 4. Similarly we define the notions of sorted and sort-closed for collections of multisets. In the following theorem, note that $Z_{\mathcal{P}}$ and $G_{\mathcal{P}}$ are defined without needing \mathcal{P} to be alcoved.

Theorem 15.1. *Let $\mathcal{P} \subset \mathbb{R}^n$ be a $(n-1)$ -dimensional polytope lying in $x_1 + x_2 + \cdots + x_n = k$ so that all points of \mathcal{P} have non-negative coordinates. Then the following are equivalent:*

- (1) \mathcal{P} is an alcoved polytope.
- (2) The set $\mathcal{I} = \{I_a \mid a \in Z_{\mathcal{P}}\}$ is sort-closed. A triangulation of \mathcal{P} is consists of the maximal simplices with vertices $\{a_1, a_2, \dots, a_n\}$ for each sorted collection $(I_{a_1}, I_{a_2}, \dots, I_{a_n})$ where $I_{a_k} \in \mathcal{I}$.
- (3) The set $C_{\mathcal{P}}$ of minimal circuits of $G_{\mathcal{P}}$ gives rise to a triangulation of \mathcal{P} : if $C = (c^{(1)}, c^{(2)}, \dots, c^{(n)}) \in C_{\mathcal{P}}$ then a maximal simplex is given by $\text{conv}(c^{(1)}, c^{(2)}, \dots, c^{(n)})$.

When these conditions hold, all the three above triangulations agree.

Proof. The proof is exactly analogous to the arguments of Theorem 6.3. The triangulation here is obtained by copying and translating the triangulations of Theorem 6.3 by an integer vector so that the corresponding simplices cover the polytope \mathcal{P} . For example, if $(c^{(1)}, c^{(2)}, \dots, c^{(n)}) \in C_{\mathcal{P}}$ then there exists $c \in \mathbb{Z}^n$ so that $c^{(i)} - c$ is a 0-1 vector for each i . Thus $(c^{(1)} - c, c^{(2)} - c, \dots, c^{(n)} - c)$ is a minimal circuit in some $G_{k,n}$ corresponding to some simplex $\Delta_{(w)}$; see Section 6. The simplex $\Delta_{(w)} + c = \text{conv}(c^{(1)}, c^{(2)}, \dots, c^{(n)})$ will correspond to the circuit $(c^{(1)}, c^{(2)}, \dots, c^{(n)})$. To show that these simplices are disjoint and cover \mathcal{P} , we change to the z -coordinates of Section 5 and see that we obtain the affine Coxeter arrangement. \square

An alcoved polytope also gives rise to a Gröbner basis $\mathcal{G}_{\mathcal{P}}$ of the associated toric ideal $\mathcal{I}_{\mathcal{P}}$. The reader will be able to write it down following Section 9.

Let us identify $w \in S_{n-1}$ with $w_1 w_2 \cdots w_{n-1} n \in S_n$ as usual. Recall that $\Delta_{(w)}$ denotes the simplex $(\psi \circ p)^{-1}(\nabla_w)$ where we view p as a map from $\{x \in \mathbb{R}^n \mid x_1 + \cdots + x_n = k\}$ to \mathbb{R}^{n-1} . For an alcoved polytope \mathcal{P} , define the polytopes $\mathcal{P}_{(w)}$

by

$$\mathcal{P}_{(w)} = \{x \in \mathbb{R}^n \mid (\Delta_{(w)} + x) \subset \mathcal{P}\}.$$

Denote by $I(\mathcal{P})$ the number $|\mathbb{Z}^n \cap \mathcal{P}| = \#Z_{\mathcal{P}}$ of lattice points in \mathcal{P} .

Theorem 15.2. *Each of the polytopes $\mathcal{P}_{(w)}$ is an alcoved polytope. The normalized volume of \mathcal{P} is given by*

$$\text{Vol}(\mathcal{P}) = \sum_w I(\mathcal{P}_{(w)})$$

where the sum is over all permutations $w \in S_{n-1}$.

Proof. The second statement was essentially shown in the proof of Theorem 15.1. Let \mathcal{P} be given by the inequalities $b_{ij} \leq x_{i+1} + \dots + x_j \leq c_{ij}$ within the hyperplane $x_1 + x_2 + \dots + x_n = l$. We check that $\mathcal{P}_{(w)}$ is an alcoved polytope. In fact this follows from the fact that $\Delta_{(w)}$ is itself an alcoved polytope and is given by some inequalities $d_{ij} \leq x_{i+1} + \dots + x_j \leq f_{ij}$ and a hyperplane $x_1 + x_2 + \dots + x_n = k$, where we pick d_{ij} and f_{ij} so that all equalities are achieved by some point in $\Delta_{(w)}$. Then $\mathcal{P}_{(w)}$ is the intersection of the inequalities $b_{ij} - d_{ij} \leq x_{i+1} + \dots + x_j \leq c_{ij} - f_{ij}$ with the hyperplane $x_1 + x_2 + \dots + x_n = l - k$, which by definition is an alcoved polytope. \square

For the multi-hypersimplices and other subpolytopes of the hypersimplex, the polytopes $\mathcal{P}_{(w)}$ are either empty or a single point.

16. ALCOVE LATTICE AND ALCOVED POLYTOPES

Define the *alcove lattice* Λ_n as the infinite graph whose vertices correspond to alcoves (i.e., regions of type A_{n-1} affine Coxeter arrangement) and edges correspond to pairs of adjacent alcoves. For example, Λ_3 is the infinite hexagonal lattice. For an alcoved polytope \mathcal{P} , define its graph $\Gamma_{\mathcal{P}}$ as the finite subgraph of Λ_n formed by alcoves in \mathcal{P} . For the graphs of Section 6, we have $\Gamma_{k,n} = \Gamma_{\Delta_{k,n}}$.

According to [LP, Sect. 14], we have the following combinatorial construction of the lattice Λ_n . Let $[\lambda_0, \dots, \lambda_{n-1}]$ denote an element of $\mathbb{Z}^n / (1, \dots, 1)\mathbb{Z}$. In other words, we assume that $[\lambda_0, \dots, \lambda_{n-1}] = [\lambda'_0, \dots, \lambda'_{n-1}]$ whenever the λ'_i are obtained from the λ_i by adding the same integer. The vertices of Λ_n can be identified with the following subset of $\mathbb{Z}^n / (1, \dots, 1)\mathbb{Z}$, see [LP]:

$$\Lambda_n = \{[\lambda_0, \dots, \lambda_{n-1}] \mid \text{the integers } \lambda_0, \dots, \lambda_{n-1} \text{ have different residues modulo } n\}.$$

Two vertices $[\lambda_0, \dots, \lambda_{n-1}]$ and $[\mu_0, \dots, \mu_{n-1}]$ of Λ_n are connected by an edge whenever there exists a pair (i, j) , $0 \leq i \neq j \leq (n-1)$, such that $\lambda_i + 1 \equiv \lambda_j \pmod n$ and $(\mu_0, \dots, \mu_{n-1}) = (\lambda_0, \dots, \lambda_{n-1}) + e_i - e_j$, where e_i, e_j are the coordinate vectors in \mathbb{Z}^n . In this construction, $\lambda_0, \dots, \lambda_{n-1}$ are the z -coordinates of the central point of the associated alcove scaled by the factor n , see [LP]. This construction immediately implies the following description of the graph of the alcoved polytope $\mathcal{P}(b_{ij}, c_{ij})$, defined as in Section 5.

Proposition 16.1. *For an alcoved polytope $\mathcal{P} = \mathcal{P}(b_{ij}, c_{ij})$, its graph $\Gamma_{\mathcal{P}}$ is the induced subgraph of Λ_n given by the subset of vertices*

$$\{[\lambda_0, \dots, \lambda_{n-1}] \in \Lambda_n \mid n \cdot b_{i,j} \leq \lambda_i - \lambda_j \leq n \cdot c_{i,j}, \text{ for } i, j \in [0, n-1]\}.$$

The vertices of the graph $\Gamma_{\mathcal{P}}$ are in bijection with the elements of $C_{\mathcal{P}}$ defined in the previous section. Let $C = (c^{(1)}, c^{(2)}, \dots, c^{(n)}) \in C_{\mathcal{P}}$ be a minimal circuit in the graph $\Gamma_{\mathcal{P}}$. The integer points $\{c^{(1)}, c^{(2)}, \dots, c^{(n)}\}$ are the vertices of an alcove A_C , in the x -coordinates. The vertex $[\lambda_1, \lambda_2, \dots, \lambda_n]$ of Λ_n associated to A_C is given by $\lambda_i = \alpha_1 + \alpha_2 + \dots + \alpha_i$ where $\alpha := (\alpha_1, \dots, \alpha_n)$ is given by $\alpha = \sum_{i=1}^n c^{(i)}$. The point $\frac{1}{n}[\lambda_1, \lambda_2, \dots, \lambda_n] \in \mathbb{R}^n / (1, \dots, 1)\mathbb{R}$ is the central point of the alcove A_C in the z -coordinates.

Example 16.2. *The k -th hypersimplex is given by inequalities (1). Thus the vertex set of the graph $\Gamma_{\Delta_{k,n}}$ is the subset of Λ_n given by the inequalities $0 \leq \lambda_1 - \lambda_0, \lambda_2 - \lambda_1, \dots, \lambda_{n-1} - \lambda_{n-2} \leq n$, and $(k-1) \cdot n \leq \lambda_{n-1} - \lambda_0 \leq k \cdot n$.*

We now give an abstract characterization of subgraphs of Λ_n corresponding to alcoved polytopes. Let us say that an induced subgraph H of some graph G is *convex* if, for any pair of vertices u, v in H , and any path P in G from u to v of minimal possible length, all vertices of P are in H . In [AP2], we will prove the following statement in the more general context of an arbitrary Weyl group.

Proposition 16.3. *An induced subgraph Γ in Λ_n is the graph of some alcoved polytope if and only if Γ is convex.*

17. MAPS OF A CIRCLE WITH n MARKED POINTS

Let S^1 be the unit circle and let $S^1_{(n)}$ denote a circle with n distinct marked points p_0, p_1, \dots, p_{n-1} arranged in clockwise order. Let \mathcal{P} be an alcoved polytope with parameters b_{ij} and c_{ij} as in Section 15. Let $M_{\mathcal{P}}$ denote the set of homotopy classes of continuous maps $f : S^1_{(n)} \rightarrow S^1$ satisfying:

- The map f is always locally bijective and locally orientation preserving. Informally, this means that f traces out S^1 in the clockwise direction and never stops.
- The images of marked points are distinct.
- For each $0 \leq i < j \leq n-1$, The number d of pre-images of $f(p_i)$ under f in the open interval (p_i, p_j) satisfies $b_{ij} \leq d < c_{ij}$.

Two maps f and g belong to the same homotopy class if and only if they can be deformed into one another by a homotopy, in the usual sense, while always satisfying the conditions above.

Proposition 17.1. *Let \mathcal{P} be an alcoved polytope. Then the simplices in the triangulation $\Gamma_{\mathcal{P}}$ are in bijection with the elements of $M_{\mathcal{P}}$.*

Proof. We think of S^1 as the interval $[0, 2\pi]$ with the two endpoints identified. For a point $x = (x_1, \dots, x_n) \in \mathcal{P}$ in the interior of some alcove of \mathcal{P} , we obtain a map $f_x : S^1_{(n)} \rightarrow S^1$ by setting $f_x(p_0) = 0$ and between the points p_i and p_{i+1} we let f_x be the unique injective map which travels a distance of $2\pi x_i$ at constant speed.

If x and y lie in the same alcove of \mathcal{P} (Theorem 15.1) then f_x and f_y are easily seen to be homotopy equivalent, as the relative positions of the points $f_x(p_i)$ will not change. Conversely, picking an interior point in each alcove of \mathcal{P} gives representatives of the homotopy classes of maps $f : S^1_{(n)} \rightarrow S^1$ satisfying the required conditions. \square

18. FINAL REMARKS

18.1. The h -vectors of alcoved polytopes. In Section 13, we gave a formula for the h -vectors of the simplicial complex $\tilde{\Gamma}_{k,n}$. Let $\tilde{\Gamma}_{\mathcal{P}}$ denote the simplicial complex associated to the collection of simplices in the alcoved triangulation of \mathcal{P} . The triangulation $\Gamma_{\mathcal{P}}$ of an alcoved polytope \mathcal{P} is regular (Theorems 15.1 and 19.1) and this implies that the h -polynomial $h(\tilde{\Gamma}_{\mathcal{P}}, t)$ is a non-negative polynomial. We have given many interpretations for the volume $h(\tilde{\Gamma}_{\mathcal{P}}, 1)$ of an alcoved polytope. It would be interesting to obtain statistics on these interpretations which give $h(\tilde{\Gamma}_{\mathcal{P}}, t)$.

In fact, the Ehrhart polynomial of \mathcal{P} equals to the Hilbert polynomial of the quotient ring $k[x_I]/\mathcal{J}_{\mathcal{P}}$ associated to the triangulation $\Gamma_{\mathcal{P}}$; see Section 19. The h -polynomial is the numerator of the associated generating function.

18.2. Relation with order polytopes. Let P be a poset naturally labelled with the numbers $[n]$. The order polytope $\mathcal{O}_P \subset \mathbb{R}^n$ is defined by the inequalities $0 \leq x_i \leq 1$ together with $x_i < x_j$ for every pair of elements $i, j \in P$ satisfying $i > j$. It is clear that \mathcal{O}_P is affinely equivalent to an alcoved polytope via the transformation $y_i = x_i - x_{i-1}$. The triangulation Γ_P of \mathcal{O}_P thus obtained is known as the *canonical triangulation* of \mathcal{O}_P .

Denote by \mathcal{L}_P the set of linear extensions of P ; see [EC1]. The simplices of Γ_P can be labelled by $w \in \mathcal{L}_P$ and this is compatible with the labelling used throughout this paper. It is known [Sta2] that we have

$$h(\tilde{\Gamma}_P, t) = \sum_{w \in \mathcal{L}_P} t^{\text{des}(w)}.$$

The descents counted by this h -vector, however, disagree with the way we have been counting descents in this paper. More precisely, we have been concerned with the number of descents $\text{des}(w^{-1})$ of the inverse permutations labelling the simplices.

18.3. Weight polytopes for type A_n and alcoved polytopes. A *weight polytope* is the convex hull of the weights which occur in some highest weight representation of a Lie algebra. For example, the vertices of the hypersimplex $\Delta_{k,n}$ are exactly the weights which occur in the k^{th} fundamental representation of \mathfrak{sl}_n .

We will identify the integral weights L of \mathfrak{sl}_n with the integer vectors (a_1, \dots, a_n) satisfying $a_1 + \dots + a_n = l$, for some fixed l . A weight polytope \mathcal{P}_{λ} is specified completely by giving the highest weight $\lambda = (\lambda_1, \dots, \lambda_n)$. We will assume all the coordinates λ_i are non-negative. A weight μ lies inside \mathcal{P}_{λ} if λ dominates μ in the usual sense: $\lambda_1 + \dots + \lambda_i \geq \mu_1 + \dots + \mu_i$ for all i .

Let $\mu, \nu \in L$ be two weights. Define $U(\mu, \nu), V(\mu, \nu) \in L$ by requiring that $U(I_{\mu}, I_{\nu}) = I_{U(\mu, \nu)}$ and $V(I_{\mu}, I_{\nu}) = I_{V(\mu, \nu)}$, in the notation of Section 15. Alternatively, $U(\mu, \nu)$ and $V(\mu, \nu)$ are determined by requiring that $\mu + \nu = U(\mu, \nu) + V(\mu, \nu)$ and $V(\mu, \nu) - U(\mu, \nu) = \sum_i b_i \alpha_i$ for some $b_i \in \{0, 1\}$, where $\alpha_i = e_{i+1} - e_i$ are the simple roots. We call \mathcal{P}_{λ} *sort-closed* if the set of weights $\mathcal{P}_{\lambda} \cap L$ is sort-closed, as in Theorem 15.1.

Proposition 18.1. *A weight polytope \mathcal{P}_{λ} is sort-closed if and only if λ is equal to $aw_i + bw_{i+1}$ for non-negative integers a, b and where $w_k = (1, \dots, 1, 0, \dots, 0)$ (k 1's) are the fundamental weights.*

Proof. The ‘if’ direction is easy as \mathcal{P}_{λ} can be specified by $x_1 + \dots + x_n = i \cdot (a+b) + b$ and the inequalities $0 \leq x_j \leq a$ for each $j \in [1, n]$ and we can use Theorem 15.1. For

the other direction, we may assume the highest weight λ is a n -tuple with highest value a and lowest value 0. Suppose λ is not of the form of the proposition, then there are two more values b, c not equal to a satisfying $a > b \geq c > 0$ so that λ is of the form $(a, \dots, a, b, \dots, b, c, \dots, c, \dots, 0)$.

Explicitly construct a pair of weights $\delta = (a, 0, b, c, \dots)$ and $\mu = (a-1, 1, b+1, c-1, \dots)$ where the tails of the two n -tuples are identical, and δ is just a permutation of the coordinates of λ . Both δ and μ are dominated by λ and hence lie in \mathcal{P}_λ . However, $U(\delta, \mu) = (a, 0, b+1, c-1, \dots)$ does not lie in \mathcal{P}_λ . \square

Sturmfels [Stu, Chapter 14] considered exactly this class of sort-closed weight polytopes. The following corollary follows immediately from Theorem 15.1 and Proposition 18.1.

Corollary 18.2. *A weight polytope in the x -coordinates with highest weight λ is alcoved if and only if λ is of the form $a\omega_i + b\omega_{i+1}$ for a and b non-negative integers. In particular, every weight polytope for A_2 is alcoved.*

18.4. Geometric motivation: degrees of torus orbits. Let $Gr_{k,n}$ denote the *grassmannian manifold* of k -dimensional subspaces in the complex linear space \mathbb{C}^n . Elements of $Gr_{k,n}$ can be represented by $k \times n$ -matrices of maximal rank k modulo left action of GL_k . The $\binom{n}{k}$ maximal minors p_I of such a matrix, where I runs over k -element subsets in $\{1, \dots, n\}$, form projective coordinates on $Gr_{k,n}$, called the *Plücker coordinates*. The map $Gr_{k,n} \rightarrow (p_I)$ gives the *Plücker embedding* of the grassmannian into the projective space $\mathbb{C}\mathbb{P}^{\binom{n}{k}-1}$. Two points $A, B \in Gr_{k,n}$ are in the same *matroid stratum* if $p_I(A) = 0$ is equivalent to $p_I(B) = 0$, for all I . The matroid \mathcal{M}_A of A has as set of bases $\left\{ I \in \binom{[n]}{k} \mid p_I(A) \neq 0 \right\}$.

The complex torus $T = (\mathbb{C} \setminus \{0\})^n$ acts on \mathbb{C}^n by stretching the coordinates

$$(t_1, \dots, t_n) : (x_1, \dots, x_n) \mapsto (t_1 x_1, \dots, t_n x_n).$$

This action lifts to an action of the torus T on the grassmannian $Gr_{k,n}$. This action was studied in [GGMS]. The authors showed that geometry of the closure of a torus orbit $X_A = \overline{T \cdot A}$ depends (only) on the matroid stratum of A . The variety X_A is a toric variety and its associated polytope is exactly the polytope $\mathcal{P}_{\mathcal{M}_A}$ associated to the matroid \mathcal{M}_A from Section 10. Our study of the volume of the polytopes $\mathcal{P}_{\mathcal{M}}$ was motivated by the well known fact (see [Ful]) that

$$\deg(X_A) = \text{Vol}(\mathcal{P}_{\mathcal{M}_A})$$

where $\deg(X_A)$ denotes the degree of X_A as a projective subvariety of $\mathbb{C}\mathbb{P}^{\binom{n}{k}-1}$ and Vol denotes the normalized volume with respect to the lattice generated by the coordinate vectors e_i . Note that by definition only *representable* matroids \mathcal{M} arise as \mathcal{M}_A in this manner. Our Theorem 9.2 gives a combinatorial description of the degree of a torus orbit closure corresponding to a stratum of a sort-closed matroid. In fact Proposition 9.3 (and Proposition 19.2) shows that these torus orbit closures are projectively normal, a fact known for all torus orbit closures; see [Whi, Dab].

It has been conjectured (see [Stu, Conjecture 13.19]) that the ideal of a projective normal toric variety is always generated by quadratic binomials. The toric varieties associated to our alcoved polytopes give more examples of this.

19. APPENDIX: COHERENT TRIANGULATIONS AND GRÖBNER BASES

We give a brief introduction to the relationship between coherent triangulations of integer polytopes and Gröbner bases. See Sturmfels [Stu] for further details.

Let k be a field and $k[x] = k[x_1, \dots, x_n]$ be the polynomial ring in n variables. A total order \prec on \mathbb{N}^n is a *term order* if it satisfies:

- The zero vector is the unique minimal element.
- For any $\underline{a}, \underline{b}, \underline{c} \in \mathbb{N}^n$, such that $\underline{a} \prec \underline{b}$ we have $\underline{a} + \underline{c} \prec \underline{b} + \underline{c}$.

One way to create a term order is by giving a weight vector $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{R}^n$. Then for sufficiently generic weight vectors a term order \prec is given by $\underline{b} \prec \underline{c}$ if and only if $\omega \cdot \underline{b} < \omega \cdot \underline{c}$. In this situation we will say that ω represents \prec .

Given a polynomial $f \in k[x]$ one defines the initial monomial $\text{in}_\prec(f)$ as the monomial $\underline{x}^{\underline{a}}$ with the largest \underline{a} under \prec . For an ideal I of $k[x]$ one defines the initial ideal $\text{in}_\prec(I)$ as the ideal generated by the initial monomials of elements of I . The monomials which do not lie in $\text{in}_\prec(I)$ are called the standard monomials. A finite subset $\mathcal{G} \subset I$ is a *Gröbner basis* for I with respect to \prec if $\text{in}_\prec(\mathcal{G})$ generates $\text{in}_\prec(I)$. The Gröbner basis is called reduced if for two distinct elements $g, g' \in \mathcal{G}$, no term of g' is divisible by $\text{in}_\prec(g)$.

Now let $\mathcal{A} = \{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n\}$ be a finite subset of \mathbb{Z}^d . We define a ring homomorphism $k[x]$ to $k[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$ by

$$x_k \mapsto t^{\underline{a}_k}.$$

The kernel $I_{\mathcal{A}}$ of this map is an ideal known as a *toric ideal*.

We now describe the relationship between Gröbner bases of $I_{\mathcal{A}}$ and coherent triangulations of the convex hull of \mathcal{A} . For any term order \prec , the initial complex $\Delta_\prec(I_{\mathcal{A}})$ of $I_{\mathcal{A}}$ is the simplicial complex defined as follows. A subset $F \subset \{1, 2, \dots, n\}$ is a face of $\Delta_\prec(I_{\mathcal{A}})$ if there is no polynomial $f \in I_{\mathcal{A}}$ such that the support of $\text{in}_\prec(f)$ is F . Thus the *Stanley-Reisner ideal* of $\Delta_\prec(I_{\mathcal{A}})$ is the radical of $\text{in}_\prec(I_{\mathcal{A}})$.

A triangulation of a set $\mathcal{A} \in \mathbb{Z}^d$ (more specifically, its convex hull) is *coherent* if one can find a piecewise-linear convex function ν on \mathbb{R}^d such that the domains of linearity are exactly the simplices of the triangulation. Alternatively, the triangulation is coherent if one can find a ‘height’ vector ω such that the projection of the ‘lower’ faces of the convex hull of $\{(\underline{a}_1, \omega_1), (\underline{a}_2, \omega_2), \dots, (\underline{a}_n, \omega_n)\}$ is exactly the triangulation. We will denote such a triangulation of \mathcal{A} by $\Delta_\omega(\mathcal{A})$. The function ν and the vector ω can be related by setting $\omega_n = \nu(\underline{a}_n)$.

The main result we will need is the following [Stu, Chapter 8]:

Theorem 19.1. *The coherent triangulations of \mathcal{A} are the initial complexes of the toric ideal $I_{\mathcal{A}}$. More precisely, if $\omega \in \mathbb{R}^n$ represents \prec for $I_{\mathcal{A}}$ then $\Delta_\prec(I_{\mathcal{A}}) = \Delta_\omega(\mathcal{A})$.*

In the case that $I_{\mathcal{A}}$ is a homogeneous toric ideal we can say more:

Proposition 19.2. *Let \mathcal{A} be such that $I_{\mathcal{A}}$ is a homogeneous toric ideal. Then the initial ideal $\text{in}_\prec(I_{\mathcal{A}})$ is square-free if and only if the corresponding regular triangulation Δ_\prec of \mathcal{A} is unimodular. Furthermore, let $Y_{\mathcal{A}}$ be the projective toric variety defined by the ideal $I_{\mathcal{A}}$. Then $Y_{\mathcal{A}}$ is projectively normal and the Hilbert polynomial of $Y_{\mathcal{A}}$ equals to the Ehrhart polynomial of the convex hull of \mathcal{A} .*

In this last case, the \prec -standard monomials correspond exactly to the simplices of the triangulation.

REFERENCES

- [Dab] R. DABROWSKI: On normality of the closure of a generic torus orbit in G/P . *Pacific J. Math.* 172 (1996), no. 2, 321-330.
- [Ful] W. FULTON: Introduction to toric varieties, Annals of Mathematics Studies, 131. The William H. Roever Lectures in Geometry. Princeton University Press, Princeton, NJ, 1993.
- [GGMS] I.M. GELFAND, R. GORESKY, R. MACPHERSON, V. SERGANOVA: Combinatorial geometries, convex polyhedra, and Schubert cells. *Adv. in Math.* 63 (1987), no. 3, 301-316.
- [Hum] J. HUMPHREYS: Reflection Groups and Coxeter Groups, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1992.
- [Hib] T. HIBI: Algebraic Combinatorics on Convex Polytopes, Carlsaw Publications, Glebe NSW, Australia, 1992.
- [Kat] M. KATZMAN: The Hilbert series of algebras of Veronese type, preprint.
- [AP2] T. LAM, A. POSTNIKOV: Alcoved Polytopes II, in preparation.
- [LP] C. LENART, A. POSTNIKOV: Affine Weyl groups in K-theory and representation theory, arXiv: [math.RT/0309207](https://arxiv.org/abs/math.RT/0309207).
- [LST] J. DE LOERA, B. STURMFELS, R. THOMAS: Gröbner bases and triangulations of the second hypersimplex. *Combinatorica* 15 (1995), no. 3, 409-424.
- [Sta1] R. STANLEY: Eulerian partitions of a unit hypercube, in Higher Combinatorics (M. Aigner, ed.), Reidel, Dordrecht/Boston, 1977, p. 49.
- [Sta2] R. STANLEY: Two poset polytopes, *Discrete Comput. Geom.* 1 (1986), no. 1, 9-23.
- [EC1] R. STANLEY: Enumerative Combinatorics, Volume 1, Cambridge University Press, 1997.
- [EC2] R. STANLEY: Enumerative Combinatorics, Volume 2, Cambridge University Press, 1999.
- [Stu] B. STURMFELS: Gröbner bases and convex polytopes, University Lecture Series, 8. American Mathematical Society, Providence, RI, 1996.
- [Whi] N. WHITE: The basis monomial ring of a matroid, *Advances in Math.* 24 (1977), no. 3, 292-297.
- [Zie] G. ZIEGLER, Lectures on Polytopes, Graduate Texts in Mathematics 152, Springer-Verlag New York, 1995.

DEPARTMENT OF MATHEMATICS, M.I.T., CAMBRIDGE, MA, 02139

E-mail address: thomasl@math.mit.edu

E-mail address: apost@math.mit.edu