Positive Grassmannian, lectures by A. Postnikov

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1 Lecture 1, 2/8/2012

1.1 Introduction

Fix $0 \leq k \leq n$, a field $\mathbf{F}$ $(\mathbb{C}, \mathbb{R}, \mathbf{F}_q)$, the **Grassmannian** $\mathbf{Gr}(k, n, \mathbf{F})$ be the manifold of k-dimensional linear subspaces in $\mathbf{F}^n$ (it has nice geometry it is projective algebraic variety/smooth,. . .)

Example 1.1. $k = 1$, $\mathbf{Gr}(1, n) = \mathbb{P}^{n-1}$ the projective space:

$$\begin{aligned}\mathbb{P}^{n-1} &= \{(x_1, \ldots, x_n) \neq (0, \ldots, 0)\} \backslash (x_1, \ldots, x_n) \sim (\lambda x_1, \ldots, \lambda x_n) \\ &= \{(x_1 : x_2 : \cdots : x_n)\}.\end{aligned}$$

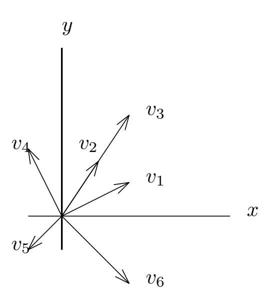
□

Take a $k \times n$ matrix A with rank k then

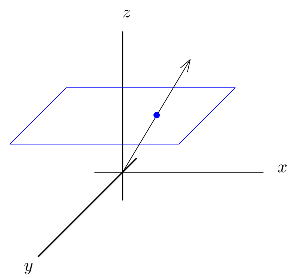
$$\mathbf{Gr}(k, n) = \{k \times n \text{ matrices of rank } k\} \backslash \text{ row operations } = {}_{GL(k)} \backslash \operatorname{Mat}^*(k, n).$$

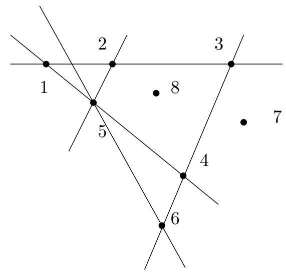
where $\mathbf{GL}(k)$ is the group of $k \times k$ invertible matrices over $\mathbf{F}$ and $\operatorname{Mat}^*(k, n)$ is the set of $k \times n$ matrices over $\mathbf{F}$ with rank k.

The row picture is considering the row space of A (this is the original definition of $\mathbf{Gr}(k, n)$). The column picture is to take the columns of A: $\delta_1, \ldots, \delta_n \in \mathbf{F}^k$. Note that $\dim \mathbf{Gr}(k, n) = k(n-k)$.

Example 1.2. $k = 2$: 

$\in \mathbf{Gr}(2, n)$.

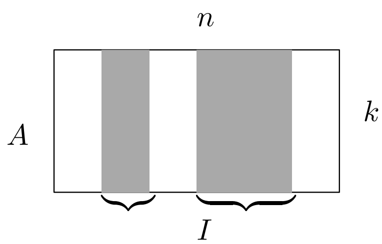
$k = 3$ 

projective picture: 

$\in \mathbf{Gr}(3, n)$. □

1.2 Plücker coordinates

Let $[n] = \{1, 2, \ldots, n\}$ and $\binom{[n]}{k} = \{I \subset [n] \mid \#I = k\}$. For a $k \times n$ matrix A and $I \subset \binom{[n]}{k}$ let $\Delta_I(A) = \det(k \times k \text{ submatrix in column set } I)$ (i.e. $\Delta_I(A)$ is a maximal minor of A). Since A has rank k, at least on $\Delta_I(A) \neq 0$.

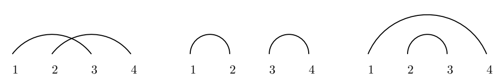


For $B \in \mathbf{GL}(k)$, $\Delta_I(B \cdot A) = \det(B) \cdot \Delta_I(A)$. Thus the $\binom{n}{k}$ minors $\Delta_I(A)$ for A in $\mathbf{Gr}(k, n)$ form projective coordinates on $\mathbf{Gr}(k, n)$.

The **Plücker embedding** is the map $\mathbf{Gr}(k, n) \mapsto \mathbb{P}^{\binom{n}{k}-1}$, $A \mapsto (\Delta_{I_1}(A) : \Delta_{I_2}(A) : \cdots)$.

Example 1.3. For $\mathbf{Gr}(2, 4)$, its dimension is $2 \cdot 4 - 2 \cdot 2 = 4$. $\mathbf{Gr}(2, 4) \to \mathbb{P}^5$, $A \mapsto (\Delta_{12} : \Delta_{13}, \Delta_{14} : \Delta_{23} : \Delta_{24} : \Delta_{34})$. Moreover, the maximal minors satisfy the following relation called a **Plücker relation**

$$\Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}.$$



□

The Grassmannian has the following decomposition: $\mathbf{Gr}(k, n) = \coprod_{\lambda \subset k \times (n-k)} \Omega_\lambda$ where λ is a Young diagram contained in the $k \times (n - k)$ rectangle, and

$$\Omega_I := \{A \in \mathbf{Gr}(k, n) \mid \Delta_I(A) \text{ is the lexicographically minimal nonzero Plücker coordinate}\}.$$

This is the **Schubert decomposition** of $\mathbf{Gr}(k, n)$.

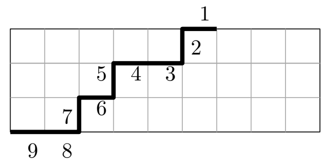
Example 1.4.

$$A = \begin{bmatrix} 0 & 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 2 & 3 \end{bmatrix} \in \Omega_{\{2,4\}} = \Omega_{\yng(2,1)}.$$

□

Identify $I \in \binom{[n]}{k}$ with the Young diagram $\lambda \subset k \times (n - k)$. The set I gives the labels of the vertical steps. Explicitly, $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ is identified with the set $I = \{i_1 < i_2 < \cdots < i_k\}$ where $\lambda_j = n - k + j - i_j$.

Example 1.5. $k = 3, n = 9$, $\lambda \sim I = \{2, 5, 7\}$



$\square$

Schubert calculus is based on the following result.

Theorem 1.6. *$\Omega_\lambda \cong \mathbf{F}^{|\lambda|}$ where $|\lambda|$ is the number of boxes of λ.*

Theorem 1.7. *If $\mathbf{F} = \mathbb{C}$ then $H^*(\mathbf{Gr}(k, n, \mathbb{C}))$ has a linear basis $[\overline{\Omega}_\lambda]$.*

Example 1.8. $\mathbf{Gr}(1, 3) = \{(x_1 : x_2 : x_3)\} = \underbrace{\{(1, x_2, x_3)\}}_{\Omega_{\square\square}} \cup \underbrace{\{(0, 1, x_3)\}}_{\Omega_{\square}} \cup \underbrace{\{(0, 0, 1)\}}_{\Omega_\varnothing}$ $\square$

1.3 Matroid Stratification

[Gelfand-Serganova stratification]

For $\mathcal{M} \subset \binom{[n]}{k}$ a **strata** is $S_\mathcal{M} = \{A \in \mathbf{Gr}(k, n) \mid \Delta_I \neq 0 \Leftrightarrow I \in \mathcal{M}\}$.

There is the following important axiom:

Exchange Axiom: For all $I, J \in \mathcal{M}$ and for all $i \in I$ there exists a $j \in J$ such that $(I \backslash \{i\}) \cup \{j\} \in \mathcal{M}$.

In fact the nonempty sets $\mathcal{M}$ that satisfy the exchange axiom are called **matroids**, the sets $I \in \mathcal{M}$ are called **bases** of $\mathcal{M}$ and k is the **rank** of the $\mathcal{M}$.

Lemma 1.9. *If $\mathcal{M}$ is not a matroid then $S_\mathcal{M} = \varnothing$.*

The best way to prove this Lemma is using the Plücker relations. The converse is not true.

A matroid $\mathcal{M}$ is a **realizable** matroid (over $\mathbf{F}$) if and only if $S_\mathcal{M} \neq \varnothing$. In general it is a hard question to characterize realizable matroids.

2 Lecture 2, 2/10/2012

From last time we saw $\mathbf{Gr}(k, n) = {}_{GL(k)}\backslash \mathrm{Mat}^*(k, n)$, Plücker coordinates $\Delta_I(A)$ for $I \in \binom{[n]}{k}$. We saw two stratifications: the Schubert and matroid stratifications.

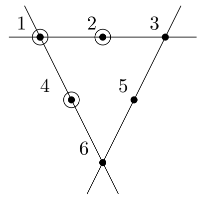
$$\mathbf{Gr}(k, n) = \coprod_\lambda \Omega_\lambda = \coprod_\mathcal{M} S_\mathcal{M},$$

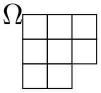
where the cells Ω_λ have a simple structure and the matroid strata $S_\mathcal{M}$ have complicated structure.

2.1 Mnëv's Universality Theorem

The strata $S_{\mathcal{M}}$ can be as complicated as any algebraic variety (even for $k = 3$).

Example 2.1. For $k = 3$ and $n = 6$, consider the point of $\mathbf{Gr}(3, 6)$ given by the projective picture:



It is in $\Omega_{1,2,4} = \Omega$  and $\mathcal{M} = \binom{[6]}{3} \setminus \{\{1, 2, 3\}, \{1, 4, 6\}, \{3, 5, 6\}\}$. $\square$

Any system of algebraic equations can be realized by a configuration like One of the main topics of this course will be to give a third decomposition of $\mathbf{Gr}(k, n)$ called the *Positroid decomposition* that sits between the Schubert decomposition and the matroid decomposition.

2.2 Schubert decomposition

Recall the Schubert decomposition $\mathbf{Gr}(k, n) = \coprod_{\lambda \subset k \times (n-k)} \Omega_\lambda$ where $\Omega_\lambda = \Omega_I$ where $I \in \binom{[n]}{k}$.

Definition 2.2. There are several ways to define a Schubert cell:

1. Δ_I consists of the elements of $\mathbf{Gr}(k, n)$ such that Δ_I is the lexicographic minimum non-zero Plücker coordinate.
2. A point in $\mathbf{Gr}(k, n)$ is in correspondence with a non-degenerate $k \times k$ matrix A modulo row operations. By Gaussian elimination such a matrix A has a canonical form: the *reduced row-echelon form* of the matrix. So Δ_I consists of the elements of $\mathbf{Gr}(k, n)$ with pivot set I.

$\square$

Example 2.3.

$$A = \begin{bmatrix} 0 & \underline{1} & * & * & 0 & * & 0 & 0 & * \\ 0 & 0 & 0 & 0 & \underline{1} & * & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \underline{1} & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \underline{1} & * \end{bmatrix},$$

Let $I = \{2, 5, 7, 8\}$ be the set of the indices of the pivot columns (the pivots are underlined). $\square$

Clearly $\Gamma_I \cong \mathbf{F}^{\#*}$ and also clear that Δ_I, where I is the set of pivots, is the lexicographically minimum nonzero Plücker coordinates.

Definition 2.4. The **Gale order** is a partial order on $\binom{[n]}{k}$ given by $I = \{i_1 < i_2 < \cdots < i_k\}$ and d $J = \{j_1 < j_2 < \cdots < j_k\}$ then $I \preceq J$ if and only if $i_1 \le j_1, i_2 \le j_2, \ldots, i_k \le j_k$. $\square$

Proposition 2.5. *For $A \in \Omega_I$, $\Delta_J(A) = 0$ unless $I \preceq J$.*

Proof. We use the reduced row echelon form. $\square$

Corollary 2.6. *The set $\mathcal{M} := \{J \in \binom{[n]}{k} \mid \Delta_J(A) \neq 0\}$ has a unique minimal element with respect to the Gale order.*

Thus, we get another equivalent definition of a Schubert cell:

3. Δ_I consists of the elements of $\mathbf{Gr}(k,n)$ such that Δ_I is the *Gale order minimum* non-zero Plücker coordinate.

Theorem 2.7 ($\mathbf{F} = \mathbb{C}$ or $\mathbb{R}$)**.** $\overline{\Omega_I} \supset \Omega_J$ *if and only if* $I \preceq J$.

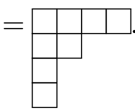
Proof. $\Rightarrow$: For J such that $I \not\preceq J$ then by Proposition 2.5 $\Delta_J(A) = 0$ and so $A \in \Omega_I$. In the closure it still holds that $\Delta_I(A) = 0$ for $A \in \overline{\Omega_I}$. (Looking at the row echelon form, pivots can only jump to the right).

$\Rightarrow$: this is left as an exercise. □

From row echelon form, remove the k pivot columns to obtain a $k \times (n-k)$ matrix.

Example 2.8. Continuing with the matrix from Example 2.3, if we remove the pivot columns we obtain:

$$\begin{bmatrix} 0 & \underline{1} & * & * & 0 & * & 0 & 0 & * \\ 0 & 0 & 0 & 0 & \underline{1} & * & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \underline{1} & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \underline{1} & * \end{bmatrix} \mapsto \begin{bmatrix} 0 & * & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix} \mapsto \text{mirror image of } \lambda =$$



Also note that λ corresponds to $I = \{2, 5, 7, 8\}$ (see Example 1.5). □

Theorem 2.9. $\Omega_\lambda \cong \mathbf{F}^{|\lambda|}$.

2.3 Classical definition of Schubert cells

We think of $\mathbf{Gr}(k,n)$ as the space of k-dimensional subspaces $V \subset \mathbf{F}^n$. Fix the complete flag of subspaces:

$$\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbf{F}^n,$$

where $V_i =< e_n, e_{n-1}, \ldots, e_{n+1-i} >$ and $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the ith elementary vector. Pick a sequence of integers $\mathbf{d} = (d_0, d_1, \ldots, d_n)$.

4. $\Omega_{\mathbf{d}} = \{V \in \mathbf{Gr}(k,n) \mid d_i = \dim(V \cap V_i) \text{ for } i = 0, \ldots, n\}$.

The conditions on $d_i \in \mathbb{Z}_{\geq 0}$ are:

$$\begin{cases} d_0 = 0, \quad d_{i+1} = d_i \text{ or } d_{i+1}, \\ d_n = k. \end{cases}$$

Proof that Definition 4 of the Schubert cell is equivalent to Definition 3. We already have $\Omega\lambda \leftrightarrow \Omega_I$. We show $\Omega_\lambda \leftrightarrow \Omega_{\mathbf{d}}$ and then show $\Omega_I \leftrightarrow \Omega_{\mathbf{d}}$.

Given $\lambda \subset k \times (n-k)$ we obtain $\mathbf{d}$ by doing the following: we start at the SW corner of the Young diagram and follow the border of the diagram until the NE corner. We label the horizontal and vertical steps, where we add 1 if we go north. See Example 2.11(a) for an illustration of this.

Given $I \subset \binom{[n]}{k}$ we obtain $\mathbf{d}$ by $d_i = \#(I \cap \{n-i+1, n-i+2, \ldots, n\})$. Note that this is precisely the dimension of the *rowspace*$(A) \cap V_i$ or equivalently the number of pivots in positions $n, n-1, \ldots, n-i+1$. □

Exercise 2.10. *Check that all these correspondences agree with each other.*

Example 2.11. For $k = 4$ and $n = 9$ let $\lambda = 3211$, from the labels of the Young diagram we get $\mathbf{d} = (0, 1, 2, 2, 3, 3, 3, 4, 4)$ (see Figure 1). And since $d_i = \#(I \cap \{n-i+1, n-i+2, \ldots, n\})$ we get that $I = \{2, 5, 7, 8\}$.

□

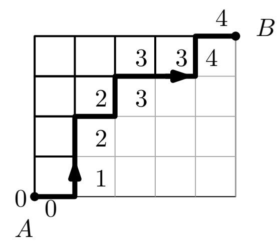


Figure 1: Illustration of correspondence between λ and $\mathbf{d}$.

2.4 Plücker embedding

Recall the definition of the Plücker map $\varphi : \mathbf{Gr}(k,n) \to \mathbb{P}^{N-1}$ where $N = \binom{n}{k}$ by $A \mapsto (\Delta_{I_1} : \Delta_{I_2} : \cdots : \Delta_{I_N})$.

Lemma 2.12. *The map φ is an embedding.*

Proof. We know that at least one Plücker coordinate is nonzero, without loss of generality say $\Delta_{12\ldots k}(A) \neq 0$. Then

$$A = \quad \widetilde{A} = [I_k \mid x_{ij}],$$

where (x_{ij}) is a $k \times (n-k)$ submatrix. From the minors we can reconstruct (x_{ij}). That is let $\Delta_I(\widetilde{A}) := \Delta_I(A)/\Delta_{12\ldots k}(A)$ so that $\Delta_{12\ldots k}(\widetilde{A}) = 1$. Then by the cofactor expansion $x_{ij} = \pm\Delta_{12\cdots i-1\,k+j\,i+1\,\ldots\,k}(\widetilde{A})$. Thus a point in the image of φ determines the matrix A in $\mathbf{Gr}(n,k)$. $\square$

3 Lecture 3, 2/15/2012

3.1 The Grassmannian $\mathbf{Gr}(n,k)$ over finite fields $\mathbf{F}_q$

Say $F = \mathbf{F}_q$ where $q = p^r$ is a power of a prime p. What can we say in this case about $\mathbf{Gr}(k,n,\mathbf{F}_q)$?

The first way is to view it as ${}_{GL(k,\mathbf{F}_q)}\backslash \operatorname{Mat}^*(k,n,\mathbf{F}_q)$. Recall that:

$$\begin{aligned}\#\operatorname{Mat}^*(k,n,\mathbf{F}_q) &= (q^n-1)(q^n-q)(q^n-q^2)\cdots(q^n-q^{n-k+1}),\\ \#GL(k,\mathbf{F}_q) &= (q^k-1)(q^k-q)\cdots(q^k-q^{k-1}).\end{aligned}$$

Then

$$\#\mathbf{Gr}(n,k,\mathbf{F}_q) = \frac{1-q^n}{1-q^k}\cdot\frac{1-q^{n-1}}{1-q^{k-1}}\cdots\frac{1-q^{n-k+1}}{1-q}.$$

We can write this more compactly using q-numbers: $[n]_q = 1+q+q^2+\cdots+q^{n-1} = \frac{1-q^n}{1-q}$, $[n]_q! = [1]_q[2]_q\cdots[n]_q$ and $\begin{bmatrix} n\\ k\end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$. Then

Theorem 3.1. $\#\mathbf{Gr}(n,k,\mathbf{F}_q) = \begin{bmatrix} n\\ k\end{bmatrix}_q$.

The second way is to use the Schubert decomposition $\mathbf{Gr}(k,n,\mathbf{F}_q) = \coprod_{\lambda\subseteq k\times(n-k)}\Omega_\lambda$ which implies

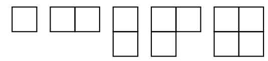
$$\#\mathbf{Gr}(k,n,\mathbf{F}_q) = \sum_{\lambda\subseteq k\times(n-k)} q^{|\lambda|}.$$

Thus

Corollary 3.2. $\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\lambda \subseteq k \times (n-k)} q^{|\lambda|}$.

Example 3.3.

$$\begin{aligned}\mathbf{Gr}(2,4,\mathbf{F}_q) &= \frac{(1-q^4)(1-q^3)}{(1-q^2)(1-q)} \\ &= 1 + q + 2q^2 + q^3 + q^4\end{aligned}$$

$\varnothing$ 

□

Problem 3.4. *Given a diagram $D \subseteq k \times (n-k)$, fix complement $\overline{D}$ to be 0. One obtains $X_D \subseteq$ $\mathbf{Gr}(k,n,\mathbf{F}_q)$. For e.g. if D is a skew Young diagram. Find $\#X_D/\mathbf{F}_q$. Note that Ricky Liu studied the combinatorics of related problems but not the # of points in $\mathbf{F}_q$.*

3.2 More on the Plücker embedding

Recall, the Plücker embedding $\varphi : \mathbf{Gr}(k,n) \mapsto \mathbb{P}^{\binom{n}{k}-1}$, $A \mapsto (\Delta_{I_1}(A) : \Delta_{I_2}(A) : \cdots)$. The **signed** Plücker coordinates are

$$\Delta_{i_1,i_2,\ldots,i_k} = \pm \Delta_{\{i_1,i_2,\ldots,i_k\}},$$

where the sign is positive if $i_1 < i_2 < \ldots < i_k$ and the sign changes whenever we switch two indices. Also $\Delta_{i-1,\ldots,i_k} = 0$ if the indices repeat. In terms of these signed coordinates the Plücker relations are: for any $i_1,\ldots,i_k,j_1,\ldots,j_k \in [n]$ and $r = 1,\ldots,k$:

$$\Delta_{i_1,\ldots,i_k,j_1,\ldots,j_k} = \sum \Delta_{i'_1,\ldots,i'_k} \Delta_{j'_1,\ldots,j'_k}, \tag{3.5}$$

where we sum over all indices $i_1,\ldots,i_k$ and $j'_1,\ldots,j'_k$ obtained from $i_1,\ldots,i_k$ and $j_1,\ldots,j_k$ by switching $i_{s_1}, i_{s_2}, \ldots, i_{s_r}$ $(s_1 < s_2 < \ldots < s_r)$ with $j_1, j_2, \ldots, j_r$.

Example 3.6. For $n = 4, k = 3$ and $r = 1$ we have $(\Delta_{32} = -\Delta_{23})$

$$\Delta_{12} \cdot \Delta_{34} = \Delta_{32} \cdot \Delta_{14} + \Delta_{13} \cdot \Delta_{24}.$$

□

Theorem 3.7 (Sylvester's Lemma)**.**

1. *The image of $\mathbf{Gr}(k,n)$ in $\mathbb{P}^{\binom{n}{k}-1}$ is the locus of common zeros of the ideal I_{kn} generated by the Plücker relations* (3.5)[1] *in the polynomial ring $\mathbb{C}[\Delta_I]$ (where we treat Δ_I as formal variables).*
2. *I_{kn} is the ideal in $\mathbb{C}[\Delta_I]$ of all polynomials vanishing on the image of $\mathbf{Gr}(k,n)$.*
3. *I_{kn} is a prime ideal.*

[$\star\star\star$ check Fulton's book for issue with r? $\star\star\star$]

We give the proof of the first part of this result. The full proof can be found in Fulton's book.
1. k-vectors $d_1,\ldots,d_k,w_1,\ldots,w_k$. Let $|v_1 \ldots v_k| := \det(v_1,\ldots,v_k)$ then

$$|v_1 \ldots v_k| \cdot |w_1 \ldots w_k| = \sum |v'_1 \ldots v'_k| \cdot |w'_1 \ldots w'_k|$$

[1] To generate the ideal one only needs the relations (3.5) for $r = 1$.

where the sum in the right hand side is obtained by switching r vectors from $v_1, \ldots, v_k$ with r vectors from $w_1, \ldots, w_k$. Let f be the difference between the left-hand side and the right-hand side (we want to show that $f = 0$). Note that (i) f is a multilinear function of $v_1, \ldots, v_k, w_k$ and (ii) f is an alternating function of $v_1, \ldots, v_k, w_k$.
Claim: If $v_i = v_{i+1}$ or $v_k = w_k$ then $f = 0$.

First we do the case $v_k = w_k$ where $r < k$ by induction on k.

$$f = |v_1 \ldots v_k| \cdot |w_1 \ldots v_k| - \sum |v_1' \ldots v_k| \cdot |w_1' \ldots v_k|$$

Assume that $v_k = [0 \ldots 01]^T = e_n$ and expand the determinants in the expression above with respect to the last column. We obtain the equation for f involving $k-1$ vectors. By induction we get $f = 0$.

Second, if $v_i = v_{i+1}$ the cancellation is easy to see as the following example shows.

Example 3.8. For $k = 3$ and $r = 1$ we get

$$\begin{aligned} f &= |v_1 v_1 v_3| \cdot |w_1 w_2 w_3| - (|w_1 v_1 v_3| \cdot |v_1 w_2 w_3| + |v_1 w_1 w_3| \cdot |v_1 w_2 w_3|) \\ &= 0 - (|w_1 v_1 v_3| \cdot |v_1 w_2 w_3| - |w_1 v_1 w_3| \cdot |v_1 w_2 w_3|) = 0. \end{aligned}$$

□

Part 1. We want to show that the image of $\varphi\mathbf{Gr}(k,n)$ in $\mathbb{P}^{\binom{n}{k}-1}$ is the zero locus of I_{kn}. Let $\{\Delta_I\}_{I \in \binom{[n]}{k}}$ be any point in $\mathbb{P}^{\binom{n}{k}-1}$ satisfying the Plücker relations. We need to find a $k \times n$ matrix A such that $\Delta_I = \Delta_I(A)$.

Suppose $\Delta_{12\ldots k} \neq 0$. We can rescale the coordinates such that $\Delta_{12\ldots k} = 1$. Let A be the matrix $[I_k \mid x_{ij}]$ where $x_{ij} = \Delta_{1\,2 \ldots i-1\, j+k\, i+1 \ldots k}$. Let $\widetilde{\Delta}_I = \Delta_I(A)$. We want tot show that $\widetilde{\Delta}_I = \Delta_I$ for all $I \in \binom{[n]}{k}$.

We know the following:

1. $\Delta_{1\,2 \ldots k} = \widetilde{\Delta}_{1\,2\,\ldots\,k} = 1$,
2. For all I such that $|I \cap \{1, 2, \ldots, k\}| = k-1$ we have $\widetilde{\Delta}_I = x_{ij}$ for some j but by construction $x_{ij} = \Delta_I$,
3. Both $\widetilde{\Delta}_I$ and Δ_I satisfy the Plücker relations.

We claim that form the three observations above it follows that $\Delta_I = \widetilde{\Delta}_I$ for all $I \in \binom{[n]}{k}$. We show this for $r = 1$ by induction on $p = |I \cap \{1, 2, \ldots, k\}|$. The base case is $p = k, k-1$.

We use the Plücker relations to expand $\Delta_{1\,2\,\ldots\,k} \cdot \Delta_{i_1\,\ldots\,i_k}$:

$$\begin{aligned} \Delta_{1\,2\,\ldots\,k} \cdot \Delta_{i_1\,\ldots\,i_k} &= \sum_{s=1}^{k} \Delta_{1\,2\,\ldots\,s-1\,i\,s+1\,\ldots k} \cdot \Delta_{s\,i_2\,\ldots\,i_k}, \\ &= \sum \widetilde{\Delta}_{1\,2\,\ldots\,s-1\,i\,s+1\,\ldots k} \cdot \widetilde{\Delta}_{s\,i_2\,\ldots\,i_k}, \\ &= \widetilde{\Delta}_{1\,2\,\ldots\,k} \cdot \widetilde{\Delta}_{i_1\,\ldots\,i_k}, \end{aligned}$$

where $\Delta_{s\,i_2\,\ldots\,i_k} = \widetilde{\Delta}_{s\,i_2\,\ldots\,i_k}$ y induction on p.

At the end since $\Delta_{1\,2\,\ldots\,k} = \widetilde{\Delta}_{1\,2\,\ldots\,k}$ we obtain $\Delta_{i_1\,\ldots\,i_k} = \widetilde{\Delta}_{i_1\,\ldots\,i_k}$ as desired. □

3.3 Matroids

Recall the notion of matroid $\mathcal{M} \subset \binom{[n]}{k}$. The elements of the matroid satisfy the **Exchange axiom**: For all $I, J \in \mathcal{M}$ and for all $i \in I$ there exists a $j \in J$ such that $(I \backslash \{i\}) \cup \{j\} \in \mathcal{M}$.

Theorem 3.9. *Pick $A \in \mathbf{Gr}(k,n)$ and let $\mathcal{M} = \{I \in \binom{[n]}{k} \mid \Delta_I(A) \neq 0\}$. Then $\mathcal{M}$ is a matroid.*

Proof. Two sets I, J and in $\mathcal{M}$ if and only if $\Delta_J \cdot \Delta_I \neq 0$. But by the Plücker relations:

$$\Delta_J \cdot \Delta_I = \sum_{j \in J} \pm \Delta_{(J\backslash j)\cup i} \cdot \Delta_{(I\backslash i)\cup j}.$$

Thus there exists a $j \in J$ such that $\Delta_{(J\backslash j)\cup i} \cdot \Delta_{(I\backslash i)\cup j} \neq 0$ which implies the stronger condition that both $(J\backslash j) \cup i$ and $(I\backslash i) \cup j$ are in $\mathcal{M}$. □

We call the stronger condition implied by the proof the **strong exchange axiom**.

Exercise 3.10. *Is the strong exchange axiom equivalent to the exchange axiom?*

4 Lecture 4, 2/17/2012

Last time we saw the bf Stronger Exchange axiom: for all $i_1, \ldots, i_r \in I$ there exists $j_1, \ldots, j_r \in J$ such that $(I\backslash\{i_1, \ldots, i_r\}) \cup \{j_1, \ldots, j_r\} \in \mathcal{M}, (J\backslash\{j_1, \ldots, j_r\}) \cup \{i_1, \ldots, i_r\} \in \mathcal{M}$.

Example 4.1 (non-realizable matroid)**.** The **Fano plane** in $\binom{[7]}{3}$ which is illustrated in Figure 2.

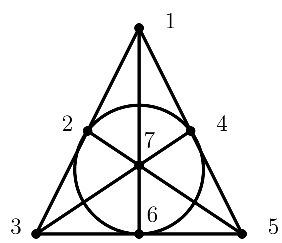


Figure 2: The Fano plane.

□

Exercise 4.2. *Check that the Fano matroid is not realizable over $\mathbb{R}$.*

4.1 Two definitions of matroids

Recall the Gale order: $\{i_1, \ldots, i_k\} \preceq \{j_1, \ldots, j_k\}$ if and only if $i_1 \leq j_1, \ldots, i_k \leq j_k$. A generalization of this order is obtained by picking a permutation $w = w_1 \cdots w_n$ in $\mathfrak{S}_n$, order the set $[n]$ by $w_1 < w_2 < \ldots < w_n$ and define a **permuted Gale order** $\preceq_w$ accordingly.

Definition 4.3. Let $\mathcal{M} \subseteq \binom{[n]}{k}$.

1. **Exchange Axiom:** for all $I, J \in \mathcal{M}$ and for all $i \in I$ there exists a $j \in J$ such that $(I\backslash\{i\}) \cup \{j\} \in \mathcal{M}$.
2. $\mathcal{M}$ is a matroid if for all w in $\mathfrak{S}_n$, $\mathcal{M}$ has a unique minimal element in $\subseteq_w$ (permuted Gale order $\subseteq_w$).

□

The second definition above is connected to the Grassmannian since in Lecture 2 we had seen that if we fix $A \in \mathbf{Gr}(n,k)$ then $\mathcal{M} := \{I \in \binom{[n]}{k} \mid \Delta_I(A) \neq 0\}$ has a unique minimal element with respect to the Gale order.

Another related result from Lecture 2 was the Schubert and matroid stratification of $\mathbf{Gr}(k,n)$:

$$\mathbf{Gr}(k,n) = \coprod_{\lambda\subseteq k\times(n-k)} \Omega_\lambda = \coprod_{\mathcal{M}} S_{\mathcal{M}},$$

where Ω_λ depends on an ordered basis but $S_{\mathcal{M}}$ depends on an unordered basis. Thus we really have $n!$-Schubert decompositions: for $w \in \mathfrak{S}_n$ we have $\mathbf{Gr}(k,n) = \coprod_{\lambda\subseteq k\times(n-k)} \Omega_\lambda^w$ where Ω_λ^w has the same definition as Γ_λ but with respect to the order $e_{w_1} < e_{w_2} < \ldots < e_{w_n}$. (We can also think of Ω_λ^w as $w^{-1}(\Gamma_\lambda)$ where $\mathfrak{S}_n$ acts on $\mathbb{C}^n$ by permuting coordinates according to w.

Theorem 4.4 (Gelfand-Goresky-MacPherson-Sergenova)**.** *Matroid stratification is the common refinement of $n!$ permuted Schubert decompositions.*

Proof. [$\star\star\star$ pending $\star\star\star$] □

Example 4.5. For $k=3$ and $n=5$ consider the matroid $\mathcal{M}$ from Figure 3. The minimal basis with respect to the Gale order is $\{1,2,4\}$. For $w = 34512$ the minimal basis with respect to the w permuted Gale order is $\{3,4,1\}$. So $S_{\mathcal{M}} \in \Omega_{1,2,4} \cap \Omega^{34512}_{3,4,1} \cap \cdots$.

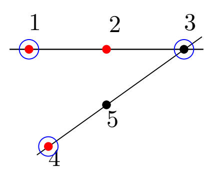


Figure 3: Matroid $\mathcal{M}$. The minimal basis with respect to the Gale order is $\{1,2,4\}$. For $w = 34512$ the minimal basis with respect to the w permuted Gale order is $\{3,4,1\}$.

□

4.2 Matroid polytopes and a third definition of matroids

We denote by $e_1,\ldots,e_n$ the coordinate vectors in $\mathbb{R}^n$. Given $I = \{i_1,\ldots,i_k\} \in \binom{[n]}{k}$ we denote by e_I the vector $e_{i_1} + e_{i_2} + \cdots + e_{i_k}$. Then for any $\mathcal{M} \subseteq \binom{[n]}{k}$ we obtain the following convex polytope

$$P_{\mathcal{M}} = \mathrm{conv}(e_I \mid I \in \mathcal{M}) \subset \mathbb{R}^n,$$

where conv means the convex hull. Note that $P_{\mathcal{M}} \subset \{x_1 + x_2 + \cdots + x_n = k\}$ so $\dim P_{\mathcal{M}} \leq n-1$.

Definition 4.6 (Gelfand-Serganova-M-S)**.** $P_{\mathcal{M}}$ is a **matroid polytope** if every edge of $P_{\mathcal{M}}$ is parallel to $e_j - e_i$, i.e. edges are of the form $[e_I, e_J]$ where $J = (I\backslash\{i\}) \cup \{j\}$. □

This gives us a third definition of a matroid.

Definition 4.7. 3. A matroid is a subset $\mathcal{M} \subseteq \binom{[n]}{k}$ such that $P_{\mathcal{M}}$ is a matroid polytope.

□

Theorem 4.8. *The three definitions of a matroid: 1. by the exchange axiom, 2. by the permuted Gale order, and 3. by matroid polytopes are equivalent.*

Exercise 4.9. *Prove this theorem.*

Example 4.10. If $\mathcal{M} = \binom{[n]}{k}$ is the uniform matroid then $P_{\mathcal{M}} = \text{conv}(e_I \mid I \in \binom{[n]}{k})$. This polytope is called the **hypersimplex** Δ_{kn}. The hypersimplex has the following property: All e_I for $I \in \binom{[n]}{k}$ are vertices of Δ_{kn} and these are all the lattice points of Δ_{kn}. □

Question 4.11. *What are the vertices of any $P_{\mathcal{M}}$?*

The answer is e_I where $I \in \mathcal{M}$ (basis of the matroid).

Example 4.12. For $k = 1$, $\Delta_{1n} = \text{conv}(e_1, e_2, \ldots, e_n)$ is the usual $(n-1)$-dimensional simplex. See Figure 4 for an illustration of Δ_{13}.

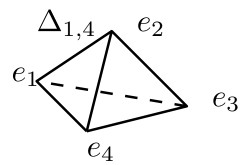


Figure 4: An illustration of Δ_{13}.

□

Example 4.13. For $k = 2$ and $n = 4$ the hypersimplex Δ_{24} has six vertices $e_{12} = (1,1,0,0), e_{13} = (1,0,1,0), ldots$ (see Figure 5(a)). The matroid polytopes are subpolytopes of Δ_{24} without new edges. In Figure 5(b) there are three subpolytopes associated with $\mathcal{M}_1 = \{12, 13, 14, 23, 34\}, \mathcal{M}_2 = \{12, 14, 23\}$ and $\mathcal{M}_3 = \{12, 23, 34, 13\}$ respectively. $\mathcal{M}_1$ and $\mathcal{M}_2$ are matroids but $\mathcal{M}_3$ is not (take $I = \{3, 4\}$, $J = \{1, 2\}$ and $i = 3$ then the Exchange Axiom fails).

□

Exercise 4.14. *Consider* $\mathbf{Gr}(n, 2n, \mathbf{F}_q)$. [⋆⋆⋆ *pending* ⋆⋆⋆]

5 Lecture 5, 2/20/2012

Last time we talked about matroid polytopes.

If $\mathcal{M} = \binom{[n]}{k}$ (uniform matroid) then the matroid polytope is the hypersimplex Δ_{kn}.

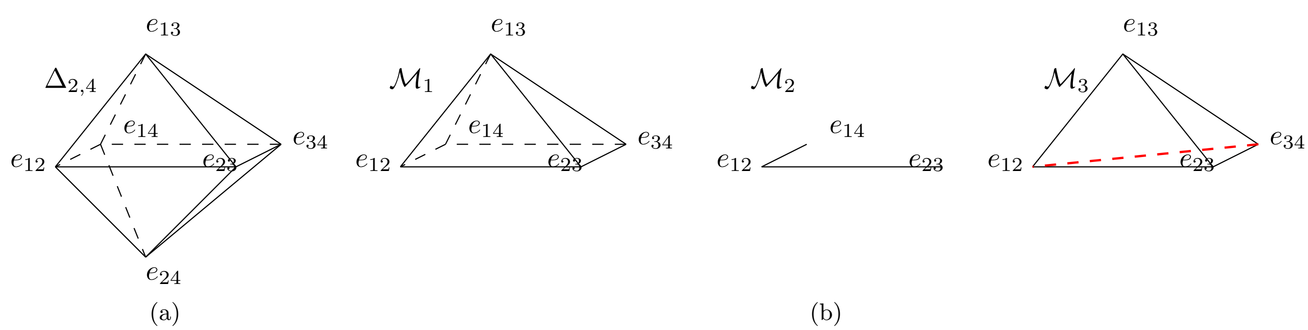


Figure 5: (a) the hypersimplex Δ_{24}, (b) three subpolytopes of Δ_{24} corresponding to $\mathcal{M}_1 = \{12, 13, 14, 23, 34\}, \mathcal{M}_2 = \{12, 14, 23\}$ and $\mathcal{M}_3 = \{12, 23, 34, 13\}$ the first two are matroid polytopes and the third one is not (it has a bad edge $[e_{12}, e_{34}]$, or in terms of $\mathcal{M}_3$ take $I = \{3, 4\}$, $J = \{1, 2\}$ and $i = 3$ then the Exchange Axiom fails).

5.1 Moment Map

First we will talk about the moment map which is important in symplectic and toric geometry.

Let $T = (\mathbb{C}^*)^n$, it acts on $\mathbb{C}^n$ by $(t_1, \dots, t_n) \cdot (x_1, \dots, x_n) \mapsto (t_1x_1, t_2x_2, \dots, t_nx_n)$. This induces a T-action on $\mathbf{Gr}(k, n, \mathbb{C})$.

Recall $\mathbf{Gr}(k, n) = {}_{GL_k}\backslash Mat(k, n)$. We claim that T acts by right multiplication by $\text{diag}(t_1, \dots, t_n)$ (we rescale column i by t_i). In terms of Plücker coordinates: $(t_1, \dots, t_n) \cdot \{\Delta_I\} \mapsto \{\prod_{i \in I} t_i \Delta_I\}$.

We define the **moment map**: $\mu : \mathbf{Gr}(k, n, \mathbb{C}) \to \mathbb{R}^n, A \mapsto (y_1, \dots, y_n)$ where $y_i = \frac{\sum_{I \ni i} |\Delta_I|^2}{\sum_I |\Delta_I|^2}$.

Example 5.1. $k = 2$ and $n = 4$, $\mathbf{Gr}(2, 4) \to \mathbb{R}^4$, $A \mapsto (y_1, y_2, y_3, y_4)$ where for instance $y_1 = \dfrac{|\Delta_{12}|^2 + |\Delta_{13}|^2 + |\Delta_{14}|^2}{\sum_I |\Delta_I|^2}$. □

Theorem 5.2 (Atiyah-Guillemin-Sternberg)**.** *(1) Image of μ is a convex polytope.*

(2) Moreover, pick any point $A \in S_{\mathcal{M}} \subset \mathbf{Gr}(k, n)$, then $\mu(T \cdot A)$ is a convex polytope (A is fixed, $T \cdot A$ is a set of matrices).

Exercise proof this case of convexity theorem.
Claim:

(1) $\mu(\mathbf{Gr}(k, n)) = \binom{[n]}{k}$.

(2) $\mu(\overline{T \cdot A}) = P_{\mathcal{M}}$ is a matroid polytope.

Idea proof of Claim.
(1) Clearly $0 \le y_i \le 1$, also $y_1 + \dots + y_n = k$. This means that $\mu(\mathbf{Gr}(k, n)) \subseteq \Delta_{kn}$. (Recall that $\Delta_{kn} = \{(y_1, \dots, y_n) \mid 0 \le y_i \le 1, y_1 + \dots + y_n = k\}$.)

Pick A_I to be the 0-1 matrix whose $k \times k$ submatrix indexed by I is the identity matrix, and the other columns are 0. This is a fixed point of T in $\mathbf{Gr}(k, n)$. Actually this is the form of all the fixed points. Thus there are $\binom{n}{k}$ such fixed points (one for each set $I \in \binom{[n]}{k}$).

Now $\Delta_J(A_I) = \delta_{I,J}$. Then $\mu(A_I) = e_I = \sum_{i \in I} e_i$ and this is a vertex of Δ_{kn}. From the convexity theorem, μ is a convex polytope, this together with the fact that $\mu(A_I) = e_I$ forces $\mu(\mathbf{Gr}(k, n))$ to be Δ_{kn}. □

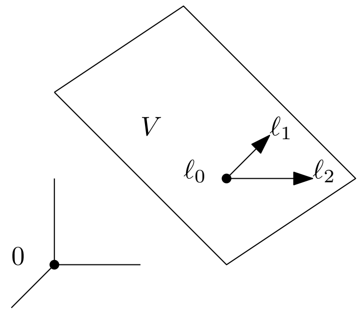
Remark 5.3. To prove the convexity result we have to show that if

$$A \overset{\text{Plücker}}{\to} \{\Delta_I\} \overset{\mu}{\to} y = (y_1, \dots, y_n),$$
$$A \cdot \text{diag}(t_1, \dots, t_n) \overset{\text{Plücker}}{\to} \{\prod_{i \in I} t_i \Delta_I\} \overset{\text{Plücker}}{\to} y' = (y'_1, \dots, y'_n),$$

then y and y' are connected by a line where every point in the line corresponds to an image of μ. □

5.2 Normalized Volumes

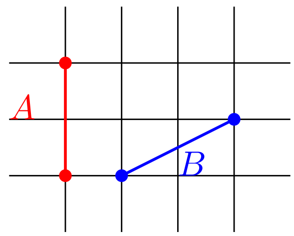
$P \subset \mathbb{R}^n$, polytope with integer vertices in $\mathbb{Z}^n$. Let $d = \dim(P)$ and let V be a d-dimensional affine subspace containing P (it may or may not contain the origin). Let $L = V \cap \mathbb{Z}^n = \ell_0 + < \ell_1, \dots, \ell_d >_{\mathbb{Z}}$, where $< \ell_1, \dots, \ell_d >_{\mathbb{Z}}$ is the set of all linear $\mathbb{Z}$-combinations of $\{\ell_1, \dots, \ell_d\}$.



Define Vol(·): volume form of V such that $\mathrm{Vol}(\ell_0 + \Pi(\ell_1, \ldots, \ell_d)) = 1$ where $\Pi(\ell_1, \ldots, \ell_d)$ is the parallelepiped spanned by $\{\ell_1, \ldots, \ell_d\}$.

Also we define **normalized volume** $\widetilde{\mathrm{Vol}} := d!\,\mathrm{Vol}$. Claim $\widetilde{\mathrm{Vol}} P \in \mathbb{Z}$ and $\widetilde{\mathrm{Vol}}$ of the standard coordinate simplex is 1.

Example 5.4. For the lines A and B, $\widetilde{\mathrm{Vol}}(A) = 2$ but $\widetilde{\mathrm{Vol}}(B) = 1$.



□

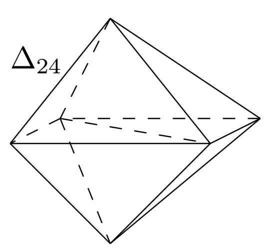
The following result justifies why normalized volumes are interesting.

Theorem 5.5. *The degree of a torus orbit $T \cdot A$ is the normalized volume of $\mu(\overline{T \cdot A}) = P_{\mathcal{M}}$.*

Example 5.6. For $\mathbf{Gr}(2, 4)$:

$$\alpha_1 t_1 t_2 + \alpha_2 t_1 t_3 + \cdots + \alpha_6 t_3 t_4 = 0$$
$$\beta_1 t_1 t_2 + \beta_2 t_1 t_3 + \cdots + \beta_6 t_3 t_4 = 0$$
$$\gamma_1 t_1 t_2 + \gamma_2 t_1 t_3 + \cdots + \gamma_6 t_3 t_4 = 0.$$

The number of solutions of this system is $\widetilde{\mathrm{Vol}}(\Delta_{24})$. We can calculate this volume by showing that we can divide Δ_{24} into four simplices of normalized volume 1. Thus $\widetilde{\mathrm{Vol}}(\Delta_{24}) = 4$. □



This motivates calculating the normalized volume of the hypersimplex. In general the normalized volume of Δ_{kn} is given by the **Eulerian number** $A_{k-1,n-1}$, where $A_{k,n} = \#\{w \in S_n \mid \mathrm{des}(w) = k\}$ where $\mathrm{des}(w)$ is the number of descents of w ($\#\{i \mid w_i > w_{i+1}\}$).

Theorem 5.7. $\widetilde{\mathrm{Vol}}(\Delta_{kn}) = A_{k-1,n-1}$.

Example 5.8.

permutations	$A_{k-1,n-1}$	$\widetilde{\mathrm{Vol}}(\Delta_{kn})$
123	$A_{0,3} = 1$	$\widetilde{\mathrm{Vol}}(\Delta_{14}) = 1$
213, 132 312, 231	$A_{1,3} = 4$	$\widetilde{\mathrm{Vol}}(\Delta_{24}) = 4$
321	$A_{23} = 1$	$\widetilde{\mathrm{Vol}}(\Delta_{34}) = 1$

□

Euler knew the following:

$$\begin{aligned} 1 + x + x^2 + x^3 + \cdots &= \frac{1}{1-x} \\ x + 2x^2 + 3x^3 + 4x^4 + \cdots &= \frac{x}{(1-x)^2} \\ 1^2x + 2^2x^2 + 3^2x^3 + 4^2x^4 + \cdots &= \frac{x+x^2}{(1-x)^3} \\ 1^3x + 2^3x^2 + 3^3x^3 + \cdots &= \frac{x+4x^2+x^3}{(1-x)^4}. \end{aligned}$$

And in general:

Proposition 5.9.

$$\sum_{r=1}^{\infty} r^n x^r = \frac{\sum_{k=0}^{n} A_{k,n} x^{k+1}}{(1-x)^{n+1}},$$

or equivalently, $A_{k-1,n-1} = [x^k](1-x)^n \sum_{r\geq 1} r^{n-1}x^r$.

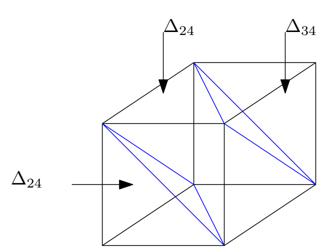
We think of the hypersimplex in two ways: as a section of the n-hypercube:

$$\Delta_{kn} = [0,1]^n \cap \{x_1 + \cdots + x_n = k\},$$

or equivalently as a slice of the $(n-1)$-hypercube:

$$\Delta_{kn} = [0,1]^{n-1} \cap \{k - 1 \leq x_1 + \cdots + x_{n-1} \leq k\}.$$

Example 5.10. We divide $[0,1]^3$ into Δ_{14}, Δ_{24} and Δ_{34}.



$\widetilde{\mathrm{Vol}} A = 2$ but $\widetilde{\mathrm{Vol}}(B) = 1$.

□

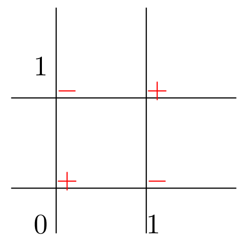
Proof. First let us take the first interpretation.

Take the positive octant $(\mathbb{R}_{\geq 0})^n$. Define $k\Delta = (\mathbb{R}_{\geq 0})^n \cap \{x_1 + \cdots + x_n = k\}$, this is a k-dilated $(n-1)$-simplex. So $\widetilde{\mathrm{Vol}}(k\Delta) = k^{n-1}$.

By inclusion-exclusion we can decompose the n-hypercube as

$$[0,1]^n = (\mathbb{R}_{>0})^n - \sum_i (e_i + (\mathbb{R}_{\geq 0})^n) + \sum_{i<j} (e_i + e_j + (\mathbb{R}_{\geq 0})^n) - \cdots$$

This is really an identity for characteristic functions of subsets (modulo set of measure 0)



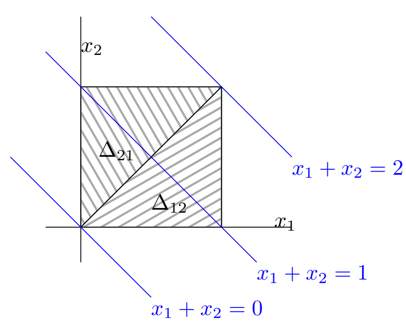
Then the volume of Δ_{kn} (using the first interpretation)

$$\begin{aligned}\widetilde{\mathrm{Vol}}(\Delta_{kn}) &= \widetilde{\mathrm{Vol}}(k\Delta) - \sum_i \widetilde{\mathrm{Vol}}((k-1)\Delta) + \sum_{i<j} \widetilde{\mathrm{Vol}}((k-2)\Delta) - \cdots \\ &= k^{n-1} - n(k-1)^{n-1} + \binom{n}{2}(k-2)^{n-1} - \cdots + (-1)^k \binom{n}{k-1}.\end{aligned}$$

From the expression $A_{k-1,n-1} = [x^k](1-x)^n \sum_{r \geq 1} r^{n-1} x^r$ from Proposition 5.9 one can show this expression above gives $A_{k-1,n-1}$. $\square$

Stanley's proof. We use the interpretation of $\Delta_{kn} = [0,1]^{n-1} \cap \{k-1 \leq x_1 + \cdots + x_{n-1} \leq k\}$. For $w \in S_{n-1}$, there is a simple triangulation of $[0,1]^{n-1}$ into $\Delta_w = \{0 < x_{w_1} < x_{w_2} < \ldots < x_{w_n} < 1\}$ of $(n-1)!$ simplices.

This is not exactly the triangulation we need, it is not compatible with the slicing of the hypercube:



Instead define $S : [0,1]^{n-1} \to [0,1]^{n-1}$ to be the following piecewise-linear volume preserving map. $(x_1, \ldots, x_{n-1}) \mapsto (y_1, \ldots, y_{n-1})$ where $x_i = \{y_1 + \cdots + y_i\} = y_1 + \cdots + y_i - \lfloor y_1 + \cdots + y_i \rfloor$ (where $\{x\}$ is the fractional part of x). The forward map is:

$$y_i = \begin{cases} x_i - x_{i-1} \text{ if } x_i \geq x_{i-1} \\ x_i - x_{i-1} + 1 \text{ if } x_i < x_{i-1}, \end{cases}$$

where we assume $x_0 = 0$. The maps S is piecewise linear map and volume preserving.

We had a trivial triangulation $\Delta_{w^{-1}} := \{0 < x_{w^{-1}(1)} < \cdots < x_{w^{-1}(n-1)} < 1\}$. We get $(n-1)!$ simplices. Then $x_{i-1} < x_i \Leftrightarrow w_{i-1} > w_i \Leftrightarrow (i-1)$ is a descent of $\widetilde{w} = 0w_1w_2\dots w_{n-1}$. Thus if $\mathrm{Des}(\widetilde{w})$ is the st of descents of $\widetilde{w}$ then

$$y_i = \begin{cases} x_i - x_{i-1} & \text{if } i-1 \notin \mathrm{Des}(\widetilde{w}), \\ x_i - x_{i-1} + 1 & \text{if } i-1 \in \mathrm{Des}(\widetilde{w}) \end{cases}$$

Then $y_1 + y_2 + \cdots y_{n-1} = x_{n-1} - x_0 + \mathrm{des}(w)$, and so $\mathrm{des}(w) \le y_1 + y_2 + \cdots + y_{n-1} \le \mathrm{des}(w) + 1$.

So $S(\Delta_{w^{-1}})$ is in the kth slice of the $(n-1)$-cube where $k = \mathrm{des}(w) + 1$. And $S(\Delta_{w^{-1}})$ is a triangulation of the $(n-1)$-cube with exactly $A_{k-1,n-1}$ simplices in the kth slice. □

6 Lecture 6, 2/24/2012

We digress a bit from the Grassmannian to talk about Matroids.

Problem 6.1. *For any matroid $\mathcal{M}$, what is the integer $\widetilde{\mathrm{Vol}} P_{\mathcal{M}}$?*

The **dual matroid** if $\mathcal{M}^* := \{[n]\backslash I \mid I \in \mathcal{M}\}$. Clearly, $P_{\mathcal{M}^*} \cong P_{\mathcal{M}}$ by the map $(x_1, \dots, x_n) \mapsto (1 - x_1, \dots, 1 - x_n)$.

6.1 Graphical Matroids $\mathcal{M}_G$

If G is a graph with labelled edges $\{1, 2, \dots, n\}$, the bases of $\mathcal{M}_G$ are the set of edges corresponding spanning trees of G. We denote by G^* the dual graph of G (when G is planar).

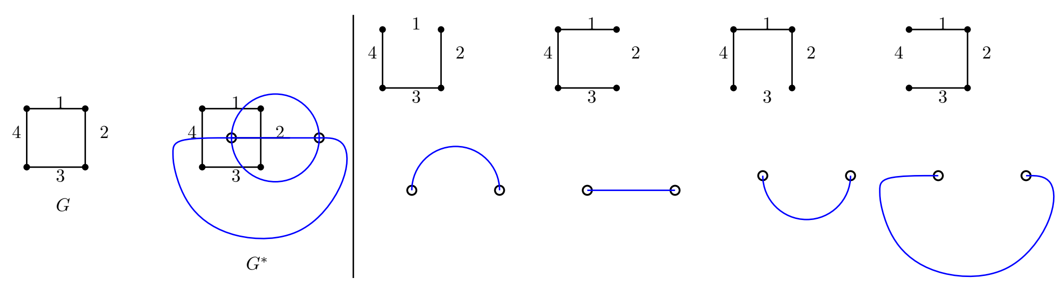


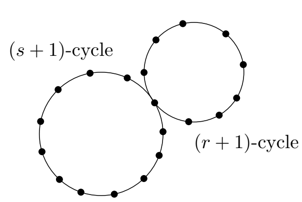
Figure 6: A graph G, its dual G^* and the spanning trees of G and of G^*.

Exercise 6.2. 1. *Check exchange axiom on spanning trees of G.*

2. *If G is a planar graph, $\mathcal{M}_{G^*} = (\mathcal{M}_G)^*$.*

Example 6.3. If G is an n-cycle then $\mathcal{M}_G \cong \Delta^{n-1}$ is an $(n-1)$-simplex. □

Example 6.4. If G is an $(r+1)$-cycle glued at a vertex with a $(s+1)$-cycle then $P_{\mathcal{M}_G} = \Delta^r \times \Delta^s$. And $\widetilde{\mathrm{Vol}}(P_{\mathcal{M}_G}) = \binom{r+s}{r}$



□

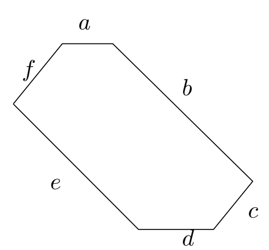
Proposition 6.5. *In full dimensions* $\mathrm{Vol}(A \times B) = \mathrm{Vol}(A) \cdot \mathrm{Vol}(B)$*, if* A *is* r*-dimensional and* B *is* s*-dimensional then* $\widetilde{\mathrm{Vol}}(A \times B) = \binom{r+s}{r} \widetilde{\mathrm{Vol}}(A) \widetilde{\mathrm{Vol}}(B)$.

Problem 6.6. *What is* $\widetilde{\mathrm{Vol}}(P_{\mathcal{M}_G}) =$? *or give families of graphs* G *with nice formulas for* $\widetilde{\mathrm{Vol}}(P_{\mathcal{M}_G}) =$?.

6.2 Generalization of Matroid polytopes

A **generalized permutahedron** is a polytope $P \subset \mathbb{R}^n$ such that any edge of P is parallel to $e_i - e_j$ for some $i \neq j$. (such a permutahedron has dimension $n - 1$)

Example 6.7. $n = 3$, the lengths of the sides satisfy the following *hexagon equation*: $a + f = c + d$, $e + f = b + c$, and $a + b = c + d$.



□

Example 6.8. The usual **permutahedron** P_n with $n!$ vertices $(w_1, \ldots, w_n)$ for all $w \in S_n$.

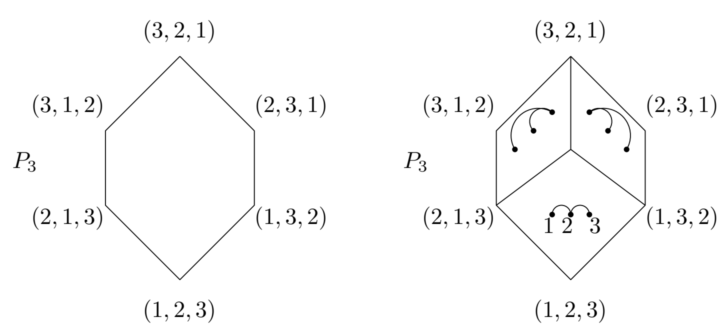


Figure 7: The permutahedron P_3, and tiling of P_3 illustrating why $\mathrm{Vol}(P_n)$ is n^{n-2}, the number of Cayley trees with n vertices.

□

Exercise 6.9. *Check that* P_n *is a generalized permutahedron.*

Exercise 6.10. $\mathrm{Vol}(P_n) = n^{n-2}$ *or equivalently* $\widetilde{\mathrm{Vol}}(P_n) = (n-1)! n^{n-2}$*. This is Cayley's formula for the number of trees with* n *labelled vertices.*

case $k = 3$. □

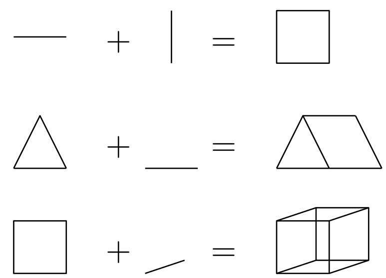


Figure 8: Examples of Minkowski sums.

6.3 Graphical Zonotopes

The **Minkowski sum** $A + B = \{x + y \mid x \in A, y \in B\}$.

A **zonotope** is a Minkowski sum of line segments. In the plane you get n-gons whose opposite sides have same length and are parallel. From the examples above, in the plane only the square is a zonotope.

If G is a graph on n vertices labelled $1, 2, \ldots, n$, the **graphical zonotope** is $Z_G = \sum_{(i,j)\in E(G)}[e_i, e_j] \cong \sum_{(i,j)\in E(G)}[0, e_j - e_i]$. In the last equation you pick an orientation of edges, however the zonotope does not depend on the orientation.

Proposition 6.11. *For the complete graph, the zonotope is the permutahedron* $Z_{K_n} = P_n$.

A **Newton polytope** of $f \in \mathbb{C}[x_1, x_2, \ldots, x_n]$, then we can write $f = \sum c_{a_1,\ldots,a_n} x_1^{a_1} \cdots x_n^{a_n}$. Then $\mathrm{New}(f) = \mathrm{conv}(\{(a_1, \ldots, a_n) \mid c_{a_1,\ldots,a_n} \neq 0\})$

Example 6.12. $\mathrm{New}(x^2y^3 + x + 27y^2) = \mathrm{conv}((2,3),(1,0),(0,2))$.

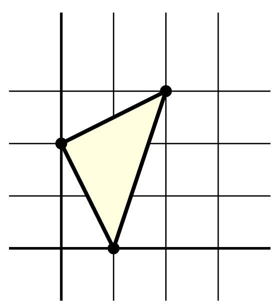


Figure 9: The newton polytope of $x^2y^3 + x + 27y^2$.

□

One of the most important features of the Newton polytope is the following property than says that we can view $\mathrm{New}(\cdot)$ as a generalized logarithm.

Proposition 6.13. $\mathrm{New}(f \cdot g) = \mathrm{New}(f) + \mathrm{New}(g)$.

Proof. The non-trivial point of this proof is that vertices are not cancelled. □

Proposition 6.14. Z_G *is a generalized permutahedron.*

using Newton polytopes. Recall the Vandermonde determinant

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{bmatrix} = \sum_{w\in S_n} (-1)^{\mathrm{sgn}(w)} x_1^{w_1-1} x_2^{w_2-1} \cdots x_n^{w_n-1} = \prod_{i<j}(x_j - x_i).$$

If we take the Newton polytope of both sides and use Property [], $\mathrm{New}(LHS) = \mathrm{conv}((w_1 - 1, \dots, w_n - 1)) = P_n$ and $\mathrm{New}(RHS) = \sum [e_i, e_j] = Z_{K_n}$. ☐

Theorem 6.15. $\mathrm{Vol}(Z_G) = \#\{$*spanning trees in* $G\}$.

Proof. We prove it by induction on the number of edges of G. Let $t(G) = \#\{\text{spanning trees in } G\}$, this numbers satisfy the following deletion-contraction relation

$$t(G) = t(G\backslash e) + t(G/e).$$

Where $G\backslash e$ is graph G with edge e deleted and G/e is the graph G with edge e contracted. One can then show that Z_G also satisfies the same relation. ☐

6.4 Chromatic polynomial of G

$\chi_G(t) = \#\{\text{proper } t\text{-colorings of vertices of } G\}$, a proper coloring is one where the vertices of any edge of G have different colors.

Theorem 6.16. $\chi_G(t)$ *is a polynomial in* t.

Proof. $\chi_G(t)$ satisfies a deletion-contraction relation $\chi_G = \chi_{G\backslash e} - \chi_{G/e}$, and show $\chi_G(t) = t^n$ if G consists of just n vertices with no edges. ☐

Problem 6.17 (open)**.** *Is there a polytope such that some statistic gives* $T_G(x, y)$. *Is there a polytopal realization of the Tutte polynomial.*

7 Lecture 7, 2/29/2012

7.1 Schubert Calculus

We start with basic facts about (co) homology: Let X be a topological space and $H_i(X)$ is the ith homology of X which is some vector space over $\mathbb{C}$. Its dual $H^i(X) := (H_i(x))^*$ is the cohomology of X. These are both topological invariant of X.

The **Betti number** is $\beta_i(X) = \dim H^i(X)$. If $H^*(X) = H^0 \oplus H^1 \oplus \cdots$, this space has a multiplicative structure (cup-product).

Suppose that X is a nonsingular complex algebraic variety and $\dim_{\mathbb{C}} X = N$ then the homology and cohomology only live in even dimension:

$$H_*(X) = H_0 \oplus H_2 \oplus \cdots \oplus H_{2N}$$
$$H^*(X) = H^0 \oplus H^2 \oplus \cdots \oplus H^{2N}.$$

The fundamental class $[X]$ is the canonical generator of H_{2N}.

Figure 10: Example of transversal and non-transversal intersections. In the first example the intersection of the tangent spaces at the points where the varieties meet is a point. In the second example, the tangent spaces are the same.

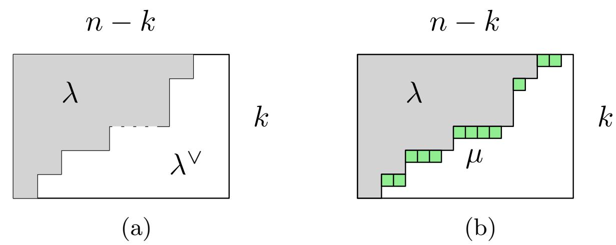


Figure 11: (a) Example of a partition λ and its complement $\lambda^\vee$, and (b) example of partitions obtained in Pieri rule.

We also consider Poincaré duality that says $H^i(X) \cong H_{2N-i}(X)$, or equivalently $H^i(X) \cong (H^{2N-i}(X))^*$. If $Y \subset X$ is an algebraic subvariety with $\dim_{\mathbb{C}} Y = m$ then $[Y] \in H_{2m}(X) \cong H^{2N-2m}(X)$ (it has codimension $2m$). If $X = \coprod_{i\in I} Y_i$ then we say that X has a cell decomposition (CW-complex), where $Y_i \cong \mathbb{C}^{m_i}$ and $\overline{Y_i}$ is an algebraic subvariety and $\overline{Y_i}\backslash Y_i$ is a union of smaller dimensional Ys.

Claim 1: Cohomology classes of $[\overline{Y_i}]$ are in $H^{2N-2m_i}(X)$ so they form a linear basis of $H^*(X)$. In particular $H^0(X)$ is spanned by $[X]$. And $H^{2N}(X)$ is spanned by [point].

Claim 2: If Y and Y' are algebraic subvarieties of X and $Y \cap Y' = Z_1 \cup \cdots \cup Z_r$ where

(i) $\operatorname{codim} Y + \operatorname{codim} Y' = \operatorname{codim} Z_i$ for all i (proper intersection)

(ii) For every generic point $z \in Z_i$, $T_z Z_i = T_z Y \cap T_z Y'$ where T_z is the *tangent space* (transversal intersection)

Then

$$[Y] \cdot [Y'] = \sum [Z_i].$$

7.2 Cohomology of $\mathbf{Gr}(k, n, \mathbb{C})$

$$\mathbf{Gr}(k,n) = \coprod_{\lambda \subseteq k\times(n-k)} \Omega_\lambda,$$

where $\Omega_\lambda \cong \mathbb{C}^{|\lambda|}$ is a Schubert cell. Let $X_\lambda := \overline{\Omega}_{\lambda^\vee}$ where $\lambda^\vee = (n-k-\lambda_k, \cdots, n-k-\lambda_1)$ is the complement of λ in $k \times (n-k)$.

Denote by $\sigma_\lambda = [X_\lambda] \in H^{2|\lambda|}(\mathbf{Gr}(k, n, \mathbb{C}))$, these are the Schubert classes. The Schubert classes do not depend on the choice of basis, just on the partition.

Theorem 7.1. *The Schubert classes σ_λ for $\lambda \subseteq k \times (n-k)$ form a linear basis of $H^*(\mathbf{Gr}(k, n, \mathbb{C}))$.*

Example 7.2. $\sigma_{\varnothing} = [\mathbf{Gr}(k,n)]$ and $\sigma_{k\times(n-k)} = [\text{point}]$. □

Remarks 7.3 (Special feature of this basis)**.** This basis is self-dual with respect to Poincaré duality. This means:

(i) $B = \{\sigma_\lambda \mid |\lambda| = i\}$ basis of $H^{2i}(\mathbf{Gr}(k,n))$,

(ii) $B* = \{\sigma_{|\mu|} \mid |\mu| = k(n-k) - i\}$ basis of $H^{2k(n-k)-2i}(\mathbf{Gr}(k,n))$. B and B^* are dual basis (the dual map is $\sigma_\lambda \mapsto \sigma_{\lambda^\vee}$).

Let $c \in \mathbb{C}$, for $\sigma \in H^{2k(n-k)}(\mathbf{Gr}(k,n))$ where $\sigma = c \cdot [\text{point}]$ then $< \sigma >:= c$.

Theorem 7.4 (Duality Theorem)**.** *For partitions λ, μ such that $|\lambda| + |\mu| = k(n-k)$ then $< \sigma_\lambda \cdot \sigma_\mu >= \delta_{\lambda,\mu^\vee}\sigma_{k\times(n-k)}$. Where the product of Schubert classes is in the cup product.*

Theorem 7.5 (Pieri Formula)**.** *Let $\sigma_r = \sigma_{\square\square\cdots\square}$ (k boxes) then*

$$\sigma_\lambda \cdot \sigma_r = \sum_\mu \sigma_\mu,$$

where the sum is over μ such that μ/λ is a horizontal r-strip.

In terms of coordinates the partitions are interlaced $n-k \geq \mu_1 \geq \lambda_1 \leq \mu_2 \geq \lambda_2 \geq \cdots \geq \mu_k \geq \lambda_k$ and $\sum(\mu_i - \lambda_i) = r$.

Example 7.6. □

7.3 Note on proofs

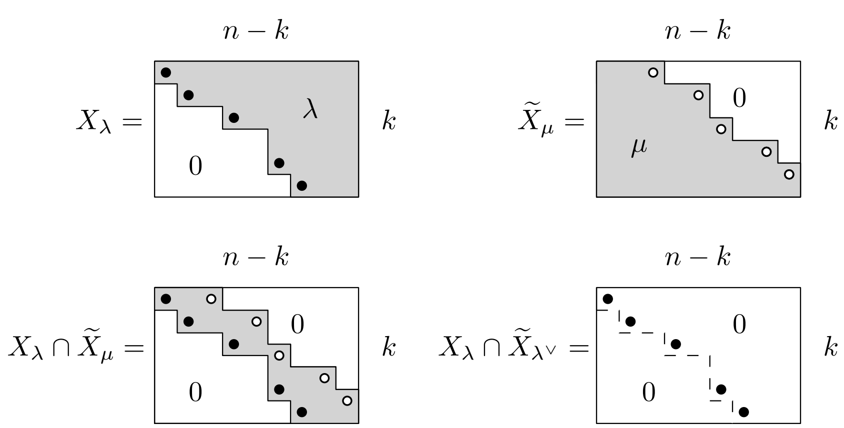
Consider $\sigma_\lambda \cdot \sigma_\mu = [X_\lambda] \cdot [X_\mu]$. We work with $X_\lambda \cap \widetilde{X}_\mu$ where X_λ corresponds to standard Schubert decomposition (ordering basis with permutation $12\cdots n$) and $\widetilde{X}_\mu$ corresponds to the opposite ordering of the coordinates (permutation $n\,n-1\cdots 21$). We do this choice of basis to obtain a transversal intersection and then use Claim 2.

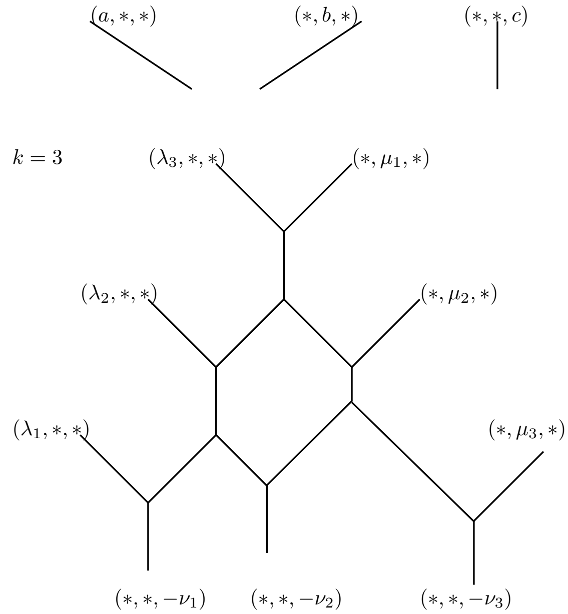
If $|\lambda| + |\mu| = k(n-k)$ and $\lambda = \mu^\vee$ then we get a point. Otherwise you can show that $X_\lambda \cap \widetilde{X}_\mu$.

Pieri formula uniquely defines the multiplicative structure of the Schubert cells.

$$\sigma_\lambda \cdot \sigma_\mu = \sum_{\nu, |\nu| = |\lambda| + |\mu|} c^\nu_{\lambda\mu}\sigma_\nu,$$

where $c^\nu_{\lambda\mu}$ are the *Littlewood Richardson coefficients*. By the duality theorem $c^\nu_{\lambda\mu} =< \sigma_\lambda \cdot \sigma_\mu \cdot \sigma_{\nu^\vee} >$, i.e. $c_{\lambda\mu\nu} := c^{\nu^\vee}_{\lambda\mu} = \#\{X_\lambda \cap \widetilde{X}_\mu \cap \widetilde{\widetilde{X}}_\nu\}$. Then $c_{\lambda\mu\nu} \in \mathbb{Z}_{\geq 0}$ and these coefficients have S_3-symmetry.





7.4 Honeycomb version of the Littlewood Richardson rule

This version was done by Knutson-Tao, it is a reformulation of a rule by Bernstein-Zelevinsky.

We work in the plane $\mathbb{R}^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=0\}$. In this plane there are three types of lines $(a,*,*), (*,b,*)$ and $(*,*,c)$ where $a,b,c \in \mathbb{Z}$.

Theorem 7.7. $c^{\nu}_{\lambda\mu} = \#\{$*integer honeycomb with fixed boundary rays*$\}$.

If we know the rational lengths ℓ_i of the internal edges we can reconstruct the honeycomb.

We can rescale the honeycomb such that $\ell_i \in \mathbb{Z}_{\geq 0}$ and also $\lambda_1 + \lambda_2 = \lambda_1 - \lambda_2$ and the lengths on a hexagon should satisfy the *hexagon condition*

8 Lecture 8, 3/2/2012

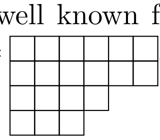
Recall from last time that $H^*(\mathbf{Gr}(k,n,\mathbb{C}))$ has a linear basis of Schubert classes Ω_λ. In this lecture we will mention the relation between $H^*(\mathbf{Gr}(k,n,\mathbb{C}))$ and symmetric functions.

8.1 Symmetric functions

Let Λ be the ring of symmetric functions. We build this ring in the following way: let $\Lambda_k = \mathbb{C}[x_1,\ldots,x_n]^{\mathfrak{S}_k}$, the symmetric polynomials with k variables and let $\Lambda = \varprojlim \Lambda_k$.

- $e_r = \sum_{1\leq i_1 < i_2 < \cdots < i_r} x_{i_1}x_{i_2}\cdots x_{i_r}$ (elementary symmetric functions)
- $h_r = \sum_{1\leq j_1 \leq j_2 \leq \cdots \leq j_r} x_{j_1}\cdots x_{j_r}$ (complete symmetric functions)

Theorem 8.1 (Fundamental theorem of symmetric functions)**.** $\Lambda = \mathbb{C}[e_1,e_2,\ldots] = \mathbb{C}[h_1,h_2,\ldots]$.

Another well known fact about Λ is it has a linear basis of Schür functions s_λ where λ is a partition $\lambda =$ .

Definition 8.2. We give two equivalent definitions of the Schür functions.

- Given a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ let $\alpha = (\alpha_1, \ldots, \alpha_k) = (\lambda_1 + k - 1, \lambda_2 + k - 2, \ldots, \lambda_k + 0)$.

$$s_\lambda(x_1, \ldots, x_k) = \begin{vmatrix} x_1^{\alpha_1} & x_2^{\alpha_1} & \cdots & x_k^{\alpha_1} \\ x_1^{\alpha_2} & x_2^{\alpha_2} & \cdots & \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\alpha_k} & x_2^{\alpha_k} & \cdots & x_k^{\alpha_k} \end{vmatrix} / \det(x_j^{k-i})$$

then $s_\lambda = \lim_{k\to\infty} s_\lambda(x_1, \ldots, x_k)$.
- $s_\lambda = \sum_{T \in SSYT(\lambda)} x^T$ where $SSYT(\lambda)$ is the set of semistandard Young tableaux

$$T = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 & 2 \\ \hline 2 & 2 & 3 & 3 & 4 & 7 \\ \hline 3 & 4 & 4 & 6 \\ \cline{1-4} 5 & 5 & 5 \\ \cline{1-3} \end{array}, \qquad x^T = x_1^3 x_2^5 x_3^3 x_4^3 x_5^3 x_6 x_7.$$

In particular $e_r = s_{1^r}$ and $h_r = s_{(r)}$.

□

From the second definition of s_λ it is easy to see the following rule.

Theorem 8.3 (Pieri formula for s_λ)**.**

$$h_r \cdot s_\lambda = \sum_\mu s_\mu.$$

where μ are partitions such that μ/λ is a horizontal r-strip.

Equivalently $e_r \cdot s_\lambda = \sum_\mu s_\mu$ where μ/λ is a vertical r-strip.

Lemma 8.4. *Suppose that we have an associative bilinear operation $*$ on Λ such that $s_r * s_\lambda = \sum_\mu s_\mu$ where μ/λ is a horizontal r-strip, then $(\Lambda, *) \cong (\Lambda, \cdot)$.*

Proof. By the fundamental theorem we know that $(\Lambda, \cdot) = \mathbb{C}[h_1, h_2, \ldots]$, since $s_r = h_r$, the Pieri-type formula essentially says that $(\Lambda, *)$ has the same product of by the algebraically independent generators h_r as $(\Lambda, \cdot)$. □

Definition 8.5. Let $\Lambda_{k,n} = \Lambda / I_{k,n}$ where $I_{k,n} :=< s_\lambda \mid \lambda \not\subset k \times (n-k) >$. □

Exercise 8.6. *Show that $I_{n,k} =< e_i, h_j \mid i > k, j > n-k >$. Show that $s_\lambda \not\subset k \times (n-k)$ form a linear basis of $I_{n,k}$.*

Theorem 8.7. *$H^*(\mathbf{Gr}(k,n)) \cong \Lambda_{k,n}$.*

Proof. Define the map $\sigma_\lambda \mapsto s_\lambda$, $s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$. □

8.2 Gleizer-Postnikov web diagrams

The GP-diagrams involve four directions instead of the three directions of the Knutson-Tao puzzles. In the latter λ, μ and ν have k parts, and in the former λ and μ have k parts and ν has $k+\ell$ parts. Note that by convention $s_{(\lambda_1,\ldots,\lambda_k,0))} = s_{(\lambda_1,\ldots,\lambda_k)}$ and $s_\lambda = 0$ if some $\lambda_i < 0$.

The horizontal edge at a height c from the x-axis is labelled $c = 2h/\sqrt{3}$. The edges going from NW-SE and from NE-SW are labelled according to their x-intercepts. See Figure 13.

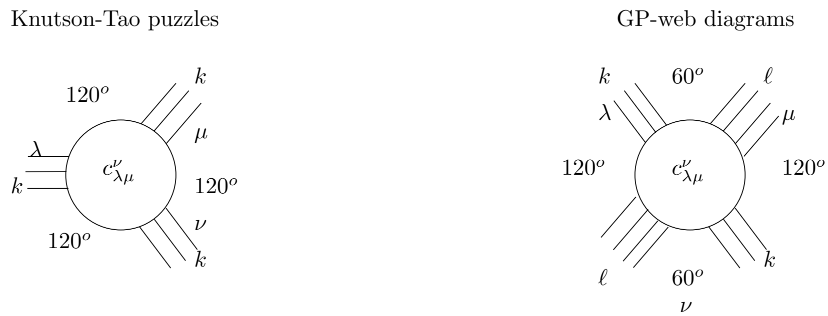


Figure 12: Schematic comparison of Knutson-Tao puzzles and Gleizer-Postnikov web diagrams.

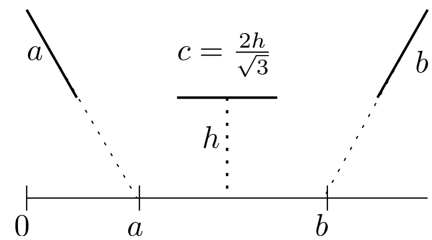


Figure 13: How to label the edges of a GP-web diagram.

Example 8.8 ($k = \ell = 1$)**.** $s_r \cdot s_s = \sum_{c \geq r-s} s_{s+c,r-c}$. We have a conservation law: a flow of $r + s$ is coming in and coming out. See Figure 14.

□

Theorem 8.9. *The Littlewood-Richardson coefficient $c^\nu_{\lambda\mu}$ is the number of web diagrams of type λ, μ and ν.*

Proof. Define $*$ product on Λ by $s_\lambda * s_\mu = \sum_\nu \#\{\text{web diagrams type } \lambda, \mu, \nu\} s_\nu$. Next we prove the Pieri rule for $*$-product.

For $\ell = 1$ we get interlacing $\nu_1 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \nu_k \geq \lambda_k \geq \nu_{k+1}$ (see Figure 15). By conservation law $|\lambda| + \mu_1 = |\nu|$. This is equivalent to saying that ν/λ is a horizontal μ_1-strip. Conversely, given ν such that ν/λ is a horizontal μ_1-strip we can build a unique web diagram. The non-trivial part showing it is associative.

□

Example 8.10. Let's verify that $c^{631}_{21,52} = 2$. See Figure 16

□

Problem 8.11. *(Open) The GP-web diagrams with six directions are infinite. However with five directions the diagrams are finite. In this case, what is the analogue of the Littlewood-Richardson coefficients.*

9 Lecture 9, 3/7/2012

A permutation w is a bijection between $[n] \to [n]$. We multiply permutations from right to left. A simple transposition is a permutation $s_i = (i, i+1)$. We can write any w as a product of simple

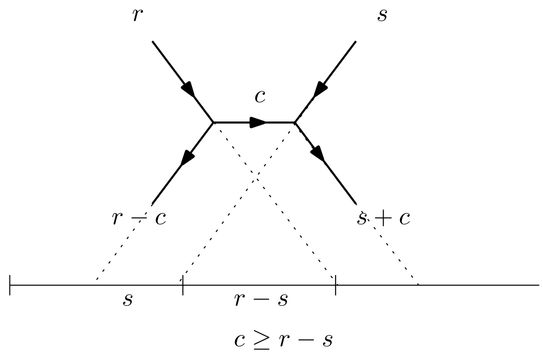


Figure 14: Example of GP-web diagrams for $k = \ell = 1$.

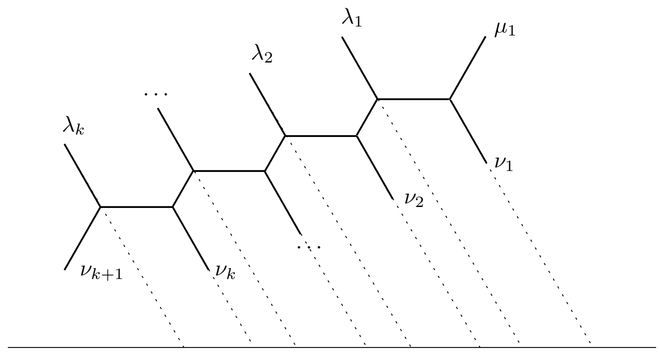


Figure 15: Illustration of the Pieri-rule for GP-web diagrams

transpositions. A *reduced decomposition* is an expression of w as a product $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ of simple transpositions of minimum possible length ℓ. The reduced decompositions of w are related by certain *moves*

$$s_i s_j = s_j s_i \quad |i-j| \geq 2, \text{(2-move)}$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{(3-move)}$$

9.1 Wiring diagrams

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 1 & 4 \end{pmatrix}, \quad w = s_4 \underline{s_2 s_1 s_2} s_3.$$

We do not allow *triple intersections* and intersections occur at different heights.

We want to transform the GP-web diagrams into wiring diagrams.

The number of such plane partitions are the Littlewood-Richardson rule.

Exercise 9.1. *Show that the classical Littlewood-Richardson rule corresponds to this rule in terms of plane partitions.*

If V is a vector space with basis $e_0, e_1, e_2, \dots$ a **Scattering matrix** (R-matrix)

$$R(c) : V \otimes V \to V \otimes V, e_r \otimes e_s = \begin{cases} e_{s+c} \otimes e_{r-c} & \text{if } c \geq r-s, \\ 0 & \text{otherwise.} \end{cases}$$

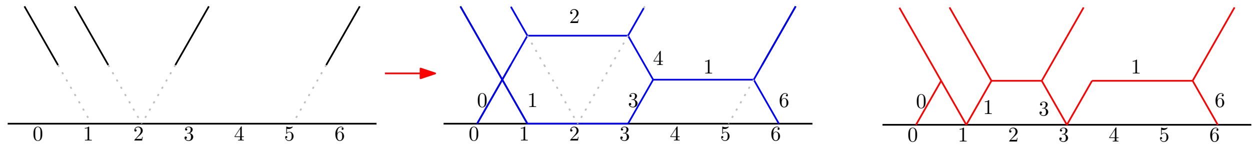


Figure 16: Example showing the two GP web diagrams when $\lambda = 21, \mu = 52$ and $\nu = 6310$.

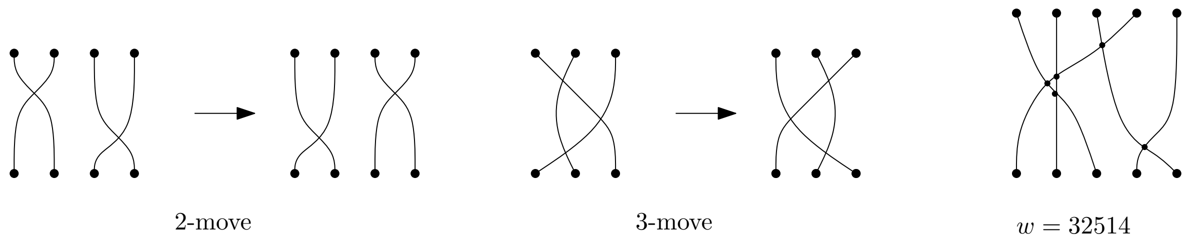


Figure 17: Illustration of 2 and 3-moves and of the wiring diagram of $w = 32514 = s_4 s_2 s_1 s_2 s_3$.

where $e_{-i} = 0$ for $i > 0$. This corresponds to picture. (e_i are levels of excitement of particles and $R(c)$ describes how they interact)

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ we map $s_\lambda \mapsto e_{\lambda_k} \otimes e_{\lambda_{k-1}} \otimes \cdots \otimes e_{\lambda_1} \in V^{\otimes k}$.

We define an operator $M_{k,\ell} : V^{\otimes k} \otimes V^{\otimes \ell} \to V^{\otimes (k+\ell)}$. Then $R_{ij}(c)$ is the operator on $V^{\otimes m}$ that acts as $R(c)$ on the ith and jth copies of V. Clearly $R_{ij}(c)$ commutes with $R_{\hat{i}\hat{j}}(c)$ if $\#\{i, j, \hat{i}, \hat{j}\} = 4$ (they act on four different copies of V).

Definition 9.2.

$$M_{k,\ell} = \sum_{(c_{ij}} \prod_{i=1,\ldots,k} \prod_{j=k+\ell,k+\ell-1,\ldots,k+1} R_{ij}(c_{ij}),$$

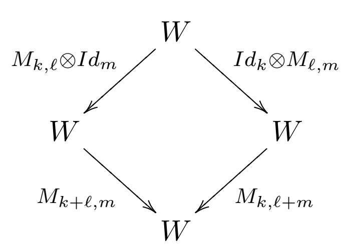
where the sum is over (c_{ij}) such that $c_{ij} \geq c_{i'j'} \geq 0$ whenever $i' \leq i < j \leq j'$. □

For example $M_{23} = \sum R_{15}(c_{15}) R_{14}(c_{14}) R_{13}(c_{13}) R_{25}(c_{25}) R_{24}(c_{24}) R_{23}(c_{23})$ (you can choose any linear extension of the poset - such number is the number of Young Tableaux on the rectangle)

Theorem 9.3 (LR-rule: R-matrix version)**.** *Given* $\lambda = (\lambda_1, \ldots, \lambda_k)$ *and* $\mu = (\mu_1, \ldots, \mu_\ell)$ *then*

$$M_{k,\ell}(e_\lambda \otimes e_\mu) = \sum_{\nu = (\nu_1, \ldots, \nu_{k+\ell})} c^{\nu}_{\lambda,\mu} e_\nu.$$

The Pieri-rule is easy, the hard part is to show associativity: if $W = V^{\otimes k} \otimes V^{\otimes \ell} \otimes V^{\otimes m}$



Proposition 9.4.

$$M_{k+\ell,m} \circ (M_{k,\ell} \otimes Id_m) = M_{k,\ell+m} \otimes (Id_k \otimes M_{\ell,m}).$$

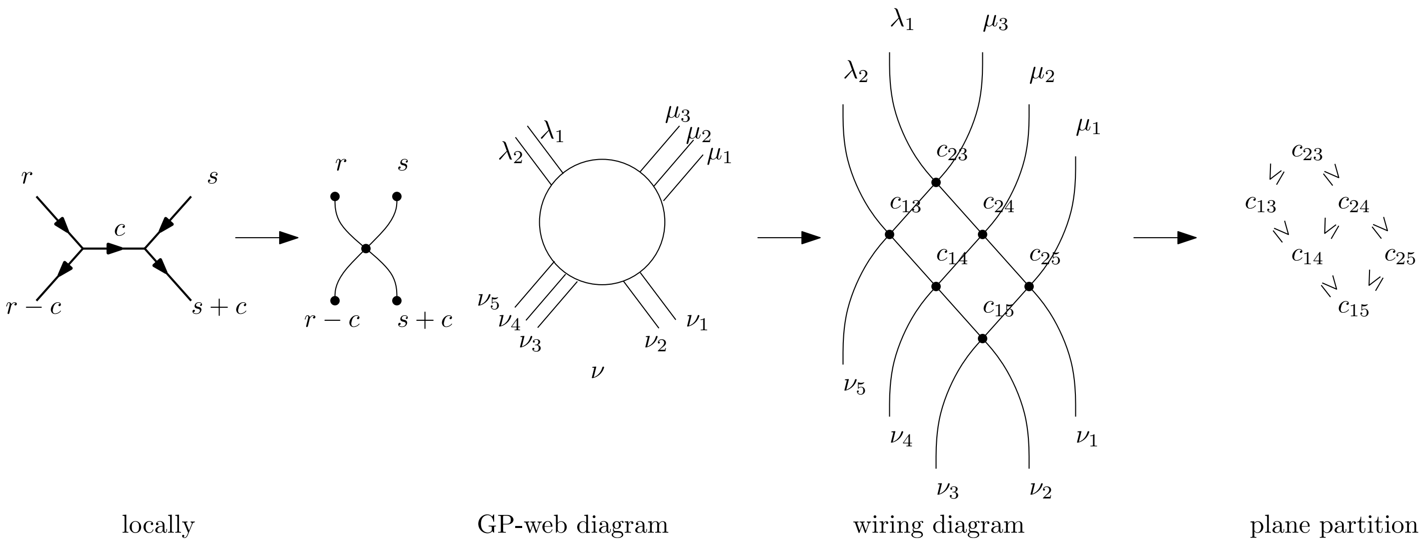


Figure 18: Illustration of how to transform a GP-web digram into a wiring diagram.

R-matrices satisfy the **Yang-Baxter Equation** which depends on two parameters α, β:

$$R_{23}(\beta)R_{13}(\alpha+\beta)R_{12}(\alpha) = R_{12}(\alpha)R_{13}(\alpha+\beta)R_{23}(\beta).$$

These diagrams satisfy a genrealized Yang-Baxter equation that depends on three parameters.

Proposition 9.5 (Generalized Yang-Baxter equation)**.**

$$R_{23}(c_{23})R_{13}(c_{13})R_{12}(c_{12}) = R_{12}(c'_{12})R_{13}(c'_{13})R_{23}(c'_{23}),$$

where

$$\begin{cases} c'_{12} = \min(c_{12}, c_{13} - c_{12}) \\ c'_{13} = c_{12} + c_{23} \\ c'_{23} = \min(c_{23}, c_{13} - c_{12}). \end{cases}$$

when $c_{13} = \alpha + \beta$ it reduces to the classical Yang-Baxter equation.

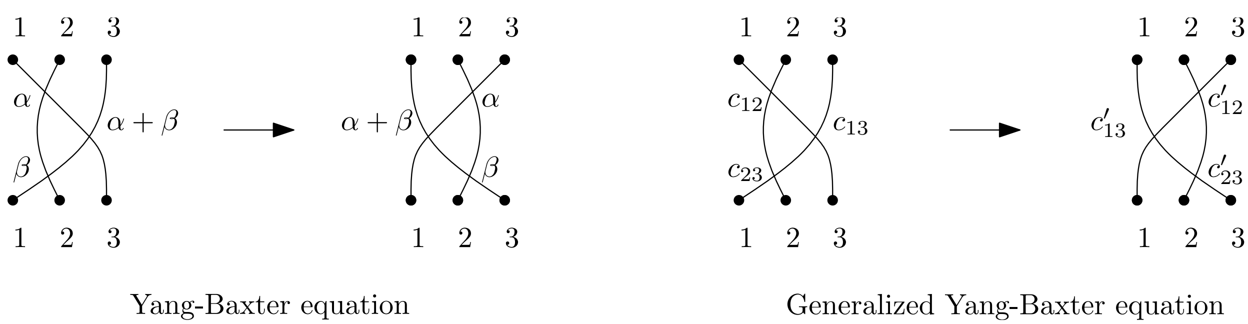


Figure 19: Illustration of the Yang Baxter equation and the generalized Yang Baxter equation.

Exercise 9.6. *Prove this generalized Yang-Baxter equation.*

This Generalized Yang-Baxter takes the wiring diagram with c_{ij} to an expression with c'_{ij}. The important point is to show that the inequalities on c_{ij} get translated to the inequalities on c'_{ij}.

To finish this prove we need to generalize this transformation of inequalities to arbitrary wiring diagrams.

9.2 String cone

[$\star\star\star$ This part needs polishing $\star\star\star$]

Let D be any wiring diagram for some reduced decomposition $w = s_{i_1} \cdots s_{i_\ell}$.

Example 9.7. We start from a wiring diagram of $w = 4213 = s_2 s_3 s_2 s_1$ and obtain a bibpartite graph (see Figure 20). □

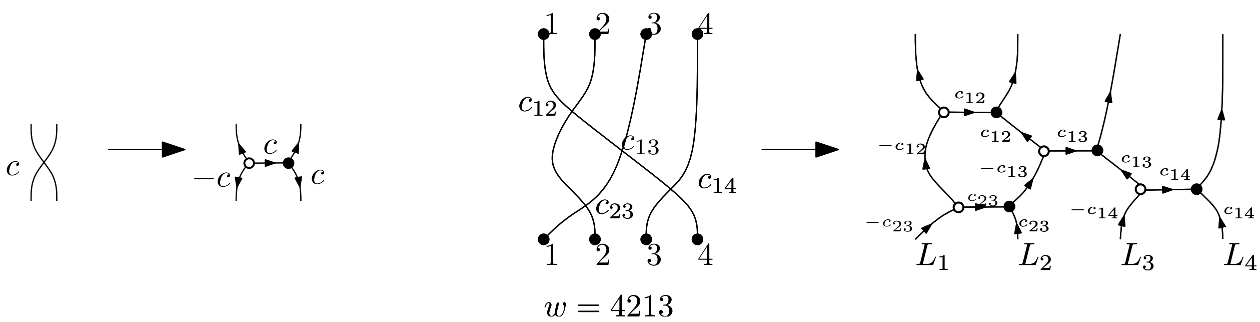


Figure 20: Illustration of how to obtain a bipartite graph from a wiring diagram of $w = 4213$.

We switch directions of some edges such that strands $L_1, L_2, \ldots, L_i$ are directed down and $L_{i+1}, \ldots, L_n$ are directed up.

$G_{D,i}$ look at directed path P from L_{i+1} to L_i, each path gives an inequality: sum of weights of edges in graph ≥ 0.

Example 9.8. Continuing from Example 9.7. For $i = 2$, there are two paths $P : L_3 \to L_2$ with the convention that $c_{ij} = -c_{ji}$. See Figure 21.

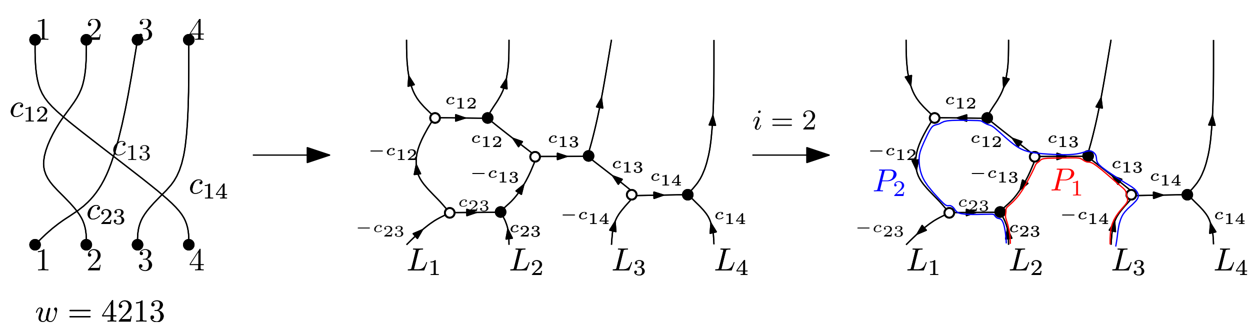


Figure 21: Paths from L_3 to L_2 in the bipartite graph obtained from a wiring diagram of $w = 4213$.

From the first path P_1, we obtain the inequality $c_{41} + (c_{13} - c_{13} + c_{13}) - c_{23} \geq 0$ which simplifies to $c_{41} + c_{13} + c_{32} \geq 0$. Form the second path P_2, we obtain the inequality $c_{41} + (c_{13} - c_{13}) + (c_{12} - c_{12} + c_{12}) + (c_{23} - c_{23}) \geq 0$ which simplifies to $c_{41} + c_{12} \geq 0$. □

Claim This cone is what we really need. Every time we apply a 3-move and transform parameters by the Generalized Yang-Baxter equation then the cone for one diagram transforms to the cone of the diagram we obtain ... transform as needed. (piecewise linear continuous map)

If we have a wiring diagram for the associativity these inequalities become very simple (we get the plane partition inequalities)

10 Lecture 10 3/09/2012

How about showing $*$-product is commutative. We know that associativity with Peiri rule shows that $*$-product of Schur functions is equivalent to normal product of Schur functions (which are commutative). Surprisingly, there is no direct way to see commutativity from the $*$-product picture.

10.1 Symmetries of Littlewood-Richardson coefficients

$c_{\lambda\mu}^{\nu} := c_{\lambda\mu\nu^\vee}$ has S_3 symmetry. It is not clear how to see this symmetry from the KT honeycomb.

Also $c_{\lambda\mu}^{\nu} = c_{\lambda'\mu'}^{\nu'}$ where λ' is the conjugate partition. There is a reformulation of KT honeycombs in terms of puzzles that makes this symmetry explicit.

Also $c_{\lambda\mu}^{\nu} = c_{\mu\lambda}^{\nu}$ and this is related to the *Schützenberger involution*. An interesting open question is to understand this symmetry in the $*$-product setting (using Yang-Baxter equation...).

10.2 Total positivity

An $m \times n$ matrix is called **totally positive** or **TP** (**totally nonnegative** or **TNN** respectively) if every minor is > 0 (≥ 0 respectively).

Example 10.1.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, a, b, c, d \geq 0, ad - bc > 0.$$

□

The following Lemma relates total positivity with combinatorics.

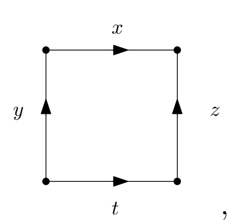
Lemma 10.2 (Lindström Lemma). *Let G be a finite acyclic directed graph with weights x_e on edges and selected vertices: $A_1, \ldots, A_n, B_1, \ldots, B_n$. Define $M = (M_{ij})$ where $M_{ij} = \sum_{p:A_i \to B_j} \prod_{e \in P} x_e$. Then*

$$\det(M) = \sum_{(P_1,\ldots,P_n), P_i : A_i \to B_{w(i)}} (-1)^{sgn(w)} \prod_{i=1}^{n} \prod_{e \in P_i} x_e,$$

where $(P_1, \ldots, P_n)$ are families of non-crossing *paths connection the As and Bs. Where non-crossing means that no pair of paths P_i and P_j have a common vertex.*

Remark 10.3. If we sum over all paths without the restriction that they are non-crossing we just get a restatement of the definition of the determinant. □

Example 10.4.



, $$M = \begin{bmatrix} x + ytz & yt \\ tz & t \end{bmatrix}, \quad \det(M) = (x + ytz)t - yt \cdot tz = xt.$$

□

Proof.

$$\det(M) = \sum_{w \in \mathfrak{S}} (-1)^{sgn(w)} \prod_i M_{i,w(i)} = \sum_{(P_1,\ldots,P_n), P_i : A_i \to B_{w(i)}} (-1)^{sgn(w)} \prod_{i=1}^{n} \prod_{e \in P_i} x_e,$$

for any family of paths connecting As with Bs. Next we use the *Involution principle.* We build a sign-reversing involution φ on families $(P_1, \ldots, P_n)$ with a crossing. Find the min. P_i that intersects a path and find first point c on P_i that intersects a path. On that path find the minimal P_j that passes through c.

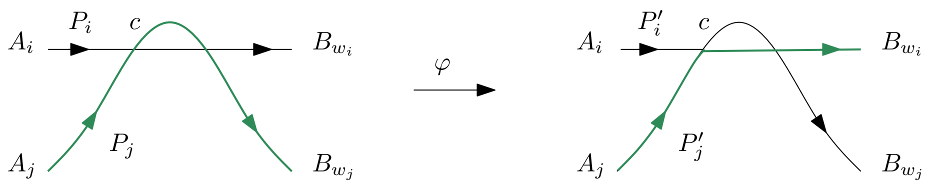


Figure 22: Illustration of sign reversing involution φ in the proof of Lindström Lemma.

We claim that this map is an involution, it preserves the weight of the path but reverses sign. □

A special case of this Lemma is related to positivity.

Corollary 10.5. *If G is a plane graph (embedded in a disk, see Figure 23) $A_1, \ldots, A_m$ are on the left-hand side of the boundary of the disk and $B_1, \ldots, B_n$ are on the right-hand side (ordered from top to bottom). Assume edge weights $x_e > 0$, then the matrix $M = (M_{ij})$ is totally nonnegative.*

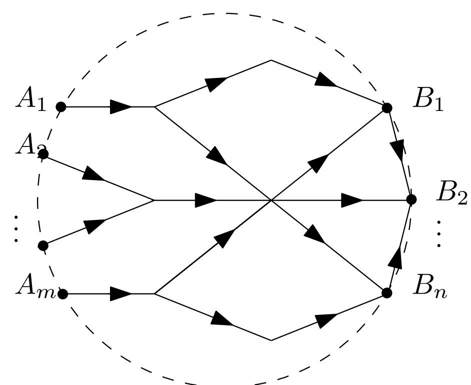
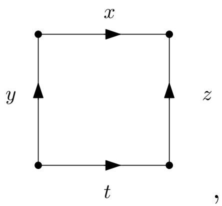


Figure 23: Plane graph G. If $x_e > 0$, then the matrix M of Lindström's Lemma is TNN.

Proof. Lindström Lemma implies that any minor is given by a nonnegative expression. That is, if graph is planar all signs associated to noncrossing paths are positive. □



, $\quad M = \begin{bmatrix} x+ytz & yt \\ tz & t \end{bmatrix}, \quad x,y,t,z > 0 \quad \det(M) = (x+ytz)t - yt \cdot tz = xt > 0.$

Thus M is totally positive. Moreover, all 2×2 totally positive matrix can be written uniquely in this way.

Claim: Any TNN matrix has this form (but not in a unique way).

The following is based on the work of Bernstein-Fomin-Zelevinsky related to previous results by Lusztig. Assume that $m = n$ and that $M \in GL_n$. Recall the LUD-decomposition $M = LUD$ where L is lower triangular with ones on the diagonal, D is a diagonal matrix, and U is upper triangular with ones on the diagonal.

It is well known that M is TNN iff L, U, D are TNN. So our first goal is to understand TNN upper triangular matrices. Let U_n be upper-triangular unipotent subgroup of GL_n. The strata of the TNN part of U_n correspond to permutations $w \in S_n$.

Example 10.6. $n = 2$, $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$ where $x \geq 0$. There are two possibilities: $x = 0$ in which case we get the identity. If $x > 0$ we get the matrix $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$. Given w, we write its wiring diagram (now drawn from left to right). See Figure 24 for an example of this correspondence for $\mathfrak{S}_2$ (in this case we are not using the fact that the graph we obtain from the wiring diagram is bipartite, this property will be important later).

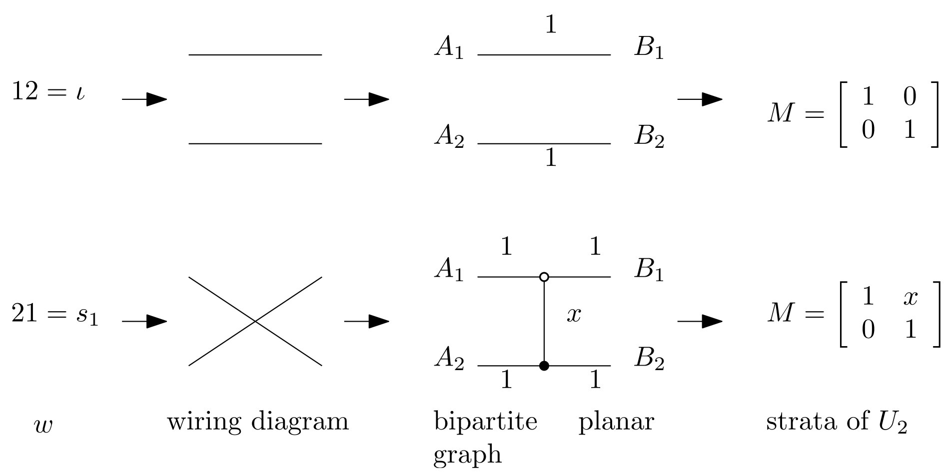


Figure 24: How permutations in S_2 correspond to strata of the TNN part of U_2.

□

For $w \in S_n$ pick any reduced decomposition $w = s_{i_1} \cdots s_{i_\ell}$. Next we decompose M into a product of certain elementary matrices. We illustrate this with an example.

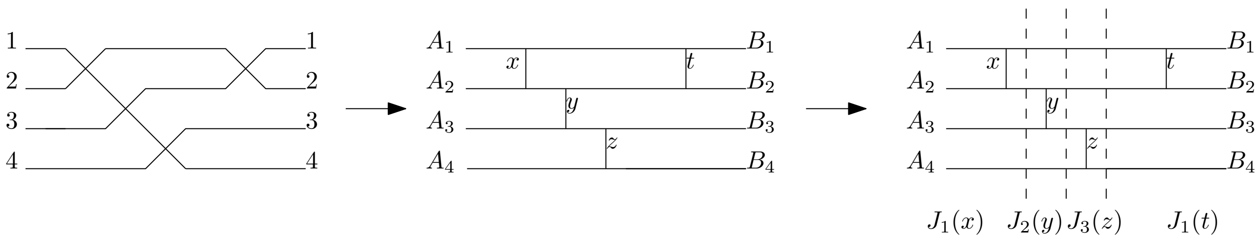


Figure 25: Example of Bruhat cell.

Example 10.7. For $w = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix} = s_1 s_3 s_2 s_1$, we have

$$M = J_1(x)J_2(y)J_3(z)J_1(t) = \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & y & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & z \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & t & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

The matrices $J_i(x)$ are called **elementary Jacobi matrices**. □

Lemma 10.8. *If $s_{i_1} \cdots s_{i_\ell}$ is a reduced decomposition of w, the set of matrices $\{J_{i_1}(x), \dots, J_{i_\ell}(x_\ell) \mid x_1, \dots, x_\ell > 0\}$ depend only on the permutation w.*

Proof. Seeing that the set of matrices does not change under a 2-move is easy. For 3-moves (see Figure 26) we do the followng:

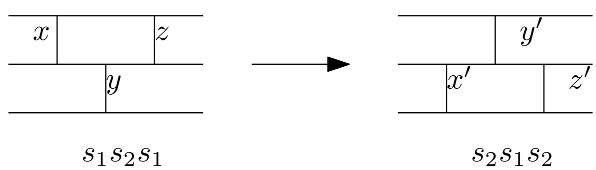


Figure 26: 3-move.

$$M = J_1(x)J_2(y)J_1(z) = \begin{bmatrix} 1 & x+z & xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & y' & y'z' \\ 0 & 1 & x'+z' \\ 0 & 0 & 1 \end{bmatrix}$$

Thus we obtain the system of equations

$$\begin{matrix} y' & = x+z \\ x'+z' & = y \\ y'z' & = xy \end{matrix} \quad \Rightarrow \quad \begin{matrix} y' & = x+z \\ z' & = \frac{xy}{x+z} \\ x'' & = y - \frac{xy}{x+z} = \frac{yz}{x+z}. \end{matrix}$$

Note that the solutions are *subtraction-free*. This means that for positive x, y, z we obtain a unique positive solution x', y', z'.

□

11 Lecture 10 3/14/12

Question 11.1. *What is the number of potential nonzero minors of an upper triangular matrices (including the "empty" minor)? Why is it the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$.*

Example 11.2. For $n = 1$ there are 2 minors, for $n = 2$ there are five minors, for $n = 3$ there are 14 such minors. □

Exercise 11.3. *Answer the question above and explain what is the refinement of Catalan numbers you get when you restrict to the size of the minor.*

Pick a reduced decomposition of $w = s_{i_1} \cdots s_{i_\ell}$. Then the **Bruhat cell** is $B_w = \{J_{i_1}(x_1)J_{i_2}(x_2)\cdots J_{i_\ell}(x_\ell) \mid x_1, \ldots, x_\ell > 0\}$ where $J_i(x)$ is the $n \times n$ upper triangular matrix with ones on the diagonal, the $(i, i+1)$ entry is x and the other entries are 0.

$$J_i(x) = \begin{bmatrix} 1 & & & \\ & 1 & & x \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

See Figure 25 for an Example of a Bruhat cell. We also showed:

Lemma 11.4. *B_w depends only on w (not on its reduced decomposition).*

11.1 Bruhat order

Definition 11.5. The **(strong) Bruhat order** on $\mathfrak{S}_n$ is the partial order on permutations with covering relations $u \lessdot w$ such that

1. $u = w(i, j)$
2. $\ell(u) + 1 = \ell(w)$.

Equivalently, $u \le w$ if any (or *some* for another equivalent definition) reduced decomposition for $w = s_{i_1} \cdots s_{i_\ell}$ contains a subword $s_{j_1} \cdots s_{j_r}$ which is a reduced decomposition for u. □

Note that it is not true that you can pick a reduced decomposition of u and add reflections to obtain a reduced decomposition of w. [⋆⋆⋆ give an example of this ⋆⋆⋆] See Figure 27 for the Hasse diagram of the Bruhat order in $\mathfrak{S}_3$.

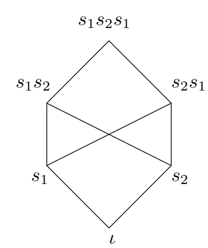


Figure 27: The strong Bruhat order in $\mathfrak{S}_3$.

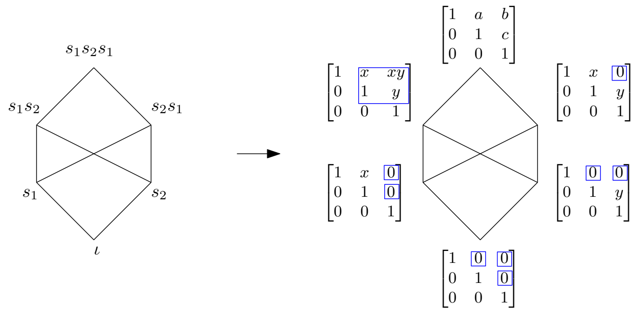
Theorem 11.6 (Bernstein-Fomin-Zelevinsky)**.**

1. *The TNN of U_n decompose as $\coprod_{w \in \mathfrak{S}_n} B_w$,*
2. *$B_w \cong \mathbb{R}^\ell_{>0}$ where $\ell = \ell(w)$, and the isomorphism is $J_{i_1}(x_1)\cdots J_{i_\ell}(x_\ell) \mapsto (x_1, x_2, \ldots, x_\ell)$.*
3. *The closure $\overline{B}_u \subseteq \overline{B}_v$ if and only if $u \le w$ in the strong Bruhat order.*

Remark 11.7. To start seeing why part 2. of the theorem above is related to the Bruhat order if $x_i = 0$ then $J_{i_j}(0) = I$ is the identity matrix. This is analogous to considering subwords (one has to check that in the closure we get $x_i \ge 0$). □

[⋆⋆⋆ say something about nonzero minors ⋆⋆⋆] □

How about the whole of GL_n?



Example 11.8.

Figure 28: Illustration of correspondence between w in the Bruhat order of $\mathfrak{S}_3$ and TNN upper-triangular matrices. The zero minors are in a blue square.

11.2 Fomin-Zelevinsky double wiring diagrams and double Bruhat cells

We take two permutations u and w and *shuffle* two reduced decompositons of these permutations.

Example 11.9. $u^{-1} = s_1s_2$ and $w^{-1} = s_2s_3s_1$. □

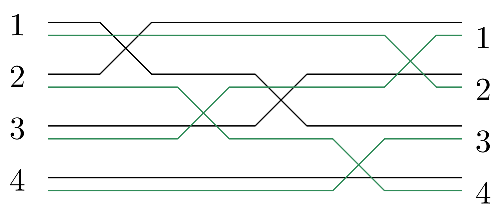


Figure 29: Example of double wiring diagram.

We can convert these double wiring diagrams into **trivalent graphs** as shown in Figure 30.

We have two types of Jacobi matrices $J_i(x)$ (x on the $(i, i+1)$ entry) as before and $J_{\bar{i}}(x)$ which is the identity matrix and x on the $(i+1, i)$ entry. We get an analogous decomposition as in the case of single Bruhat cells.

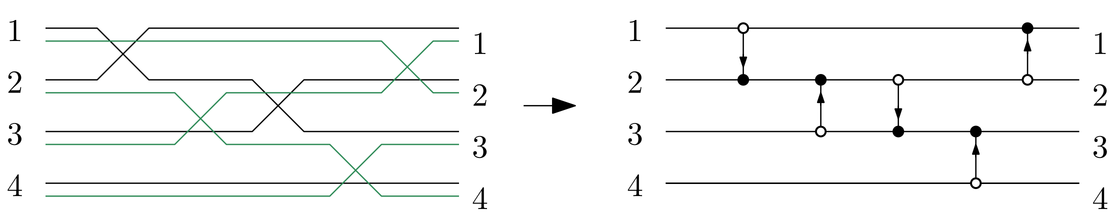


Figure 30: Going from double wiring diagram to trivalent graph.

Example 11.10. Continuing from Example 11.9, $M = J_1(x_1)J_{\bar{2}}(x_2)J_2(x_3)J_{\bar{3}}(x_4)J_{\bar{1}}(x_5)$. □

Definition 11.11. For $u, w \in \mathfrak{S}_n$ with reduced decompositions $u = s_{i_1}s_{i_2}\cdots s_{i_{\ell(u)}}$ and $w = s_{j_1}s_{j_2}\cdots s_{j_{\ell(w)}}$ the **double Bruhat cell** is

$$B_{u,w} = \{J_{i_1}(x_1)J_{\overline{j_1}}(t_1)J_{i_1}(x_2)J_{\overline{j_1}}\cdots \mid x_1, x_2, \ldots, x_{\ell(u)}, t_1, t_2, \ldots, t_{\ell(w)} > 0\}$$

. □

Theorem 11.12 (Lusztig, Fomin-Zelevinsky)**.**

1. *The TNN of GL_n decompose as* $\coprod_{u,w\in\mathfrak{S}_n} B_{u,w}$*,*
2. $B_{u,w} \cong \mathbb{R}_{>0}^{\ell(u)+\ell(w)+n}$*, and the isomorphism is* $J_{i_1}(x_1)\cdots J_{i_\ell}(x_\ell) \mapsto (x_1, x_2, \ldots, x_{\ell(u)}, t_1, t_2, \ldots, t_{\ell(w)})$*.*
3. *The closure* $\overline{B}_{u,w} \subseteq \overline{B}_{v,z}$ *if and only if* $u \leq v$ *and* $w \leq z$ *in the strong Bruhat order.*

11.3 Totally nonnegative Grassmannian

Definition 11.13. $\mathbf{Gr}_{\geq 0}(k, n, \mathbb{R})$ are the elements in $\mathbf{Gr}(k, n, \mathbb{R})$ such that all Plücker coordinates $\Delta_I \geq 0$ (we use only maximal minors). □

In this setting the matroid strata are

$$S_{\mathcal{M}}^{>0} = \{A \in \mathbf{Gr}(k, n) \mid \Delta_I(A) > 0 \text{ for } I \in \mathcal{M}, \Delta_J(A) = 0 \text{ for } J \notin \mathcal{M}\}.$$

We also have that $\mathbf{Gr}_{\geq 0}(k, n) = \coprod S_{\mathcal{M}}^{>0}$.

Recall that for $\mathbf{Gr}(k, n)$ the matroid stratification can get complicated (recall Lecture **??**, Mnëv's Universality Theorem,...) but the TNN Grassmannian has a "nice" stratification.

11.4 Relation to classical total positivity

Let B be a $k \times (n-k)$ matrix. From this matrix we obtain a $k \times n$ matrix A such that there is a one-to-one correspondence between minors of B (of all sizes) and maximal minors of A (and with the same sign).

$$B = \begin{bmatrix} b_{11} & \cdots & b_{1\,n-k} \\ \vdots & & \vdots \\ b_{k1} & \cdots & b_{k\,n-k} \end{bmatrix}, \quad A = \left[\begin{array}{ccccc|cccc} 1 & & & & & \pm b_{k1} & \pm b_{k2} & \cdots & \pm b_{k\,n-k} \\ & \ddots & & & & & & \cdots & \\ & & 1 & & & b_{31} & b_{32} & \cdots & b_{3\,n-k} \\ & & & 1 & & -b_{21} & -b_{22} & \cdots & -b_{2\,n-k} \\ & & & & 1 & b_{11} & b_{12} & \cdots & b_{1\,n-k} \end{array}\right]$$

So the classical total positivity embeds on the TNN Grassmannian. Moreover, there is a symmetry feature in the latter (take first column and place it at the end and change sign by $(-1)^{k-1}$. This operation does not change the TNN Grassmannian. There is no such operation in the classical setting.

Thanks: to Darij Grinberg, Nan Li and Tom Roby for comments, proofreading and board photos!