PROBLEM SET 1 due on March 15, 2004

Problem 1. Let P be either the Young lattice $\mathbb{Y}$ or the Fibonacci lattice $\mathbb{F}$. Let λ be an element of P of rank n, i.e., the length of a shortest path from $\hat{0}$ to λ is n. Let $N(\lambda, m)$ be the number of oscillating paths from $\hat{0}$ to λ of length m. Prove bijectively that

$$N(\lambda, n+2k) = \binom{n+2k}{2k} \cdot (2k-1)!! \cdot N(\lambda, n),$$

for $k \geq 0$. Here $(2k-1)!! = (2k-1)(2k-3)\cdots 3 \cdot 1$.

Hint: The number $\binom{n+2k}{2k} \cdot (2k-1)!!$ is the number of partial matchings in the graph K_{n+2k} with k edges.

Problem 2. Let f_λ be the number of standard Young tableaux of shape λ. Show that

$$\sum_{\lambda \,:\, |\lambda|=n} f_\lambda = \#\{w \in S_n \mid w^2 = id\}.$$

Problem 3. For any $n, k \geq 0$, calculate the number

$$A_{nk} = \sum_{\lambda \subseteq n^k} \#SSYT(\lambda, k) \times \#SSYT(\lambda', n),$$

where the sum is over partitions $\lambda = (\lambda_1, \dots, \lambda_k)$, $n \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$, whose Young diagram fits inside the $k \times n$-rectangle, λ' is the conjugate partition (with transposed Young diagram), and $\#SSYT(\lambda, k)$ is the number of semi-standard Young tableaux of shape λ with entries in $\{1, \dots, k\}$. Can you find a bijective proof?

Problem 4. Let Z_{n-1} be the center of the group algebra $\mathbb{C}[S_{n-1}]$, let $Z_{n-1,1} = \{a \in \mathbb{C}[S_n] \mid aw = wa \text{ for any } w \in S_{n-1}\}$ be the centralizer of the subalgebra $\mathbb{C}[S_{n-1}]$ in $\mathbb{C}[S_n]$. Let $X_n = (1,n) + (2,n) + \cdots + (n-1,n)$ be the last Jucys-Murphy element in $\mathbb{C}[S_n]$. Show that the centralizer $Z_{n-1,1}$ is the subalgebra of $\mathbb{C}[S_n]$ generated by Z_{n-1} and X_n.

Problem 5. For $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, the Schur polynomial s_λ can be written, using Gelfand-Zetlin patterns, as

$$s_{(\lambda_1,\lambda_2,\lambda_3)}(t_1, t_2, t_3) = \sum_{\mu_1,\mu_2,\nu_1} t_1^{\nu_1} t_2^{\mu_1+\mu_2-\nu_1} t_3^{\lambda_1+\lambda_2+\lambda_3-\mu_1-\mu_2},$$

where the sum is over μ_1, μ_2, ν_1 such that $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3$ and $\mu_1 \geq \nu_1 \geq \mu_2$. Use this formula to show that $s_{(\lambda_1,\lambda_2,\lambda_3)}$ is a symmetric polynomial in t_1, t_2, t_3.

Problem 6. Recall that a horizontal k-strip is a set-theoretic difference of two Young diagrams that consists of k boxes such that each column contains at most one box. Let D_k and U_k, $k \geq 0$, be the operators acting on the space $\mathbb{C}^{\mathbb{Y}}$ of formal linear combinations of Young diagrams by

$$U_k : \lambda \longmapsto \sum_{\mu \,:\, \mu/\lambda \text{ is a horizontal } k\text{-strip}} \mu$$

$$D_k : \lambda \longmapsto \sum_{\mu \,:\, \lambda/\mu \text{ is a horizontal } k\text{-strip}} \mu$$

Use a bijective argument to show that the operators U_k and D_l satisfy the following commutation relations:

$$U_k \cdot U_l = U_l \cdot U_k, \qquad D_k \cdot D_l = D_l \cdot D_k,$$

$$D_k \cdot U_l = \sum_{j=0}^{\min(k,l)} U_{l-j} \cdot D_{k-j}.$$

Problem 7. Let U be a linear space and let $\mathrm{Sym}^N(U) = \{S_N\text{-invariant elements of } U^{\otimes N}\}$ be its N-th symmetric power. Show that

$$\mathrm{Sym}^N(U) = \text{linear span of } \{u \otimes \cdots \otimes u \mid u \in U\} \subset U^{\otimes N}.$$

Problem 8. Let $X = (x_{ij})$, $i = 1, \ldots, k$, $j = 1, \ldots, n$, be the $k \times n$-matrix with variable entries x_{ij}. Let $\mathbb{C}[X]$ be the polynomial ring in the $k \cdot n$ variables x_{ij}. The group $GL_k \times GL_n$ acts on polynomials $f(X) \in \mathbb{C}[X]$ by $(A, B) : f(X) \mapsto f(AXB^{-1})$, for $A \in GL_k$ and $B \in GL_n$. Show that these two actions centralize each other (using only this definition). In other words, show that every liner operator $C : \mathbb{C}[X] \to \mathbb{C}[X]$ that commutes with the action of GL_k is a linear combination of operators that came from the action of GL_n, i.e., the operator C has the form $C : f(X) \mapsto f(XB_1^{-1}) + \cdots + f(XB_r^{-1})$, for some $B_1, \cdots, B_r \in GL_n$.

Problem 9. Let $X = (x_{ij})$ be as above. Let $\Lambda[X]$ be the exterior algebra in the $k \cdot n$ variables x_{ij}. The group $GL_k \times GL_n$ acts on elements $f(X) \in \Lambda[X]$ by $(A, B) : f(X) \mapsto f(AXB^{-1})$, for $A \in GL_k$ and $B \in GL_n$. Show that this $GL_k \times GL_n$-module decomposes into the irreducible representations as

$$\Lambda[X] \simeq \sum_{\lambda \subseteq n^k} V_\lambda^{(k)} \otimes V_{\lambda'}^{(n)},$$

where the sum is over partitions λ whose Young diagram fits inside the $k \times n$-rectangle, λ' is the conjugate partition, and $V_\lambda^{(k)}$ denotes the irreducible representation of GL_k with highest weight λ.

Problem 10. Fix two parameters x, y. For a partition λ and a partition μ, whose Young diagram is obtained from λ by adding a box, i.e., $\lambda \nearrow \mu$, define

$$p_{x,y}(\lambda, \mu) = \frac{(c(\mu/\lambda) + x) \cdot (c(\mu/\lambda) + y)}{(|\lambda| + xy)} \cdot \frac{f_\mu}{(|\lambda| + 1) \cdot f_\lambda},$$

where f_λ is the number of standard Young tableaux of shape λ and $c(\mu/\lambda)$ is the content of the added box. Show that, for any λ, we have

$$\sum_{\mu \,:\, \lambda \nearrow \mu} p_{x,y}(\lambda, \mu) = 1.$$