7 OPEN PROBLEMS IN COMBINATORICS

Problem 1 (see Catalan addendum[1] 6.C3) Start with a monomial $\mathbf{x}$ in the variables x_{ij}, $i < j$, and repeatedly apply the following reduction rule

$$x_{ij}x_{jk} \to x_{ik}x_{ij} + x_{jk}x_{ik} \qquad \text{for } i < j < k$$

in any order until unable to do so. For example, for $\mathbf{x} = x_{12}x_{23}x_{24}$, we have

$$x_{12}x_{23}x_{24} \to x_{13}x_{12}x_{24} + x_{23}x_{13}x_{24} \to x_{13}x_{14}x_{12} + x_{13}x_{24}x_{14} + x_{23}x_{13}x_{24}.$$

(A) *Show that the number of terms $N(\mathbf{x})$ in the final result depends only on monomial $\mathbf{x}$ and does not depend on the order in which the reduction rule is applied.*

For example, $N(x_{12}x_{23}x_{24}) = 3$. Notice that, in general, the resulting polynomial depends on the order in which reductions are applied.

(B) *Show that $N(x_{12}x_{23}x_{34}\cdots x_{n\,n+1})$ is equal to the Catalan number C_n.*

(C) *For $n \geq 1$, show that*

$$N\left(\prod_{1\leq i<j\leq n+1} x_{ij}\right) = C_1 \cdot C_2 \cdots C_n$$

(product of the Catalan numbers).

(D) (open problem) *Find a combinatorial proof of* (C).

Problem 2 (Laurent phenomenon[2] & Somos sequences) For a positive integer k and a polynomial $f(u_1, \ldots, u_{k-1})$ in $k-1$ variables. Let us define the sequence $\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$, infinite in both directions, such that $a_1 = x_1, a_2 = x_2, \ldots, a_k = x_k$, where $x_1, \ldots, x_k$ are variables, and all other a_i's are recursively given by

$$a_i a_{i+k} = f(a_{i+1}, \ldots, a_{i+k-1}) \qquad \text{for any } i.$$

Let us say that this is a *Somos-type* recurrence relation if all a_i's are Laurent polynomials in $x_1, \ldots, x_k$ with positive integers coefficients.

Here are a few examples of Somos-type relations:

$$a_i a_{i+2} = 1 + a_i^2 \qquad \textit{(Prove that these are even Fibonacci numbers.)}$$

$$a_i a_{i+3} = 1 + a_{i+1}a_{i+2} \qquad \textit{(Find general formula.)}$$

$$a_i a_{i+4} = a_{i+1}a_{i+3} + a_{i+2}^2 \qquad \textit{(Somos-4)}$$

(open problem) *Describe all Somos-type recurrence relations or find a general class of Somos-type relations.*

[1] Catalan addendum is available at `http://www-math.mit.edu/~rstan/ec/`

[2] see S. Fomin, A. Zelevinsky: The Laurent phenomenon, *Adv. Applied Math.* 28 (2002), 119-144. Available at `http://www.math.lsa.umich.edu/~fomin/papers.html`

Problem 3 (Gluings of the $4n$-gon) A surface of *genus* n is topologically a sphere with n handles. For example, a genus 0 surface is a usual sphere, genus 1 surface is a torus, etc. It is known[3] that the number of ways to glue sides of the $4n$-gon into a surface of genus n equals

$$\frac{1 \cdot 3 \cdot 5 \cdots (4n-1)}{2n+1}.$$

(open problem) *Find a combinatorial proof of this formula.*

Problem 4 Let $\nu_2(m)$ be the exponent of the largest power of 2 dividing number m.

(A) *Show that, for the Catalan number C_n, the number $\nu_2(C_n)+1$ is equal to the sum of digits in the binary form of $n+1$.*

Let D_n be the expansion coefficients of the continued fraction

$$\cfrac{1}{1-\cfrac{1^2x}{1-\cfrac{3^2x}{1-\cfrac{5^2x}{1-\cfrac{7^2x}{1-\cdots}}}}} = \sum_{n\geq 0} D_n x^n = 1 + x + 10x^2 + 325x^3 + 22150x^4 + \cdots.$$

Conjecture $\nu_2(D_n) = \nu_2(C_n)$, *where C_n is the Catalan number.*

(B) (open problem) *Prove the conjecture.*

Problem 5 (see [EC2][4], Problem 7.68(e)) For a permutation $w \in S_n$, let $\kappa(w)$ denotes the number of cycles in w.

(open problem) *Find a combinatorial proof of the formula*

$$\frac{1}{n!}\sum_{u,v\in S_n} q^{\kappa(uvu^{-1}v^{-1})} = \sum_{|\lambda|=n}\prod_{t\in\lambda}(q+c(t)),$$

where the sum is over all partitions λ of n, the product is over all boxes t in the Young diagram of λ, and $c(t)=j-i$ for a box t with coordinates (i,j). ($c(t)$ is called content of t.)

Problem 6 (Triangulations of the product of two simplices) For $m,n \geq 1$, the product of two simplices $P_{m,n} = \Delta^{m-1}\times\Delta^{n-1}$ is the convex hull of the points $v_{ij} = e_i + e_{m+j}$, $i = 1,\ldots,m$ and $j = 1,\ldots,n$, where $e_1,\ldots,e_{m+n}$ are coordinate vectors. The dimension of the polytope $P_{m,n}$ is $m+n-2$.

(A) *Show that the convex hull of a subset S of the vertices v_{ij} is a simplex of maximal dimension $m+n-2$ if and only if the vertices v_{ij} in S correspond to edges (i,j) of a spanning tree in the complete bipartite graph $K_{m,n}$.*

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[3] J. Harer, D. Zagier, *"The Euler characteristic of the moduli space of curves,"* Invent. Math. **85** (1986), no. 3, 457–485.

[4] R. Stanley, *Enumerative Combinatorics*, vol. 2., Cambridge University Press, 1999.

Thus there are exactly $m^{n-1}n^{m-1}$ such simplices. A collection of such simplices of maximal dimension is called a *triangulation* of $P_{m,n}$ if and only if (i) the union of the simplices is $P_{m,n}$ and (ii) the simplices do not have common interior points.

(B) *Show that $P_{1,n}$ has 1 triangulation and $P_{2,n}$ has $n!$ triangulations.*

(C) (open problem) *Find the number of triangulations of $P_{3,n}$ and describe all these triangulations. More generally, describe all triangulations of $P_{m,n}$.*

Problem 7 (Laurent phenomenon for planar graphs) Let G be a planar oriented graph such that, for each vertex v, the outdegree of v equals the indegree v and the incoming edges interlace with the outgoing edges when we go around the vertex v. We will call such graphs *interlaced.*

For a vertex v of degree is 4 in G (indegree(v) = outdegree(v) = 2) with adjacent vertices v_1, v_2, v_3, v_4 (in the clockwise order), let us define *mutation* $\widetilde{G} = M_v(G)$ of the graph G as the interlaced graph obtained G by switching the directions of the 4 edges at the vertex v and changing the edges between adjacent vertices so that v_i is connected with v_j in $\widetilde{G}$ if and only if v_i is not connected with v_j in G, for any pair $(i,j) = (1,2),(2,3),(3,4),(4,1)$. (The directions of new edges are uniquely determined by the interlacing condition.) All other edges are remained intact.

Let us assign the commuting variables $x_{v,G}$ to all vertices v in an interlaced graph G. Suppose that these variables change when we mutate graphs according to the following rule:

$$x_{v,\widetilde{G}} = \frac{x_{v_1,G} \cdot x_{v_3,G} + x_{v_2,G} \cdot x_{v_4,G}}{x_{v,G}},$$

$$x_{w,\widetilde{G}} = x_{v,G} \qquad \text{for any vertex } w \neq v,$$

where $\widetilde{G} = M_v(G)$.

Conjecture. *Fix an interlaced graph G and fix the variables $x_v = x_{v,G}$. Let G_2 be any graph obtained from G by a sequence of mutations. Then the variables $y_v = x_{v,G_2}$ are expressed by Laurent polynomials with nonnegative coefficients in the variables x_v. These polynomials depend only on the graphs G and G_2 and do not depend on choice of a chain of mutations that connects G and G_2.*

Apriori, we can only say that the y_v can be written as *rational expressions* in the x_v.

(open problem) *Prove the conjecture.*