Symplectic actions of two-tori on four-manifolds

Alvaro Pelayo
MIT

Assumptions:

- (M, σ) symplectic $2n$-manifold.
- T torus.
- T acts effectively, symplectically.
- M compact, connected.

Terminology:

- σ *symplectic*: $\sigma \in \Omega^2(M)$, $\quad d\sigma = 0$, $\quad$ non-degenerate.
- *Effectively*:
$$\bigcap_{x \in M} T_x = \{e\},$$
where
$$T_x := \{t \in T \mid t \cdot x = x\}$$
is *stabilizer at* x.
- *Symplectically*: by symplectomorphisms, i.e. diffeomorphisms
$$\phi\colon M \to M, \quad \phi^*\sigma = \sigma.$$

Symplectic Hamiltonian Actions:

- **Definition.** $T \times M \to M$ is *Hamiltonian* if $\exists\, \mu \colon M \to \mathfrak{t}^*$,

$$\mathrm{d}\langle \mu, X \rangle = \mathrm{i}_{X_M} \sigma, \quad X \in \mathfrak{t}.$$

$$X_M := \text{v.f. infinitesimal action of } X \text{ on } M.$$

- **Atiyah-Guillemin-Sternberg** (1982):

$$\mu(M) = \text{convex hull } \{\mu(\text{fixed point set})\}.$$

- **Delzant** (1988): If $\dim(T) = n$,
  - $\mu(M)$ Delzant polytope (simple, edge-$\mathbb{Q}$, smooth).
  - $\mu(M)$ classifies (M, σ, T) (uniqueness & existence).

Example.

- $(S^2,\, \sigma = \mathsf{d}\theta \wedge \mathsf{d}h)$.
- Rotational action $\mathbb{R}/\mathbb{Z}$.
- Hamiltonian, momentum map $\mu \colon S^2 \to \mathbb{R}$

$$\mu(\theta,\, h) = h.$$

- Polytope $\Delta = [-1,\, 1]$.

Example.

- $(\mathbb{CP}^n,\, \lambda \cdot \sigma_{\mathsf{FS}})$, rotational $\mathbb{T}^n$-action.
- Hamiltonian, momentum map components

$$\mu_k^{\mathbb{CP}^n}(z) = \frac{\lambda |z_k|^2}{\sum_{i=0}^n |z_i|^2}.$$

- Polytope $\Delta =$ convex hull $\{\, 0,\, \lambda e_1, \ldots, \lambda e_n \,\}$.

Talk Goal.

To classify symplectic T-actions on (M, σ) if:

Case 1: *principal T-orbits are coisotropic.*

Case 2: $\exists$ dim T*-dimensional, symplectic T-orbit*.

Case 3: dim $T = 2$, dim $M = 4$.

Case 1: Principal T-orbits coisotropic.

- *Principal orbit type*:

$$M_{\mathsf{reg}} = \{\, x \in M \;\mid\; T_x = \{e\}\,\}.$$

- $C \subset M$ is *coisotropic* if

$$(\mathsf{T}_x C)^{\sigma_x} \subset \mathsf{T}_x C, \quad \forall x \in C.$$

- $C \subset M$ is *Lagrangian* if

$$\sigma|_C = 0, \quad \dim C = \frac{\dim M}{2}$$

(Special coisotropic when $\subset$ is $=$).

Example 1 (*Kodaira-Thurston*).

- $(j_1, j_2) \in \mathbb{Z}^2$ acts on $(\mathbb{R}/\mathbb{Z})^2$:

$$\begin{pmatrix} 1 & j_2 \\ 0 & 1 \end{pmatrix}$$

- $(\mathbb{R}^2 \times_{\mathbb{Z}^2} (\mathbb{R}/\mathbb{Z})^2,\ \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \mathrm{d}x_2 \wedge \mathrm{d}y_2)$.

- $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ acts symplectically, freely on (x_1, y_2).

- All orbits principal, Lagrangian.

- $\nexists$ Kaehler structure.

Example 2 (Not Hamiltonian, not free).

- $((\mathbb{R}/\mathbb{Z})^2 \times S^2,\ \mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}\theta \wedge \mathrm{d}h)$.

- $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ acts symplectically.

- No fixed points: not Hamiltonian.

- Not free.

- Lagrangian principal orbits.

Theorem (*Duistermaat, P.-*).

$M \simeq G \times_H M_{\mathsf{h}}$, where:

- $G \times_H M_{\mathsf{h}}$ fibration over G/H:
  - *Fiber* $= (M_{\mathsf{h}}, T_{\mathsf{h}})$ symplectic toric.
  - *Base* $= G/H$, (T/T_{h})-bundle over torus $(G/H)/T$.
- G is 2-step Nilpotent Lie.
- $H \leq G$ closed (holonomy).

Theorem (*Duistermaat, P.-*).

Symplectic T-actions with coisotropic principal orbits are classified by:

1) Antisymmetric bilinear $\sigma^{\mathfrak{t}}\colon \mathfrak{t} \times \mathfrak{t} \to \mathbb{R} : \sigma|_{T-\mathsf{orbits}}$.

2) Hamiltonian torus T_{h} & polytope Δ.

3) Period lattice P of maximal subgroup of $N \leq \mathfrak{l}^*$ acting on M/T, where $\mathfrak{l} = \ker(\sigma^{\mathfrak{t}})$.

4) – Bilinear $c\colon N \times N \to \mathfrak{l}$ encoding Chern class of

$$M_{\mathsf{reg}} \to M_{\mathsf{reg}}/T,$$

– a connection holonomy invariant

$$[\tau\colon P \to T]_{\exp(\mathcal{A})} \in \mathrm{Hom}_c(P, T)/\exp \mathcal{A}.$$

Invariants of Kodaira-Thurston's $M = \mathbb{R}^2 \times_{\mathbb{Z}^2} (\mathbb{R}/\mathbb{Z})^2$:

1) Bilinear form: $\sigma^{\mathrm{t}} = 0$.

2) – Hamiltonian torus: $T_{\mathsf{h}} = \{[0, 0]\}$.

– Delzant polytope: $\Delta = \{(0, 0)\}$.

3) Period lattice $P = \mathbb{Z}^2$.

4) – Bilinear $c\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, $\quad c(e_1, e_2) = e_1$.

– Holonomy invariant

$$\tau \in \mathrm{Hom}_c(\mathbb{Z}^2, (\mathbb{R}/\mathbb{Z})^2), \quad \tau_{e_1} = \tau_{e_2} = [0, 0].$$

In this case:

- $G = (\mathbb{R}/\mathbb{Z})^2 \times \mathbb{R}^2$.
- $M_{\mathsf{h}} = \{p\}$.
- $H = \{[0, 0]\} \times \mathbb{Z}^2$.

Hence:

$$G \times_H M_{\mathsf{h}} \;\simeq\; G/H \;\simeq\; \mathbb{R}^2 \times_{\mathbb{Z}^2} (\mathbb{R}/\mathbb{Z})^2.$$

Case 2:

- **dim** $M = 2n$, **dim** $T = 2n - 2$

- $(2n-2)$**-dim. sympl.** T**-orbits**

- $C \subset M$ is symplectic if $\sigma|_C$ is symplectic.

Example 4 (free action).

- $(\mathbb{R}/\mathbb{Z})^2 \times S^2$.
- $(\mathbb{R}/\mathbb{Z})^2$ acts freely, symplectically (left factor).
- T-orbits symplectic 2-tori.
- $M/T = S^2$.

Example 5 (non-free action).

- $S^2 \times_{\mathbb{Z}/2\,\mathbb{Z}} (\mathbb{R}/\mathbb{Z})^2$.
- $\mathbb{Z}/2\,\mathbb{Z}$ on S^2 rotates horizontally 180^o, and antipodal action on first factor of $(\mathbb{R}/\mathbb{Z})^2$.
- $(\mathbb{R}/\mathbb{Z})^2$ acts non–freely, symplectically (right factor).
- T-orbits symplectic 2-tori.
- $M/T = S^2/(\mathbb{Z}/2\,\mathbb{Z})$.

Theorem (*P.*-).

$M \simeq \widetilde{M/T} \times_\Gamma T$ where

- $\widetilde{M/T}$ universal cover, $\Gamma = \pi_1^{\mathrm{orb}}(M/T)$,
- Γ acts on T by monodromy $\mu\colon \Gamma \to T$ of $\{(\mathsf{T}_x(T\cdot x))^{\sigma_x}\}_{x \in M}$.

Theorem (*P.*-).

Symplectic T^{2n-2} on $2n$-manifolds with symplectic, $(2n-2)$-dimensional T-orbits are classified by:

1) Non–degenerate antisymmetric bilinear $\sigma^{\mathfrak{t}}\colon \mathfrak{t} \times \mathfrak{t} \to \mathbb{R}$: $\sigma|_{T-\text{orbits}}$.

2) Fuchsian signature $(g;\, \vec{o})$ encoding M/T.

3) Symplectic area λ of M/T.

4) Monodromy invariant:

$$\mathcal{G}_{(g,\,\vec{o})} \cdot ((\mu_{\mathsf{h}}(\alpha_i),\, \mu_{\mathsf{h}}(\beta_i))_{i=1}^{g},\ \ (\mu_{\mathsf{h}}(\gamma_k))_{k=1}^{n}),$$

where

- $\mathcal{G}_{(g;\,\vec{o})} := \{ \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \mid A \in \mathrm{Sp}(2g,\, \mathbb{Z}),\ \ D \in \mathcal{M}\mathsf{S}_n^{\vec{o}} \}$,
- homology monodromy $\mu_{\mathsf{h}}\colon \mathrm{H}_1^{\mathrm{orb}}(M/T,\, \mathbb{Z}) \to T$,
- $\{\alpha_i, \beta_i\} \subset \mathrm{H}_1^{\mathrm{orb}}(M/T,\, \mathbb{Z})$ symplectic basis maximal free subgroup,
- $\{\gamma_k\} \subset \mathrm{H}_1^{\mathrm{orb}}(M/T,\, \mathbb{Z})$ geometric torsion basis.

Invariants of $M = S^2 \times_{\mathbb{Z}/2\,\mathbb{Z}} (\mathbb{R}/\mathbb{Z})^2$:

1) Bilinear form: $\sigma^{\mathbb{R}^2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

2) Fuchsian signature: $(g;\, \vec{o}) = (0;\, 2,\, 2)$.

3) Symplectic area of $S^2/(\mathbb{Z}/2\,\mathbb{Z})$: 1 (half of S^2 area).

4) Monodromy invariant: $\mathcal{G}_{(0;\,2,\,2)} \cdot (\mu_{\mathsf{h}}(\gamma_1),\, \mu_{\mathsf{h}}(\gamma_2)) =$

$$\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\} \cdot ([1/2,\, 0],\, [1/2,\, 0]).$$

γ_i small loops around poles of S^2.

Then:

- $M/T = S^2/(\mathbb{Z}/2\,\mathbb{Z})$.
- $\pi_1^{\mathsf{orb}}(M/T,\, p_0) = \langle \gamma_1 \,|\, \gamma_1^2 = 1\rangle \simeq \mathbb{Z}/2\,\mathbb{Z}$.
- $\mu\colon \langle \gamma_1 \,|\, \gamma_1^2 = 1\rangle \to T = (\mathbb{R}/\mathbb{Z})^2$, given by
$$\mu(\gamma_1) = [1/2,\, 0].$$

And:

$$\begin{aligned} \widetilde{M/T} \times_{\pi_1^{\mathsf{orb}}(M/T,\, p_0)} T &= S^2 \widetilde{/(\mathbb{Z}/2\,\mathbb{Z})} \times_{\pi_1^{\mathsf{orb}}(S^2/(\mathbb{Z}/2\,\mathbb{Z}),\, p_0)} (\mathbb{R}/\mathbb{Z})^2 \\ &\simeq S^2 \times_{\mathbb{Z}/2\,\mathbb{Z}} (\mathbb{R}/\mathbb{Z})^2 = M. \end{aligned}$$

Case 3: dim $M = 4$, dim $T = 2$.

Theorem (*P.*-). One and only one occurs:

1) (M, σ) symplectic toric 4-manifold.

2) $(M, \sigma) \simeq (\mathbb{R}/\mathbb{Z})^2 \times S^2$.

3) $(M, \sigma) \simeq (T \times \mathfrak{t}^*)/\iota(P)$, where:

- Standard form, standard T-action on $T \times \mathfrak{t}^*$.
- Discrete cocompact $P \subset \mathfrak{t}^*$,
- $\iota(P) \leq T \times \mathfrak{t}^*$ discrete, cocompact, subgroup for non standard structure, in terms of:
  * Chern class: certain bilinear $c \colon \mathfrak{t}^* \times \mathfrak{t}^* \to \mathfrak{t}$,
  * Holonomy invariant:

$$[\tau \colon P \to T]_{\exp(\mathcal{A})} \in \mathrm{Hom}_c(P, T)/\exp \mathcal{A},$$

4) $(M, \sigma) \simeq \widetilde{\Sigma} \times_{\pi_1^{\mathrm{orb}}(\Sigma, p_0)} T$ where

- Symplectic form, T-action induced by product ones.
- Σ good orbisurface.
- $\pi_1^{\mathrm{orb}}(\Sigma, p_0)$ acts on $\widetilde{\Sigma} \times T$ diagonally.

Quick Idea of Proofs.

A few words on proofs of Case 2, Case 3:

Proof of Case 2 involves studying:

- Symplectic normal forms, integrable distributions on M.
- Orbifold coverings of orbit spaces of M/T.
- Symplectic forms on orbit spaces.
- Geometric isomorphisms $\mathrm{H}_1^{\mathrm{orb}}(\Sigma) \to \mathrm{H}_1^{\mathrm{orb}}(\Sigma)$.

Proof of Case 3 uses as stepping stones:

- Case 1.
- Methods from the proof of Case 2.
- A key lemma:

  a) Either all T-orbits symplectic 2-tori.

  b) Or all 2-dimensional T-orbits Lagrangian 2-tori.
- Most proof: unfolding b) into 1), 2), 3) in statement.

<u>Sketch of Proof of Case 1</u>:

Step 1. *The orbit space.*

- M/T is polyhedral $\mathfrak{l}^*$-parallel space, $\mathfrak{l} := \ker(\sigma^{\mathfrak{t}})$.
- As $\mathfrak{l}^*$-parallel spaces,
$$M/T \simeq \Delta \times S$$
Δ Delzant polytope, S torus.
- $\Delta =$ Hamiltonian part, $S =$ free part.

Step 2. *A nice connection.*

- $\exists$ "nice admissible" connection for $\pi\colon M_{\text{reg}} \to M_{\text{reg}}/T$,
$$\xi \in \mathfrak{l}^* \mapsto L_\xi \in \mathcal{X}^\infty(M_{\text{reg}}),$$
  - Simple $[L_\xi, L_\eta]$:
    * $[L_\xi, L_\eta] = c(\xi, \eta)_M, \quad \xi, \eta \in N := (\mathfrak{l}/\mathfrak{t}_{\mathsf{h}})^*$.

      (c encodes Chern class of $\pi\colon M_{\text{reg}} \to M_{\text{reg}}/T$).
    * zero otherwise.
  - Simple $\sigma(L_\xi, L_\eta)$.
- L_ξ, $\xi \in N$, $\exists$ smooth extensions to M.
- L_ξ, $\xi \notin N$, singular on $M \setminus M_{\text{reg}}$.

Step 3. *Distribution by symplectic toric manifolds.*

- Integrable distribution on M

$$D_x := \text{span}\{\, L_\eta(x),\ Y_M(x) \mid Y \in \mathfrak{t}_\mathsf{h},\ \eta \in C \,\},$$

where:

$$C \oplus N = \mathfrak{l}^*.$$

- Integral manifolds T-symplectomorphic $(M_\mathsf{h}, \sigma_\mathsf{h}, T_\mathsf{h})$.

- Definition of H, G, and $G \times_H M_\mathsf{h}$ from $\xi \mapsto L_\xi$.

- H involves holonomy of $\xi \mapsto L_\xi$.

- T-symplectomorphism $G \times_H M_\mathsf{h} \to M$:

$$((t, \xi), x) \mapsto t \cdot \mathrm{e}^{L_\xi}(x).$$

End of Sketch of Proof.

APPLICATIONS & NEW DIRECTIONS:

I. Non-Kähler Manifolds.

- **Examples**: Thurston (78), McDuff (84), Gompf (94) ...
- **Principle**: Large symmetry $\rightarrow$ Kähler.
- **Examples with symmetry**: Tolman (98), Woodward (00) ...

Theorem (Lin, P.–)

If $n \geq 2$, $\exists$ ∞ homotopically inequivalent non-Kähler symplectic M^{2n} with:

(i) $b_1(M^{2n}) = 2n - 1$.

(ii) a free symplectic $\mathbb{T}^{2n-2}$-action with coisotropic orbits.

For each finite cyclic F, $\exists$ ∞ non isomorphic M^{2n} with

$$\mathrm{H}_1(M^{2n}, \mathbb{Z}) \simeq F \times \mathbb{Z}^{2n-1}.$$

II. Symplectic Topology: Symplectomorphism Groups.

M^4 with $\pi_1(M^4) = \{0\}$.

Theorem (Karshon-Kessler-Pinsonnault, '06)

$|\{$conjugacy classes of 2-tori in **Sympl**$(M^4)\}| < \infty$.

Theorem (Pinsonnault, '06)

$|\{$conjugacy classes of maximal tori in **Sympl**$(M^4)\}| < \infty$.

Proof Ingredients:

- $\mathrm{H}^1(M^4, \mathbb{R}) = 0 \rightarrow$ any symplectic action is Hamiltonian.
- Hamiltonian $\mathbb{T}^2$-actions on M^4 classified (Delzant).
- Hamiltonian S^1-actions on M^4 classified (Karshon).
- Both "soft" and "hard" techniques.

OUR MOTIVATION:

M^4 with $\pi_1(M^4) \neq \{0\}$.

Question. Under which assumptions on our invariants is

$|\{$conjugacy classes of maximal tori in **Sympl**$(M^4)\}| < \infty$?

Expect: **"soft" and "hard" methods**.

Note: Contact case, diffeomorphism case of theorems? False.

III. Symplectic Topology: Ball Embeddings.

Problem: for fixed r, when is

$$\mathcal{A}_r := \{f\colon \mathbb{B}_r^{2n} \to M^{2n} \,|\, f \text{ symplectic}\}$$

connected?

Answers:

- ($n = 2$) remarkable results:
  Biran, McDuff-Polterovich, Lalonde-Pinsonault.
- (*) If M^{2n} toric wrt $\mathbb{T}^n$, f equivariant,

  $$\mathcal{A}_r = \text{step function in terms of } r.$$
- M^{2n} toric and f partially equivariant (wrt subtorus)?

Open Direction. [Non-Hamiltonian analogue of (*)]

- Assume M^4 admits symplectic 2-torus action.
- M^4 non-toric.
- f equivariant.
- How many components does $\mathcal{A}_r$ have?

Want more?

See www-math.mit.edu/$\sim$apelayo

- Symplectic torus actions with coisotropic principal orbits (JJ Duistermaat, A Pelayo). *Ann. Inst. Fourier*, 89 pp.

- Symplectic actions of two-tori on four-manifolds. (A Pelayo). *Submitted*, 81 pp.

- Geography of non-Kähler symplectic torus actions. (Y Lin, A Pelayo). *Submitted*, 9 pp.

The end. THANK YOU!