18.212 Problem Set 2 (due Monday, April 10, 2023)

Each problem is 10 points.

Problem 1. Construct a bijection between 213-avoiding permutations of size n and Dyck paths with $2n$ steps.

Problem 2. For any n, calculate the number of permutations $w \in S_n$ that avoid 2 patterns 123 and 4321 (i.e., w should be both 123-avoiding and 4321-avoiding).

Problem 3. Let U and D be the up and down operations acting on the space $\mathbb{R}[\mathbb{Y}]$ of formal linear combinations of Young diagrams. Show that the coefficient of $\emptyset$ (the empty Young diagram) in $(U+D)^{2n}(\emptyset)$ equals $(2n-1)!! := 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)$.

Problem 4. Let $A(n)$ be the number of partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l)$ of n such that λ_i is not divisible by 3 for any i. Let $B(n)$ be the number of partitions $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_l)$ of n such that $\mu_i \neq \mu_{i+2}$ for any i. (In other words, μ cannot have 3 equal parts.) Prove bijectively that $A(n) = B(n)$.

Problem 5. In class, we defined the Fibonacci lattice $\mathbb{F}$ and labelled its elements by 12-compositions, i.e., compositions $c = (c_1, \ldots, c_l)$ with all parts c_i equal 1 or 2. This construction of $\mathbb{F}$ is based on a recursive procedure. Find an explicit non-recursive description of the covering relation $c \lessdot c'$ in $\mathbb{F}$, where c and c' are 12-compositions.

Problem 6. The *q-Pochhammer symbol* is defined as follows:

$$(x;q)_n := (1-x)(1-xq)(1-xq^2)\cdots(1-xq^{n-1}),$$

and $(x;q)_0 = 1$. Prove the identity

$$(x;q)_n = \sum_{k=0}^{n} q^{k(k-1)/2} \left[\begin{array}{c} n \\ k \end{array} \right]_q (-x)^k.$$

Problem 7. Let $(q;q)_n := (1-q)(1-q^2)\cdots(1-q^n)$, and also define $(q;q)_\infty := \prod_{k=1}^{\infty}(1-q^k)$.

Let us fix a nonnegative integer r. Prove the identity

$$\frac{1}{(q;q)_\infty} = \sum_{k\geq 0} \frac{q^{k(k+r)}}{(q;q)_k\,(q;q)_{k+r}}.$$

Hint for problems 6 and 7: Try to interpret the identities in terms of partitions.

Bonus problems:

Problem 8. Prove the identity:

$$\prod_{k\geq 1} \frac{(1-q^k)}{(1+q^k)} = 1 + 2\sum_{n\geq 1}(-1)^n\, q^{n^2}.$$

Problem 9. The lattice of non-crossing partitions NC_n is the poset whose elements are non-crossing set partitions of $[n]$ ordered by refinement. (NC_n is a subposet of the partition lattice Π_n.) Find the number of saturated chains $(\hat{0} \lessdot a_1 \lessdot a_2 \lessdot \cdots \lessdot a_{n-2} \lessdot \hat{1})$ in NC_n.