PROBLEM SET 2 due on April 27, 2004

Problem 1. Let W be a reflection group of rank r with connected Coxeter graph. Calculate the number of elements of length 2 in W as a function of r.

Problem 2. Let W be the Weyl group of type B_n. Its elements are represented by signed permutations of size n. Give a combinatorial interpretation of the length $\ell(w)$ for $w \in W$ in terms of signed permutations. Prove the formula $\sum_{w\in W} q^{\ell(w)} = [2]_q[4]_q \cdots [2n]_q$ combinatorially.

Problem 3. For root systems of types A_r, B_r, C_r and D_r, calculate the index $[P : Q]$ of the root lattice Q in the weight lattice P directly from the definitions.

Problem 4. Let Φ be an irreducible crystallographic root system. Let $\alpha_1, \dots, \alpha_r \in \Phi$ be simple roots, and let $\theta = c_1\alpha_1 + \cdots + c_r\alpha_r \in \Phi$ be the highest root. Prove that the order $[P : Q]$ of the root lattice Q in the weight lattice P equals the number $f = 1 + \#\{i \mid c_i = 1\}$.

Problem 5. Classify all simple connected graphs with vertices weighted by positive integers c_v such that (1) for any vertex v, the weight c_v is $1/2$ of the sum of weights of all neighbors of v; (2) there is a vertex of weight 1. Prove that only graphs in your list satisfy these properties.

Problem 6. Let T_n be the set of trees on $n+1$ labelled vertices with edges labelled by $1, \dots, n$. So $|T_n| = n!\,(n+1)^{n-1}$. (Why?) Let $b_1, \dots, b_{n-1}$ be the maps $T_n \to T_n$ defined as follows. (1) If the edges labelled i and $i+1$ are not adjacent then b_i switches the labels of these edges. (2) If the edges labelled i and $i+1$ are adjacent then b_i transforms these edges as follows:

and leaves the remaining edges.

(a) Show that the operations $b_1, \dots, b_{n-1}$ satisfy the *braid relations*

$$b_i\, b_{i+1}\, b_i = b_{i+1}\, b_i\, b_{i+1} \qquad \text{and} \qquad b_i\, b_j = b_j\, b_i, \quad \text{if } |i-j| \geq 2.$$

(b) According to (a), the braid group Br_n acts on the set T_n. Describe the orbits of this action.
(c) Describe action of the element $b_1 b_2 \cdots b_{n-1} \in Br_n$ on T_n explicitly.
(d) Calculate the index of the subgroup $H = \{\pi \in Br_n \mid \pi(T) = T \text{ for any } T \in T_n\}$ in Br_n.

Problem 7. For a root system $\Phi \in V$ with a choice of simple roots $\alpha_1, \dots, \alpha_r$, the *divided difference operators* $\partial_i : \mathbb{Q}[V^*] \to \mathbb{Q}[V^*]$ are given by $\partial_i = f \mapsto (f - s_i(f))/\alpha_i$. Also $\partial_w = \partial_{i_1} \cdots \partial_{i_l}$ for a reduced decomposition $w = s_{i_1} \cdots s_{i_l} \in W$. Show that

$$\partial_{w_\circ}\left(\frac{1}{|W|}\prod_{\alpha>0}\alpha\right) = 1,$$

where $w_\circ$ is the longest element in W.

Easier variant: Prove this claim for type A root systems.

Problem 8. For a reduced decomposition $w = s_{i_1} \cdots s_{i_l}$ in the symmetric group S_n, let us say that (a, b, c) is an *allowed triple* if $1 \leq a < b < c \leq l$; $(s_{i_a}, s_{i_b}, s_{i_c}) = (s_i, s_{i\pm1}, s_i)$ for some i; and all s_{i_j}, for $a < j < c$, $j \neq b$, commute with s_i.

(a) Find the number of permutations $w \in S_n$ such that any (equivalently, at least one) reduced decomposition for w has no allowed triples. Can you characterize this class of permutations in terms of pattern avoidance?

(b) Generalize this class elements to any Coxeter group and calculate their numbers for types B_n, and D_n. (These elements are called *fully commutative elements.*)

(c) Prove that any reduced decomposition of the longest permutation $w_\circ \in S_n$ has at least $n-2$ allowed triples.

Problem 9. Suppose that Φ is of type A_{n-1}. In this case, the weights that occur in V_ρ with nonzero multiplicities are exactly the integer points inside the permutohedron $\mathrm{Conv}((w_1, \ldots, w_n) \mid w \in S_n)$.

(a) Show that the number of weights that occur in V_ρ equals the number of forests on n labelled vertices.

(b) Derive a similar combinatorial interpretation for the number of weights that occur in $V_{k\rho}$, where k is a positive integer.

(c) For a dominant weight $\lambda = (\lambda_1, \ldots, \lambda_n) \in P^+$, let $n(\lambda)$ be the number of weights that occur in V_λ. What can you say about the function $n(\lambda)$? Is it true that $n(\lambda)$ is a polynomial in the variables $\lambda_1, \ldots, \lambda_n$?

Problem 10. For a permutation $w \in S_n$, the *code* $c(w) = (c_1, \ldots, c_n)$ of w is defined by $c_i = \#\{j \mid j > i,\ w_j < w_i\}$. Prove that $w \mapsto c(w)$ is a bijection between the set of 132-avoiding permutations in S_n and the set $\mathcal{C}_n = \{(c_1, \ldots, c_n) \mid c_1 \leq c_2 \leq \cdots \leq c_n;\ 1 \leq c_i \leq n-i \text{ for } i = 1, \ldots, n\}$. Note that the set $\mathcal{C}_n$ is the set of partitions whose Young diagram fits inside the staircase partition $(n-1, n-2, \ldots, 0)$.