18.212 Problem Set 3 (due Wednesday, May 10, 2023)

Each problem is 10 points.

Problem 1. A *3-valent tree* is a tree T such that, for any vertex v in T, the degree $\deg_T(v)$ equals 3 or 1. For $n = 2, 3, 4$, the numbers of 3-valent spanning trees in K_n are $1, 0, 4$, respectively.

Find a closed-form expression for the number of 3-valent spanning trees of the complete graph K_n, for any n.

Problem 2. Let $G_n = (V, E)$ be the graph on the set of vertices $V = \{1, 2, \ldots, 2n\}$ such that two vertices $i, j \in V$, $i \neq j$, are connected by an edge in G if an only if the product $i \cdot j$ is even. Find a closed-form expression for the number of spanning trees of the graph G_n.

Problem 3. Consider the resistor network given by the complete graph K_n such that the resistance of every edge of K_n equals 1 Ohm. Find the effective resistance between the first and the last vertex in this network.

Problem 4. Consider the chip-firing game on the $(2N+1)$-chain graph graph $G = (V, E)$, where $V = \{-N, -N+1, \ldots, N-1, N\}$ and $E = \{\{i, i+1\} \mid -N \leq i < N\}$, with the sink vertex $q = N$.

Let c_{init} be the initial chip configuration given by

$$c_{\text{init}} = (c_{-N}, \ldots, c_N) = (0, \ldots, 0, n, 0, \ldots, 0).$$

(Here $c_0 = n$ and all other c_i's are 0.) Assume that $N > n$.

Start with the initial configuration c_{init} and fire vertices of the graph until we get a stable chip configuration c_{stab}.

(a) Find c_{stab}.

(b) Prove that the number of firings one needs to perform to get from c_{init} to c_{stab} is independent of the choice of order of firings.

(c) Find the number of firings one needs to perform to get from c_{init} to c_{stab}.

For example, for $n = 4$, one possible choice of firings of vertices is $0,\ 0,\ -1,\ 1,\ 0$. (First, we fire vertex 0, then fire vertex 0 again, then fire vertex -1, etc.) This produces the stable configuration $c_{\text{stab}} = (0,\ldots,0,1,1,0,1,1,0,\ldots,0)$. The number of firings is 5.

Problem 5. Define a *symmetric labelled tree* as a tree $T = (V, E)$ on the set vertices $V = \{-n, -n+1, \ldots, -1, 0, 1, \ldots, n\}$ such that if $\{i, j\} \in E$ then $\{-i, -j\} \in E$. There is 1 symmetric tree on 3 vertices, and 5 symmetric trees on 5 vertices.

Find a closed-form expression for the number of symmetric trees on $2n + 1$ vertices, for any n.

Problem 6. Recall that a *complete binary tree* is a rooted tree such that each non-leaf vertex has exactly two children designated as the left child and the right child. The number of (unlabelled) complete binary trees with $2n + 1$ vertices equals the Catalan number C_n.

Define an *increasing complete binary tree* as a complete binary tree with vertices labelled by $1, 2, \ldots, 2n+1$ such that (a) the root is labelled by 1; and (b) for any vertex v, the labels of vertices in the shortest path from the root to v increase.

A permutation $w = w_1, w_2, \ldots, w_m$ in S_m is called *alternating* if $w_1 < w_2 > w_3 < w_4 > \cdots$.

Prove that the number of increasing complete binary trees on $2n+1$ vertices equals the number $A(2n + 1)$ of alternating permutations in S_{2n+1}.

(Note that the structure of an increasing complete binary tree has more data than the structure a spanning tree of K_{2n+1}. Namely, for $i, j \in [2n + 1]$, we need to know whether the vertex labelled i is the left (resp., right) child of the vertex labelled j. For example, there are two different increasing complete binary trees on 3 vertices. But both of these trees correspond to the same spanning tree of K_3.)

Problem 7. An *extreme point* of a set $S \subset \mathbb{R}^N$ is a point $p \in S$ such that there does not exist a line segment $[a, b] \subset S$ (with $a \neq b$) with midpoint $(a + b)/2 = p$. (If S is a convex polytope, then its extreme points are exactly its vertices.)

Find the number of extreme points of the set of $n \times (n+1)$-matrices $A = (a_{ij})$ with real entries a_{ij} such that

- $a_{ij} \geq 0$, for any i, j;
- all row sums of A are equal to $n + 1$;
- all column sums of A are equal to n.

(Here we consider the space of all real $n \times (n+1)$-matrices as $\mathbb{R}^{n(n+1)}$.)

For example, for $n = 2$, such matrices have the form

$$A = \begin{pmatrix} x & y & 3-x-y \\ 2-x & 2-y & x+y-1 \end{pmatrix},$$

where x, y satisfy the linear inequalites $x \geq 0$, $y \geq 0$, $3-x-y \geq 0$, $2-x \geq 0$, $2-y \geq 0$, $x+y-1 \geq 0$. These inequalities describe the hexagon in the xy-plane with the vertices $(1,0)$, $(0,1)$, $(2,0)$, $(0,2)$, $(2,1)$, $(1,2)$. So the set of such matrices has 6 extreme points:

$$\begin{pmatrix} 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

Problem 8. Find a closed-form expression for the deteminant of the $n \times n$ matrix

$$\begin{pmatrix} C_3 & C_4 & C_5 & \cdots & C_{n+2} \\ C_4 & C_5 & C_6 & \cdots & C_{n+3} \\ C_5 & C_6 & C_7 & \cdots & C_{n+4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{n+2} & C_{n+3} & C_{n+4} & \cdots & C_{2n+1} \end{pmatrix},$$

where $C_m := \frac{1}{m+1}\binom{2m}{m}$.