

# The Picard group of the moduli space of curves with level structures

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## Abstract

For  $4 \nmid L$  and  $g$  large, we calculate the integral Picard groups of the moduli spaces of curves and principally polarized abelian varieties with level  $L$  structures. In particular, we determine the divisibility properties of the standard line bundles over these moduli spaces and we calculate the second integral cohomology group of the level  $L$  subgroup of the mapping class group (in a previous paper, the author determined this rationally). This entails calculating the abelianization of the level  $L$  subgroup of the mapping class group, generalizing previous results of Perron, Sato, and the author. Finally, along the way we calculate the first homology group of the mod  $L$  symplectic group with coefficients in the adjoint representation.

## 1 Introduction

Let  $\mathcal{M}_g$  be the moduli space of genus  $g$  Riemann surfaces (see [25] for a survey). This is a quasiprojective orbifold whose orbifold fundamental group is the *mapping class group*  $\text{Mod}_g$ , that is, the group of isotopy classes of orientation preserving diffeomorphisms of a closed orientable genus  $g$  topological surface  $\Sigma_g$  (see [19] for a survey). Denote by  $\text{Pic}(\mathcal{M}_g)$  the *Picard group* of  $\mathcal{M}_g$ , that is, the set of algebraic line bundles on  $\mathcal{M}_g$  (as an orbifold; see §2.2 for the precise definition). This forms an abelian group under tensor products, and in the '60's Mumford [46] showed that the first Chern class map gives an isomorphism  $\text{Pic}(\mathcal{M}_g) \cong H^2(\text{Mod}_g; \mathbb{Z})$ . Verifying a conjecture of Mumford, Harer [24] later proved that  $H^2(\text{Mod}_g; \mathbb{Z}) \cong \mathbb{Z}$  for  $g$  sufficiently large.

**Level structures.** Associated to any finite-index subgroup of  $\text{Mod}_g$  is a finite cover of  $\mathcal{M}_g$ . An important family of such subgroups is obtained as follows. For any  $L \geq 2$ , denote by  $\mathbb{Z}_L$  the integers mod  $L$ . The group  $\text{Mod}_g$  acts on  $H_1(\Sigma_g; \mathbb{Z}_L)$  and preserves the algebraic intersection form. This is a nondegenerate alternating form, so we obtain a representation  $\text{Mod}_g \rightarrow \text{Sp}_{2g}(\mathbb{Z}_L)$ . The *level  $L$  subgroup* of  $\text{Mod}_g$ , denoted  $\text{Mod}_g(L)$ , is the kernel of this representation. The group  $\text{Mod}_g(L)$  thus consists of all mapping classes that act trivially on  $H_1(\Sigma_g; \mathbb{Z}_L)$ . The associated finite cover of  $\mathcal{M}_g$  is the *moduli space of curves with level  $L$  structures*, denoted  $\mathcal{M}_g(L)$ . This is the space of pairs  $(S, B)$ , where  $S$  is a genus  $g$  Riemann surface and  $B = \{a_1, b_1, \dots, a_g, b_g\}$  is a *symplectic basis* for  $H_1(\Sigma_g; \mathbb{Z}_L)$ . By a symplectic basis we mean that  $B$  forms a basis for  $H_1(\Sigma_g; \mathbb{Z}_L)$  and

$$i(a_i, b_j) = \delta_{i,j} \quad \text{and} \quad i(a_i, a_j) = i(b_i, b_j) = 0$$

for  $1 \leq i, j \leq g$ , where  $i(\cdot, \cdot)$  is the algebraic intersection pairing.

**Picard groups of  $\mathcal{M}_g(L)$ .** The spaces  $\mathcal{M}_g(L)$  play an important role in the study of  $\mathcal{M}_g$ . One reason for this is that for  $L \geq 3$ , the spaces  $\mathcal{M}_g(L)$  are fine moduli spaces rather than merely coarse ones. It is thus of some interest to determine their Picard groups. The first step in this direction was taken by Hain [22]. A key step in Mumford's argument demonstrating that  $\text{Pic}(\mathcal{M}_g) \cong H^2(\text{Mod}_g; \mathbb{Z})$  consisted in proving that  $H^1(\text{Mod}_g; \mathbb{Z}) = 0$ . Hain showed that we also have  $H^1(\text{Mod}_g(L); \mathbb{Z}) = 0$  for  $g \geq 3$ . Using this, he deduced that there is an injection  $\text{Pic}(\mathcal{M}_g(L)) \hookrightarrow H^2(\text{Mod}_g(L); \mathbb{Z})$ . In a recent paper [55], the author proved that  $H^2(\text{Mod}_g(L); \mathbb{Z})$  has rank 1 for  $g \geq 5$ . Using this, it is easy to show (see §2.2 and §7) that there is a short exact sequence

$$0 \longrightarrow \text{Hom}(H_1(\text{Mod}_g(L); \mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \longrightarrow \text{Pic}(\mathcal{M}_g(L)) \longrightarrow \mathbb{Z} \longrightarrow 0 \quad (1)$$

coming from the universal coefficients exact sequence for  $H^2(\text{Mod}_g(L); \mathbb{Z})$ . The goal of this paper is to answer the following two questions suggested by this calculation.

1. What is  $H_1(\text{Mod}_g(L); \mathbb{Z})$ ? For our answer, see Theorem F below. We remark that determining this group is Problem 5.23 in [18].
2. What specific line bundles give the various terms in (1)? For our answer, see Theorem A below.

For technical reasons, we will make the assumption that  $4 \nmid L$ , though we do not believe that this hypothesis is necessary for our theorems. See the end of the introduction for more details about this.

**The standard line bundle on moduli space.** We begin with the second question. The elements of the torsion subgroup  $\text{Hom}(H_1(\text{Mod}_g(L); \mathbb{Z}), \mathbb{Q}/\mathbb{Z})$  come from flat line bundles (see §2.2), so we focus on the  $\mathbb{Z}$  term. The first step is to describe a generator for  $\text{Pic}(\mathcal{M}_g) \cong \mathbb{Z}$ . Let  $\mathcal{H}_g$  denote the *Hodge bundle*. This is the complex vector bundle over  $\mathcal{M}_g$  whose fiber over the Riemann surface  $S$  is the space of holomorphic 1-forms on  $S$ . By Riemann-Roch,  $\mathcal{H}_g$  is  $g$ -dimensional, so we have a complex line bundle  $\mathcal{L}_g = \wedge^g \mathcal{H}_g$ . Arbarello-Cornalba [1] showed that  $\text{Pic}(\mathcal{M}_g) = \langle \mathcal{L}_g \rangle$ .

**Divisibility.** Let  $\mathcal{L}_g(L) \in \text{Pic}(\mathcal{M}_g(L))$  be the pullback of  $\mathcal{L}_g$ . It was recently shown by G. Farkas [20] that for  $L$  even, the line bundle  $\mathcal{L}_g(L)$  is divisible by 4 (see the remark after Theorem 7.7 below for how to extract this from Farkas's paper). Our first result shows that even modulo torsion, this is the best that one can do when  $4 \nmid L$ .

**Theorem A** (Divisibility of  $\mathcal{L}_g(L)$ ). *For  $g \geq 5$  and  $L \geq 2$  such that  $4 \nmid L$ , the group  $\text{Pic}(\mathcal{M}_g(L))$  is generated modulo torsion by  $\frac{1}{n} \mathcal{L}_g(L)$ , where  $n = 1$  if  $L$  is odd and  $n = 4$  if  $L$  is even.*

Our proof of Theorem A is topological. The key result, which is equivalent to Theorem A modulo some standard technical machinery (see §7), is the following theorem.

**Theorem B** (Image of second homology group of  $\text{Mod}_g(L)$ ). *For  $g \geq 5$  and  $L \geq 2$  such that  $4 \nmid L$ , the image of  $H_2(\text{Mod}_g(L); \mathbb{Z})$  in  $H_2(\text{Mod}_g; \mathbb{Z}) \cong \mathbb{Z}$  is  $n\mathbb{Z}$ , where  $n = 1$  if  $L$  is odd and  $n = 4$  if  $L$  is even.*

**Principally polarized abelian varieties.** Let  $\mathcal{A}_g$  be the moduli space of principally polarized abelian varieties (see §5). This is a quasiprojective orbifold with orbifold fundamental group  $\mathrm{Sp}_{2g}(\mathbb{Z})$ , and there is a map  $\mathcal{M}_g \rightarrow \mathcal{A}_g$  that takes a Riemann surface to its Jacobian. At the level of fundamental groups, this corresponds to the homomorphism  $\mathrm{Mod}_g \rightarrow \mathrm{Sp}_{2g}(\mathbb{Z})$  coming from the action of  $\mathrm{Mod}_g$  on  $H_1(\Sigma_g; \mathbb{Z})$ . It is known that the map  $\mathcal{M}_g \rightarrow \mathcal{A}_g$  induces an isomorphism  $\mathrm{Pic}(\mathcal{A}_g) \rightarrow \mathrm{Pic}(\mathcal{M}_g)$ . To prove Theorems A and B, we must prove analogous results for appropriate covers of  $\mathcal{A}_g$ .

The *level  $L$  subgroup* of  $\mathrm{Sp}_{2g}(\mathbb{Z})$ , denoted  $\mathrm{Sp}_{2g}(\mathbb{Z}, L)$ , is the kernel of the map  $\mathrm{Sp}_{2g}(\mathbb{Z}) \rightarrow \mathrm{Sp}_{2g}(\mathbb{Z}_L)$ . Associated to  $\mathrm{Sp}_{2g}(\mathbb{Z}, L)$  is the cover  $\mathcal{A}_g(L)$  of  $\mathcal{A}_g$ , known as the *moduli space of principally polarized abelian varieties with level  $L$  structures*. Work of Kazhdan [38] shows that  $H^1(\mathrm{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z}) = 0$  for  $g \geq 2$ , and Borel [7, 8] proved that  $H^2(\mathrm{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z})$  has rank 1 for  $g \geq 3$ . Applying these two results (see §2.2 and §5), the same techniques used to derive (1) yield an exact sequence

$$0 \longrightarrow \mathrm{Hom}(H_1(\mathrm{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Pic}(\mathcal{A}_g(L)) \longrightarrow \mathbb{Z} \longrightarrow 0. \quad (2)$$

We remark that  $H_1(\mathrm{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z})$  has been calculated by Sato [56] (see §3.1 below; for odd  $L$ , this was also determined independently by Perron [49] and the author [54]).

**Siegel modular forms.** There is a line bundle  $\overline{\mathcal{L}}_g \in \mathrm{Pic}(\mathcal{A}_g)$  whose holomorphic sections are Siegel modular forms of weight 1 and level 1. It is known that  $\mathrm{Pic}(\mathcal{A}_g) = \langle \overline{\mathcal{L}}_g \rangle$  and that  $\overline{\mathcal{L}}_g$  pulls back to  $\mathcal{L}_g \in \mathrm{Pic}(\mathcal{M}_g)$ . Let  $\overline{\mathcal{L}}_g(L) \in \mathrm{Pic}(\mathcal{A}_g(L))$  be the pullback of  $\overline{\mathcal{L}}_g$ . For  $n \in \mathbb{Z}$ , holomorphic sections of  $n \cdot \overline{\mathcal{L}}_g(L)$  are the Siegel modular forms of weight  $n$  and level  $L$ . For  $L$  even, the classical theta-nulls are Siegel modular forms of weight  $1/2$  and level  $L$  in an appropriate sense, so for  $L$  even we can divide  $\overline{\mathcal{L}}_g(L)$  by 2. The following theorem says that if  $4 \nmid L$ , then even modulo torsion this is the best one can do.

**Theorem C** (Divisibility of  $\overline{\mathcal{L}}_g(L)$ ). *For  $g \geq 4$  and  $L \geq 2$  such that  $4 \nmid L$ , the group  $\mathrm{Pic}(\mathcal{A}_g(L))$  is generated modulo torsion by  $\frac{1}{n} \overline{\mathcal{L}}_g(L)$ , where  $n = 1$  if  $L$  is odd and  $n = 2$  if  $L$  is even.*

*Remark.* Work of Deligne [14] shows that  $\overline{\mathcal{L}}_g(L)$  is never divisible by more than 2. However, one cannot deduce from Deligne's work that  $\overline{\mathcal{L}}_g(L)$  is not divisible by more than 2 modulo torsion.

*Remark.* It follows from work of Howe and Weissauer that all Siegel modular forms of sufficiently small weight come from theta series (see [21] for a precise formulation and history of this result). From this, it follows that for  $L$  odd there do not exist Siegel modular forms of weight  $1/2$  and level  $L$ . One can view Theorem C as showing that this remains true even if we do not demand that our modular forms be holomorphic.

Just like for  $\mathcal{M}_g$ , our proof of Theorem C is topological and the key step is to prove the following theorem.

**Theorem D** (Image of second homology group of  $\mathrm{Sp}_{2g}(\mathbb{Z}, L)$ ). *For  $g \geq 4$  and  $L \geq 2$  such that  $4 \nmid L$ , the image of  $H_2(\mathrm{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z})$  in  $H_2(\mathrm{Sp}_{2g}(\mathbb{Z}); \mathbb{Z}) \cong \mathbb{Z}$  is  $n\mathbb{Z}$ , where  $n = 1$  if  $L$  is odd and  $n = 2$  if  $L$  is even.*

**The adjoint representation of the symplectic group.** The key technical result that goes into proving Theorem D, which we believe to be of independent interest, is as follows. We first introduce some notation. Denoting the  $n \times n$  zero matrix by  $\mathbb{O}_n$  and the  $n \times n$  identity matrix by  $\mathbb{I}_n$ , let  $\Omega_g$  be

the matrix  $\begin{pmatrix} \mathbb{O}_g & \mathbb{I}_g \\ -\mathbb{I}_g & \mathbb{O}_g \end{pmatrix}$ . By definition, for a commutative ring  $R$ , the group  $\mathrm{Sp}_{2g}(R)$  consists of  $2g \times 2g$  matrices  $X$  with entries in  $R$  that satisfy  $X^t \Omega_g X = \Omega_g$ . We will denote by  $\mathfrak{sp}_{2g}(R)$  the additive group of all  $2g \times 2g$  matrices  $A$  with entries in  $R$  that satisfy  $A^t \Omega_g + \Omega_g A = 0$ . It is easy to see that  $\mathrm{Sp}_{2g}(R)$  acts on  $\mathfrak{sp}_{2g}(R)$  by conjugation.

**Theorem E.** *For  $g \geq 3$  and  $L \geq 2$  such that  $4 \nmid L$ , we have  $H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_L); \mathfrak{sp}_{2g}(\mathbb{Z}_L)) = 0$ .*

*Remark.* For  $L$  a prime greater than 5, this was originally proven by Völklein [62].

*Remark.* For  $L$  odd, the  $\mathrm{Sp}_{2g}(\mathbb{Z}_L)$ -representation  $\mathfrak{sp}_{2g}(\mathbb{Z}_L)$  is isomorphic to its dual (see §4.2.2), and it follows that  $H^1(\mathrm{Sp}_{2g}(\mathbb{Z}_L); \mathfrak{sp}_{2g}(\mathbb{Z}_L)) = 0$  as well. This self-duality is false for  $L$  even, and in fact computer calculations indicate that  $H^1(\mathrm{Sp}_{2g}(\mathbb{Z}_L); \mathfrak{sp}_{2g}(\mathbb{Z}_L))$  can be nonzero for  $L$  even. We also remark that similar computer calculations suggest that the condition  $4 \nmid L$  in Theorem E is unnecessary.

**Torsion in the Picard group.** We now discuss the torsion in  $\mathrm{Pic}(\mathcal{M}_g(L))$ , i.e. the abelian group  $H_1(\mathrm{Mod}_g(L); \mathbb{Z})$ . Let  $\Sigma_{g,n}$  denote a compact orientable genus  $g$  surface with  $n$  boundary components and let  $\mathrm{Mod}_{g,n}$  denote the group of isotopy classes of orientation preserving diffeomorphisms of  $\Sigma_{g,n}$  that act as the identity on  $\partial \Sigma_{g,n}$ . It is easiest to state our theorem for  $H_1(\mathrm{Mod}_{g,1}(L); \mathbb{Z})$ , where by  $\mathrm{Mod}_{g,1}(L)$  we mean the kernel of the action of  $\mathrm{Mod}_{g,1}$  on  $H_1(\Sigma_{g,1}; \mathbb{Z}_L)$  (see Theorem 7.8 for the closed case).

The answer is closely related to some beautiful work of Dennis Johnson on the *Torelli group*  $\mathcal{I}_{g,1}$ , which is the kernel of the action of  $\mathrm{Mod}_{g,1}$  on  $H_1(\Sigma_{g,1}; \mathbb{Z})$ . Let  $H = H_1(\Sigma_{g,1}; \mathbb{Z})$ . Johnson proved that

$$H_1(\mathcal{I}_{g,1}; \mathbb{Z}) \cong B_2(2g) \oplus \wedge^3 H,$$

where the two indicated terms are as follows (see §6 for a more complete description).

- $\wedge^3 H$  comes from the *Johnson homomorphism*, which is a surjective homomorphism  $\tau : \mathcal{I}_{g,1} \rightarrow \wedge^3 H$  arising from the action of  $\mathcal{I}_{g,1}$  on the second nilpotent truncation of  $\pi_1(\Sigma_{g,1}; \mathbb{Z})$ .
- $B_2(2g)$  is the space of boolean (i.e. square-free) polynomials over  $\mathbb{Z}_2$  in  $2g$  generators whose degree is at most 2. These  $2g$  generators correspond to a basis for  $H_1(\Sigma_{g,1}; \mathbb{Z}_2)$ . This portion of  $H_1(\mathcal{I}_{g,1}; \mathbb{Z})$  comes from work of Birman-Craggs [5] and Johnson [34] and is related to the Rochlin invariant of homology 3-spheres. The corresponding abelian quotient of  $\mathcal{I}_{g,1}$  is known as the *Birman-Craggs-Johnson homomorphism*.

We have a short exact sequence

$$1 \longrightarrow \mathcal{I}_{g,1} \longrightarrow \mathrm{Mod}_{g,1}(L) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}, L) \longrightarrow 1.$$

To understand  $H_1(\mathrm{Mod}_{g,1}(L); \mathbb{Z})$ , one must first investigate the extent to which the Johnson and Birman-Craggs-Johnson homomorphisms extend over  $\mathrm{Mod}_{g,1}(L)$ . The following results are known (see §6.4 for more details).

- Let  $H_L = H_1(\Sigma_{g,1}; \mathbb{Z}_L)$ . A “mod  $L$ ” Johnson homomorphism of the form  $\mathrm{Mod}_{g,1}(L) \rightarrow \wedge^3 H_L$  was defined independently by Broaddus-Farb-Putman [9], by Perron [49], and by Sato [56].

- Sato [56] proved that “most” of the Birman-Craggs-Johnson homomorphism can be extended over  $\text{Mod}_{g,1}(L)$  if  $L$  is even. More precisely, observe that  $B_2(2g)$  contains a constant term  $B_0(2g) \cong \mathbb{Z}_2$ . Sato proved that for  $L$  even,  $H_1(\text{Mod}_{g,1}(L); \mathbb{Z})$  contains a term of the form  $B_2(2g)/B_0(2g)$ . Sato also proved that the  $B_0(2g)$  piece of  $H_1(\mathcal{I}_{g,1}; \mathbb{Z})$  does *not* exist in  $H_1(\text{Mod}_{g,1}(2); \mathbb{Z})$ , and asked whether a similar thing was true for general even  $L$ .

Our final theorem says that the above together with  $H_1(\text{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z})$  (which as we noted above is known; see §3.1) gives the entire abelianization of  $\text{Mod}_{g,1}(L)$  if  $4 \nmid L$ . It answers Sato’s question in the affirmative in these cases.

**Theorem F** (Abelianization of  $\text{Mod}_g(L)$ ). *Fix  $g \geq 5$  and  $L \geq 2$  such that  $4 \nmid L$ . Setting  $H_L = H_1(\Sigma_{g,1}; \mathbb{Z}_L)$ , there is a short exact sequence*

$$0 \longrightarrow K_{g,1} \longrightarrow H_1(\text{Mod}_{g,1}(L); \mathbb{Z}) \longrightarrow H_1(\text{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z}) \longrightarrow 0,$$

where  $K_{g,1} = \wedge^3 H_L$  if  $L$  is odd and  $K_{g,1} = (B_2(2g)/B_0(2g)) \oplus \wedge^3 H_L$  if  $L$  is even.

*Remark.* For  $L$  odd, this was previously and independently proven by the author [54] and Sato [56] and by Perron [49] modulo 2-torsion. Sato also proved it for  $L = 2$ .

*Remark.* Fix  $L$  such that  $L$  is even but  $4 \nmid L$ . It follows from the proof of Theorem F that the term  $B_0(2g)$  of  $H_1(\mathcal{I}_{g,1}; \mathbb{Z})$  is in the kernel of the natural map  $H_1(\mathcal{I}_{g,1}; \mathbb{Z}) \rightarrow H_1(\text{Mod}_{g,1}(L); \mathbb{Z})$ . As we shall see, this purely group-theoretic vanishing phenomena is a consequence of the fact that  $\mathcal{L}_g(L) \in \text{Pic}(\mathcal{M}_g(L))$  is divisible by 4 instead of merely 2, like  $\overline{\mathcal{L}}_g(L) \in \text{Pic}(\mathcal{A}_g(L))$ .

*Remark.* To remove the condition  $4 \nmid L$  from the hypotheses of Theorems A–F while still using the basic techniques of this paper, one would have to remove the condition  $4 \nmid L$  not only from the hypotheses of Theorem E, but also from the hypotheses of Theorems 3.7 and 4.1 below. Computer calculations indicate that this is probably possible for Theorems E and 4.1, but I am not sure if it is possible for Theorem 3.7 (see the remark following Theorem 3.7 for more details on this).

**Outline of paper and remarks.** In §2, we will discuss some preliminary results and definitions about group cohomology and orbifolds. The proofs of Theorems C–E come next : in §3, we discuss some basic results about the symplectic group, in §4, we prove Theorem E, and in §5, we introduce  $\mathcal{A}_g$  and prove Theorems C and D. In §6, we then discuss some results about the Torelli subgroup of the mapping class group needed for the remaining theorems of the paper. Finally, in §7 we introduce  $\mathcal{M}_g$  and prove Theorems A–B and F.

**A naturality convention.** Throughout this paper, the following will occur several times. Let  $\Gamma$  be a group, let  $G$  be a normal subgroup of  $\Gamma$ , and let  $\phi : G \rightarrow M$  be a homomorphism, where  $M$  is an abelian group with a natural  $\Gamma$  action. For instance,  $\Gamma$  might be  $\text{Sp}_{2g}(\mathbb{Z})$  and  $M$  might be  $H_1(\Sigma_g; \mathbb{Z})$ . We will say that  $\phi$  is  $\Gamma$ -equivariant if  $\phi(yxy^{-1}) = y \cdot \phi(x)$  for all  $x \in G$  and  $y \in \Gamma$ .

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## 2 Preliminaries

### 2.1 Group cohomology

We begin by reviewing some facts about group cohomology and establishing some notation. Good references include [10] and [41].

**Degree zero.** Let  $G$  be a group and  $M$  be a  $G$ -module. The *invariants* of  $M$ , denoted  $M^G$ , is the submodule  $\{x \in M \mid g \cdot x = x \text{ for all } g \in G\}$ . The *coinvariants* of  $M$ , denoted  $M_G$ , is the quotient  $M/K$ , where  $K$  is the submodule spanned by the set  $\{g \cdot x - x \mid x \in M, g \in G\}$ . We have  $H^0(G; M) = M^G$  and  $H_0(G; M) = M_G$ .

**Duality.** For  $M$  an abelian group, define  $M^* = \text{Hom}(M, \mathbb{Q}/\mathbb{Z})$ . Observe that if  $M$  is a module over  $\mathbb{Z}_L$ , then  $M^* \cong \text{Hom}(M, \mathbb{Z}_L)$ . We have the following duality between twisted homology and cohomology.

**Lemma 2.1** ([10, Proposition VI.7.1]). *Let  $G$  be a group and let  $M$  be a  $G$ -module. Then there is a natural isomorphism  $H^k(G; M^*) \cong (H_k(G; M))^*$  for every  $k \geq 0$ .*

**The Hochschild-Serre spectral sequence.** Let

$$1 \longrightarrow K \longrightarrow G \longrightarrow Q \longrightarrow 1$$

be a short exact sequence of groups and let  $M$  be a  $G$ -module. The homology Hochschild-Serre spectral sequence is a first quadrant spectral sequence converging to  $H_*(G; M)$  whose  $E^2$  page is of the form

$$E_{p,q}^2 = H_p(Q; H_q(K; M)).$$

The edge groups of this spectral sequence have the following interpretations.

**Theorem 2.2** ([41, Proposition 10.2]).

- $E_{p,0}^\infty$ , a subgroup of  $E_{p,0}^2 \cong H_p(Q; H_0(K; M)) \cong H_p(Q; M_K)$ , is equal to  $\text{Im}(H_p(G; M) \rightarrow H_p(Q; M_K))$ .
- $E_{0,q}^\infty$ , a quotient of  $E_{0,q}^2 \cong H_0(Q; H_q(K; M)) \cong (H_q(K; M))_Q$ , is isomorphic to  $\text{Im}(H_q(K; M) \rightarrow H_q(G; M))$ .

A standard consequence of the Hochschild-Serre spectral sequence is the 5-term exact sequence

$$H_2(G; M) \longrightarrow H_2(Q; M_K) \longrightarrow (H_1(K; M))_Q \longrightarrow H_1(G; M) \longrightarrow H_1(Q; M_K) \longrightarrow 0.$$

A similar spectral sequence exists in group cohomology.

Several times we will use the following consequence of the 5-term exact sequence.

**Lemma 2.3.** *Let  $G$  be a group and  $M$  be a  $G$ -module. Assume that  $G \cong G_1 \oplus G_2$ , that  $G_2 < G$  acts trivially on  $M$ , and that  $H_1(G_2; \mathbb{Z}) = 0$ . Then  $H_1(G; M) \cong H_1(G_1; M)$  and  $H^1(G; M) \cong H^1(G_1; M)$ .*

*Proof.* An immediate consequence of the 5-term exact sequence of the short exact sequence

$$1 \longrightarrow G_2 \longrightarrow G \longrightarrow G_1 \longrightarrow 1$$

with coefficients in  $M$ . □

**Equivariant homology.** To define Chern classes of line bundles over orbifolds, we will need some basic results about equivariant homology. A standard reference is [10, Chapter VII]. Throughout this section,  $X$  is a smooth manifold and  $G$  is a discrete group acting properly discontinuously on  $X$  by orientation preserving diffeomorphisms.

*Remark.* In [10], it is assumed that  $X$  has a  $G$ -equivariant triangulation. In our situation, a theorem of Illman [30] says that such a triangulation always exists and is essentially unique.

We start with the following definition.

**Definition.** Let  $EG$  be the universal cover of a fixed classifying space for  $G$ . The group  $G$  acts freely on  $X \times EG$ , and we will denote by  $X \times_G EG$  the result of the *Borel construction*, i.e. the quotient  $(X \times EG)/G$ . We define

$$H_G^*(X; \mathbb{Z}) = H^*(X \times_G EG; \mathbb{Z}) \quad \text{and} \quad H_*^G(X; \mathbb{Z}) = H_*(X \times_G EG; \mathbb{Z}).$$

*Remark.* It is easy to see that  $H_G^*(X; \mathbb{Z})$  and  $H_*^G(X; \mathbb{Z})$  do not depend on the choice of  $EG$ .

We will need the following two results about  $H_G^*(X; \mathbb{Z})$ .

**Lemma 2.4.** *Let  $X$  be a smooth manifold and  $G$  be a group acting properly discontinuously on  $X$  by orientation-preserving diffeomorphisms.*

1. *If  $X$  is  $n$ -connected, then  $H_G^k(X; \mathbb{Z}) \cong H^k(G; \mathbb{Z})$  and  $H_k^G(X; \mathbb{Z}) \cong H_k(G; \mathbb{Z})$  for  $0 \leq k \leq n$ .*
2. *For all  $k \geq 1$ , we have a short exact sequence*

$$0 \longrightarrow \text{Hom}(H_{k-1}^G(X; \mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \longrightarrow H_G^k(X; \mathbb{Z}) \longrightarrow \text{Hom}(H_k^G(X; \mathbb{Z}), \mathbb{Z}) \longrightarrow 0.$$

The first conclusion is proven exactly like [10, Proposition VII.7.3], and the second conclusion is a translation of the universal coefficients theorem.

## 2.2 Orbifolds

### 2.2.1 Topological theory

We will not bother to set up a proper category of orbifolds, but simply give definitions and theorems adapted to the needs of this paper (see [61, Chapter 13] for a more general discussion). We begin with the following definitions.

**Definition.** An *orbifold* consists of a pair  $(X, G)$ , where  $X$  is a simply-connected smooth manifold and  $G$  is a group acting properly discontinuously on  $X$  by diffeomorphisms. An orbifold  $(X, G)$  is a *trivial orbifold* if  $G$  acts freely. If  $(X_1, G_1)$  and  $(X_2, G_2)$  are orbifolds, then a map  $(X_1, G_1) \rightarrow (X_2, G_2)$  consists of a map  $f : X_1 \rightarrow X_2$  and a homomorphism  $\phi : G_1 \rightarrow G_2$  such that  $f(g \cdot x) = \phi(g) \cdot f(x)$  for all  $x \in X_1$  and  $g \in G_1$ . We will say that a map  $(X_1, G_1) \rightarrow (X_2, G_2)$  is a *cover* if  $X_1 = X_2$ , the map  $f : X_1 \rightarrow X_2$  is the identity, and the homomorphism  $\phi : G_1 \rightarrow G_2$  is injective. The cover is *finite* if  $G_1$  is a finite index subgroup of  $G_2$ .

*Remark.* The standard definition of an orbifold involves a local definition. Our orbifolds are what are usually called *good orbifolds*.

*Remark.* If  $(X, G)$  is a trivial orbifold, then  $G$  acts as a group of covering transformations on  $X$ . In this case, the notion of a cover reduces to the standard notion of a cover of the quotient  $X/G$ .

We now define the topological Picard group of an orbifold.

**Definition.** Let  $(X, G)$  be an orbifold. The *topological Picard group*  $\text{Pic}_{\text{top}}(X, G)$  of  $(X, G)$  is the set of oriented  $G$ -equivariant complex line bundles on  $X$  (up to the obvious equivalence). The set  $\text{Pic}_{\text{top}}(X, G)$  forms an abelian group under tensor products.

*Remark.* A map  $(X_1, G_1) \rightarrow (X_2, G_2)$  induces a homomorphism  $\text{Pic}_{\text{top}}(X_2, G_2) \rightarrow \text{Pic}_{\text{top}}(X_1, G_1)$ .

If  $(X, G)$  is a trivial orbifold, then  $\text{Pic}_{\text{top}}(X, G)$  is the usual group  $\text{Pic}_{\text{top}}(X/G)$  of complex line bundles on the quotient  $X/G$ , and it is well-known that in this case the first Chern class map  $c_1 : \text{Pic}_{\text{top}}(X/G) \rightarrow H^2(X/G; \mathbb{Z})$  is an isomorphism. The extension of this to more general orbifolds is as follows.

**Lemma 2.5.** *Let  $(X, G)$  be an orbifold. There is then a natural injection  $\text{Pic}_{\text{top}}(X, G) \hookrightarrow H_G^2(X; \mathbb{Z})$ .*

This is a well-known result, but we have been unable to locate a proof of it in precisely this form (for instance, Conner [12] gives a proof in the case that  $G$  is finite, in which case the indicated injection is an isomorphism). We thus give a quick sketch of a proof.

*Proof of Lemma 2.5.* Let  $EG$  be the universal cover of a classifying space for  $G$ . The group  $G$  thus acts freely and properly discontinuously on  $EG \times X$ . The space  $EG$  need not be a smooth manifold, so strictly speaking  $(EG \times X, G)$  is not an orbifold, but during this proof we will relax our definition and regard  $(EG \times X, G)$  as a trivial orbifold. The projection  $\pi : EG \times X \rightarrow X$  induces a map of orbifolds  $(EG \times X, G) \rightarrow (X, G)$ . Below we will prove that the induced map  $\pi^* : \text{Pic}_{\text{top}}(X, G) \rightarrow \text{Pic}_{\text{top}}(EG \times X, G)$  is injective. Since  $\text{Pic}_{\text{top}}(EG \times X, G)$  is a trivial orbifold, we have an isomorphism

$$\text{Pic}_{\text{top}}(EG \times X, G) \cong H^2((EG \times X)/G; \mathbb{Z}) \cong H_G^2(X; \mathbb{Z}),$$

and the lemma will follow.

To see that the map  $\pi^* : \text{Pic}_{\text{top}}(X, G) \rightarrow \text{Pic}_{\text{top}}(EG \times X, G)$  is injective, consider  $L \in \ker(\pi^*)$ . Since  $EG$  is contractible, we have  $\text{Pic}_{\text{top}}(X) \cong \text{Pic}_{\text{top}}(EG \times X)$ . Thus  $L$  is a trivial line bundle  $X \times \mathbb{C}$  with a group action. However, it is easy to see that since the group action on  $\pi^*(L)$  is trivial, the group action on  $L$  must be trivial; i.e.  $L = 0$ , as desired.  $\square$

We close this section by discussing the flat line bundles over an orbifold  $(X, G)$  (see [23, Exercise 14.1]). Since  $X$  is simply connected, the first conclusion of Lemma 2.4 says that  $H_1^G(X; \mathbb{Z}) \cong H_1(G; \mathbb{Z})$ . The second conclusion of Lemma 2.4 thus gives us an injection  $\text{Hom}(H_1(G; \mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \hookrightarrow H_G^2(X; \mathbb{Z})$ . Consider  $\phi \in \text{Hom}(H_1(G; \mathbb{Z}), \mathbb{Q}/\mathbb{Z})$ . The line bundle whose first Chern class is  $\phi$  is the trivial line bundle  $X \times \mathbb{C}$  with the following  $G$ -action. Consider  $g \in G$ . Then  $\phi(g) \in \mathbb{Q}$  is well-defined up to  $\mathbb{Z}$ , and for  $(x, z) \in X \times \mathbb{C}$ , we have  $g \cdot (x, z) = (g \cdot x, e^{2\pi i \phi(g)} z)$ . This is known as the *flat line bundle* over  $(X, G)$  with monodromy  $\phi$ .

## 2.2.2 Quasiprojective orbifolds

We now discuss quasiprojective orbifolds. For more information, see [23].

**Definition.** Let  $(X, G)$  be an orbifold. We will say that  $X/G$  is a *quasiprojective orbifold* if it satisfies the following properties.

- $X$  is a complex manifold and  $G$  acts by biholomorphisms.
- There is a finite-index subgroup  $G' < G$  that acts freely on  $X$  such that if  $G'' < G'$  is a finite index subgroup, then the complex manifold  $X/G''$  is a quasiprojective variety. We will say that  $(X, G'')$  is a *quasiprojective finite cover* of  $(X, G)$ .

*Remark.* Observe that by definition, all finite covers of quasiprojective orbifolds are quasiprojective orbifolds.

**Definition.** If  $(X, G)$  is a quasiprojective orbifold with a quasiprojective finite cover  $(X, G')$ , then an *algebraic line bundle* on  $(X, G)$  is a  $G$ -equivariant holomorphic line bundle on  $X$  that induces an algebraic line bundle on the quasiprojective variety  $X/G'$ . It is easy to see that this notion is independent of the choice of quasiprojective finite cover. The set of algebraic line bundles on  $(X, G)$  forms the *Picard group* of  $(X, G)$ , which we will denote by  $\text{Pic}(X, G)$ . The set  $\text{Pic}(X, G)$  forms an abelian group under tensor products. We will denote by  $\text{Pic}^0(X, G)$  the kernel of the natural homomorphism  $\text{Pic}(X, G) \rightarrow \text{Pic}_{\text{top}}(X, G)$ .

*Remark.* It is clear that the flat line bundles over a quasiprojective orbifold are always algebraic.

If  $X$  is a projective variety with  $H^1(X; \mathbb{Z}) = 0$ , then a standard result says that  $\text{Pic}^0(X) = 0$ , so we have an injection  $c_1 : \text{Pic}(X) \rightarrow H^2(X; \mathbb{Z})$ . The following is the extension of this to quasiprojective orbifolds.

**Theorem 2.6** ([23, Theorem 14.3]). *Let  $(X, G)$  be a quasiprojective orbifold with  $H_G^1(X; \mathbb{Z}) = 0$ . Then  $\text{Pic}^0(X, G) = 0$ .*

We conclude this section with the following lemma, which summarizes the manner in which we will use the above results. To make sense of its statement, recall that the first conclusion of Lemma 2.4 says that if  $(X, G)$  is an orbifold and  $X$  is contractible, then  $H_G^k(X; \mathbb{Z}) \cong H^k(G; \mathbb{Z})$ .

**Lemma 2.7.** *Let  $(X, G)$  be a quasiprojective orbifold with  $X$  contractible and let  $G' < G$  be a finite-index subgroup. Assume that the following hold.*

- $H_1(G; \mathbb{Z}) = 0$  and  $H^1(G'; \mathbb{Z}) = 0$ .
- $H_2(G; \mathbb{Z}) = \mathbb{Z}$  and  $H_2(G'; \mathbb{Q}) = \mathbb{Q}$ .
- There exists  $\lambda \in \text{Pic}(X, G)$  such that  $c_1(\lambda) \in H^2(G; \mathbb{Z})$  generates  $H^2(G; \mathbb{Z})$  (we remark that the universal coefficients theorem implies that  $H^2(G; \mathbb{Z}) \cong \mathbb{Z}$ ).
- There exists some  $n \geq 1$  such that the pullback  $\tilde{\lambda} \in \text{Pic}(X, G')$  of  $\lambda$  is divisible by  $n$  and the image of  $H_2(G'; \mathbb{Z})$  in  $H_2(G; \mathbb{Z}) \cong \mathbb{Z}$  is  $m\mathbb{Z}$  for some  $0 \leq m \leq n$ .

*Then the following hold.*

- $\text{Pic}(X, G') \cong H^2(G; \mathbb{Z})$ .
- $\text{Pic}(X, G')$  is generated modulo torsion by  $\frac{1}{n}\tilde{\lambda}$ .

- The image of  $H_2(G'; \mathbb{Z})$  in  $H_2(G; \mathbb{Z}) \cong \mathbb{Z}$  is  $n\mathbb{Z}$ .

*Proof.* By Theorem 2.6 and Lemma 2.5 and the assumptions of the lemma, we have an isomorphism  $\text{Pic}(X, G) \cong H^2(G; \mathbb{Z})$  and an injection  $c_1 : \text{Pic}(X, G') \hookrightarrow H^2(G'; \mathbb{Z})$ . Via flat bundles we see that  $c_1(\text{Pic}(X, G'))$  contains the torsion subgroup of  $H^2(G'; \mathbb{Z})$ . The remainder of the conclusions of the lemma are immediate consequences of the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \text{Hom}(H_1(G; \mathbb{Z}), \mathbb{Q}/\mathbb{Z}) & \longrightarrow & H^2(G; \mathbb{Z}) & \longrightarrow & \text{Hom}(H_2(G; \mathbb{Z}), \mathbb{Z}) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{Hom}(H_1(G'; \mathbb{Z}), \mathbb{Q}/\mathbb{Z}) & \longrightarrow & H^2(G'; \mathbb{Z}) & \longrightarrow & \text{Hom}(H_2(G'; \mathbb{Z}), \mathbb{Z}) \longrightarrow 0
\end{array}$$

of universal coefficient exact sequences.  $\square$

### 3 The symplectic group

The key to proving our theorems about  $\mathcal{A}_g(L)$  is the exact sequence

$$1 \longrightarrow \text{Sp}_{2g}(\mathbb{Z}, L) \longrightarrow \text{Sp}_{2g}(\mathbb{Z}) \longrightarrow \text{Sp}_{2g}(\mathbb{Z}_L) \longrightarrow 1.$$

To analyze it, we will need some results about  $\text{Sp}_{2g}(\mathbb{Z}, L)$  and  $\text{Sp}_{2g}(\mathbb{Z}_L)$  that are collected in §3.1–3.2.

*Remark.* The fact that the map  $\text{Sp}_{2g}(\mathbb{Z}) \rightarrow \text{Sp}_{2g}(\mathbb{Z}_L)$  is surjective is nontrivial. For a proof, see [48, Theorem 1].

#### 3.1 The abelianization of $\text{Sp}_{2g}(\mathbb{Z}, L)$

For  $L \geq 2$ , we define an  $\text{Sp}_{2g}(\mathbb{Z})$ -equivariant homomorphism  $\phi : \text{Sp}_{2g}(\mathbb{Z}, L) \rightarrow \mathfrak{sp}_{2g}(\mathbb{Z}_L)$  as follows (see [49, 54, 56] for more details). An element  $M \in \text{Sp}_{2g}(\mathbb{Z}, L)$  can be written in the form  $\mathbb{I}_{2g} + L \cdot A$  for some matrix  $A$ . Define  $\phi(M)$  to equal  $A$  modulo  $L$ . It is easy to see that  $\phi(M) \in \mathfrak{sp}_{2g}(\mathbb{Z}_L)$  and that the resulting map is a homomorphism whose kernel is  $\text{Sp}_{2g}(\mathbb{Z}, L^2)$ . It is also not hard to show that  $\phi$  is surjective. Summing up, we have a short exact sequence

$$1 \longrightarrow \text{Sp}_{2g}(\mathbb{Z}, L^2) \longrightarrow \text{Sp}_{2g}(\mathbb{Z}, L) \longrightarrow \mathfrak{sp}_{2g}(\mathbb{Z}_L) \longrightarrow 1. \quad (3)$$

*Remark.* The construction of  $\phi$  is inspired by a similar homomorphism constructed by Lee and Szczarba [40] for congruence subgroups of  $\text{SL}_n(\mathbb{Z})$ . They also proved an analogue of Theorem 3.2 below. We also remark that Newman and Smart identified  $\text{Sp}_{2g}(\mathbb{Z}, L)/\text{Sp}_{2g}(\mathbb{Z}, L^2)$  as an abelian group (but not as an  $\text{Sp}_{2g}(\mathbb{Z})$ -module) in [48].

We pause now to prove the following lemma, which will be need in §3.2.

**Lemma 3.1.** *Fix  $g \geq 3$ , a prime  $p$ , and some  $k \geq 1$ . We then have an  $\text{Sp}_{2g}(\mathbb{Z})$ -equivariant isomorphism  $\text{Sp}_{2g}(\mathbb{Z}, p^k)/\text{Sp}_{2g}(\mathbb{Z}, p^{k+1}) \cong \mathfrak{sp}_{2g}(\mathbb{Z}_p)$ .*

*Remark.* The isomorphism class of  $\text{Sp}_{2g}(\mathbb{Z}, p^k)/\text{Sp}_{2g}(\mathbb{Z}, p^{k+1})$  as an abelian group was determined by Newman and Smart in [48]. Lemma 3.1 identifies it as an  $\text{Sp}_{2g}(\mathbb{Z})$ -module.

*Proof of Lemma 3.1.* By exact sequence (3), we have  $\mathrm{Sp}_{2g}(\mathbb{Z}, p^k) / \mathrm{Sp}_{2g}(\mathbb{Z}, p^{2k}) \cong \mathfrak{sp}_{2g}(\mathbb{Z}_{p^k})$ . Moreover, the image of  $\mathrm{Sp}_{2g}(\mathbb{Z}, p^{k+1})$  in  $\mathfrak{sp}_{2g}(\mathbb{Z}_{p^k})$  is  $p \cdot \mathfrak{sp}_{2g}(\mathbb{Z}_{p^k})$ . The result follows.  $\square$

For  $L$  odd and  $g \geq 3$ , the following result (which was proven independently by Perron, Sato, and the author) shows that  $\phi$  gives the entire abelianization of  $\mathrm{Sp}_{2g}(\mathbb{Z}, L)$ .

**Theorem 3.2** (Perron [49], Putman [54], Sato [56]). *Fix  $g \geq 3$  and  $L > 2$  such that  $L$  is odd. We then have  $[\mathrm{Sp}_{2g}(\mathbb{Z}, L), \mathrm{Sp}_{2g}(\mathbb{Z}, L)] = \mathrm{Sp}_{2g}(\mathbb{Z}, L^2)$ .*

**Corollary 3.3.** *Fix  $g \geq 3$  and  $L > 2$  such that  $L$  is odd. We then have an  $\mathrm{Sp}_{2g}(\mathbb{Z})$ -equivariant isomorphism  $H_1(\mathrm{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z}) \cong \mathfrak{sp}_{2g}(\mathbb{Z}_L)$ .*

Now assume that  $L$  is even. In this case, the abelianization of  $\mathrm{Sp}_{2g}(\mathbb{Z}, L)$  is slightly larger than  $\mathfrak{sp}_{2g}(\mathbb{Z}_L)$ . Define  $\mathrm{Sp}_{2g}(\mathbb{Z}, L, 2L)$  to be the set of block matrices  $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$  in  $\mathrm{Sp}_{2g}(\mathbb{Z}, L)$  such that the diagonal entries of  $B$  and  $C$  are equal to 0 modulo  $2L$ . Igusa [28, Lemma 1.i] showed that  $\mathrm{Sp}_{2g}(\mathbb{Z}, L, 2L)$  is a normal subgroup of  $\mathrm{Sp}_{2g}(\mathbb{Z})$ . He also proved that

$$\mathrm{Sp}_{2g}(\mathbb{Z}, L) / \mathrm{Sp}_{2g}(\mathbb{Z}, L, 2L) \cong H_1(\Sigma_g; \mathbb{Z}_2). \quad (4)$$

This  $\mathrm{Sp}_{2g}(\mathbb{Z})$ -equivariant isomorphism takes  $\mathbb{I}_g + L \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z}, L)$  to the  $2g$ -dimensional vector whose entries are the diagonals of  $B$  and  $C$  modulo 2 (this is not stated in quite this form in [28], but it is implicit in the proof of Lemma 1.i from that paper).

Using (4), we can take the quotient of exact sequence (3) by  $\mathrm{Sp}_{2g}(\mathbb{Z}, L^2, 2L^2)$  and get an exact sequence

$$1 \longrightarrow H_1(\Sigma_g; \mathbb{Z}_2) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}, L) / \mathrm{Sp}_{2g}(\mathbb{Z}, L^2, 2L^2) \longrightarrow \mathfrak{sp}_{2g}(\mathbb{Z}_L) \longrightarrow 1.$$

Since  $\mathrm{Sp}_{2g}(\mathbb{Z}, L)$  acts trivially on  $H_1(\Sigma_g; \mathbb{Z}_2)$ , we deduce that  $\mathrm{Sp}_{2g}(\mathbb{Z}, L) / \mathrm{Sp}_{2g}(\mathbb{Z}, L^2, 2L^2)$  is abelian. Sato proved that we have obtained the abelianization of  $\mathrm{Sp}_{2g}(\mathbb{Z}, L)$ .

**Theorem 3.4** (Sato [56]). *Fix  $g \geq 3$  and  $L \geq 2$  such that  $L$  is even. We then have*

$$[\mathrm{Sp}_{2g}(\mathbb{Z}, L), \mathrm{Sp}_{2g}(\mathbb{Z}, L)] = \mathrm{Sp}_{2g}(\mathbb{Z}, L^2, 2L^2).$$

**Corollary 3.5.** *Fix  $g \geq 3$  and  $L \geq 2$  such that  $L$  is even. We then have a short exact sequence*

$$0 \longrightarrow H_1(\Sigma_g; \mathbb{Z}_2) \longrightarrow H_1(\mathrm{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z}) \longrightarrow \mathfrak{sp}_{2g}(\mathbb{Z}_L) \longrightarrow 0$$

of  $\mathrm{Sp}_{2g}(\mathbb{Z})$ -modules.

### 3.2 Four results about $\mathrm{Sp}_{2g}(\mathbb{Z}_L)$

We will need four results about  $\mathrm{Sp}_{2g}(\mathbb{Z}_L)$ , the first two of which are the following group cohomology calculations.

**Lemma 3.6.** *For  $g \geq 3$  and  $L \geq 2$ , we have  $H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_L); \mathbb{Z}) = 0$ .*

*Proof.* The group  $\mathrm{Sp}_{2g}(\mathbb{Z}_L)$  is the surjective image of  $\mathrm{Sp}_{2g}(\mathbb{Z})$ , which satisfies  $H_1(\mathrm{Sp}_{2g}(\mathbb{Z}); \mathbb{Z}) = 0$  (see Theorem 5.1 below). The lemma follows.  $\square$

**Theorem 3.7** (M. Stein, [58, Theorem 2.13 and Proposition 3.3.a]). *For  $g \geq 3$  and  $L \geq 2$  such that  $4 \nmid L$ , we have  $H_2(\mathrm{Sp}_{2g}(\mathbb{Z}_L); \mathbb{Z}) = 0$ .*

*Remark.* For  $L$  prime, Theorem 3.7 was originally proven by Steinberg [59]. The calculations in [59] and [58], which deal with arbitrary Chevalley groups, are closely related to Milnor's definition of  $K_2$  of a ring. In the general context of Chevalley groups, the condition  $4 \nmid L$  is necessary; indeed, for large  $n$  we have  $H_2(\mathrm{SL}_n(\mathbb{Z}_L); \mathbb{Z})$  equal to 0 if  $4 \nmid L$  and  $\mathbb{Z}_2$  if  $4 \mid L$ . This latter result is due to Dennis; see [44, Corollary 10.8] and [16, §12].

The other two lemmas we need allow us to express  $\mathrm{Sp}_{2g}(\mathbb{Z}_L)$  in terms of  $\mathrm{Sp}_{2g}(\mathbb{Z}_p)$  for  $p$  prime. The first is the following, which is due to Newman and Smart.

**Lemma 3.8** (Newman-Smart, [48, Theorem 5]). *Fix  $g \geq 1$  and  $L \geq 2$ . Write  $L = p_1^{k_1} \cdots p_n^{k_n}$ , where the  $p_i$  are distinct primes. We then have*

$$\mathrm{Sp}_{2g}(\mathbb{Z}_L) \cong \bigoplus_{i=1}^n \mathrm{Sp}_{2g}(\mathbb{Z}_{p_i^{k_i}}).$$

The second is as follows.

**Lemma 3.9.** *Fix  $g \geq 3$ , a prime  $p$ , and some  $k \geq 1$ . We then have a short exact sequence*

$$1 \longrightarrow \mathfrak{sp}_{2g}(\mathbb{Z}_p) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}_{p^{k+1}}) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}_{p^k}) \longrightarrow 1. \quad (5)$$

*Moreover, if  $(p, k) \notin \{(2, 1), (3, 1)\}$ , then this exact sequence does not split.*

*Remark.* The condition  $(p, k) \notin \{(2, 1), (3, 1)\}$  is not necessary, but the indicated result is sufficient for our purposes and has an easier proof.

*Proof of Lemma 3.9.* By Lemma 3.1, we can obtain (5) by quotienting the exact sequence

$$1 \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}, p^k) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}_{p^k}) \longrightarrow 1,$$

by  $\mathrm{Sp}_{2g}(\mathbb{Z}, p^{k+1})$ . We must only prove that (5) does not split if  $(p, k) \notin \{(2, 1), (3, 1)\}$ .

Assume that (5) does split, and let  $\phi : \mathrm{Sp}_{2g}(\mathbb{Z}_{p^k}) \rightarrow \mathrm{Sp}_{2g}(\mathbb{Z}_{p^{k+1}})$  be a splitting. For an integer  $L$ , we will during this proof denote by  $\mathbb{I}_L$  and  $\mathbb{O}_L$  the  $2g \times 2g$  identity and zero matrices with entries in  $\mathbb{Z}_L$ , respectively (in the rest of this paper, the subscripts refer to the dimensions of the matrices). Also, we will denote by  $E_L$  the  $2g \times 2g$  matrix with entries in  $\mathbb{Z}_L$  with a 1 at position  $(1, g+1)$  and 0's elsewhere. Observe that  $\mathbb{I}_{p^k} + E_{p^k} \in \mathrm{Sp}_{2g}(\mathbb{Z}_{p^k})$  and that  $(\mathbb{I}_{p^k} + E_{p^k})^{p^k} = 1$ . There exists a matrix  $A$  with entries in  $\mathbb{Z}_{p^{k+1}}$  such that  $\phi(\mathbb{I}_{p^k} + E_{p^k}) = \mathbb{I}_{p^{k+1}} + E_{p^{k+1}} + p^k A$ . Observe that

$$\begin{aligned} \mathbb{I}_{p^{k+1}} &= (\mathbb{I}_{p^{k+1}} + E_{p^{k+1}} + p^k A)^{p^k} \\ &= \mathbb{I}_{p^{k+1}} + p^k (E_{p^{k+1}} + p^k A) + \binom{p^k}{2} (E_{p^{k+1}} + p^k A)^2 + \cdots + (E_{p^{k+1}} + p^k A)^{p^k}. \end{aligned}$$

Since  $p^k (E_{p^{k+1}} + p^k A) = p^k E_{p^{k+1}} \neq 0$ , we will obtain a contradiction if we show that all the higher order terms in this sum vanish.

Consider  $\binom{p^k}{n} (E_{p^{k+1}} + p^k A)^n$  for some  $2 \leq n \leq p^k$ . We can expand out  $(E_{p^{k+1}} + p^k A)^n$  as a sum of terms each of which for some  $0 \leq m \leq n$  is a product of  $m$  factors of  $E_{p^{k+1}}$  and  $(n-m)$  factors of  $p^k A$ ,

taken in some order. If one of these terms contains 2 factors of  $p^k A$ , then it must vanish. Also, since  $E_{p^{k+1}}^2 = 0$ , if one of these terms contains two adjacent factors of  $E_{p^{k+1}}$ , then it must vanish. If  $n \geq 4$ , then we conclude that  $(E_{p^{k+1}} + p^k A)^n = 0$ . If  $2 \leq n \leq 3$ , then we might not have  $(E_{p^{k+1}} + p^k A)^n = 0$ , but each nonzero term in it must contain a factor of  $p^k A$ , so we deduce that  $(E_{p^{k+1}} + p^k A)^n$  is a multiple of  $p^k$ . Since  $(p, k) \notin \{(2, 1), (3, 1)\}$ , we must have  $p^k \geq 4$ , so  $2 \leq n < p^k$ . We then have that  $\binom{p^k}{n}$  is divisible by  $p$ , so  $\binom{p^k}{n}(E_{p^{k+1}} + p^k A)^n$  is a multiple of  $p^{k+1}$ ; i.e. it vanishes.  $\square$

## 4 The twisted first homology groups of $\mathrm{Sp}_{2g}(\mathbb{Z}_L)$

This section has three subsections. In §4.1, we show that  $H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_L); H_1(\Sigma_g; \mathbb{Z}_2)) \cong \mathbb{Z}_2$  for  $g \geq 2$  and  $L$  even with  $4 \nmid L$ , slightly generalizing a result of Pollatsek [51]. In §4.3, we prove Theorem E, which asserts that  $H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_L); \mathfrak{sp}_{2g}(\mathbb{Z}_L)) = 0$  for  $g \geq 3$  and  $L \geq 2$  such that  $4 \nmid L$ . This is preceded by §4.2, where a number of preliminary results are collected.

### 4.1 The first homology group of $\mathrm{Sp}_{2g}(\mathbb{Z}_L)$ with coefficients $H_1(\Sigma_g; \mathbb{Z}_2)$

Assume that  $G$  is a group and  $M$  is a  $G$ -vector space over a field of characteristic not equal to 2. Furthermore, assume that there is some element in the center of  $G$  that acts by  $-1$  on  $M$ . It is an easy exercise in elementary group cohomology (whose details we leave to the reader since this result is only needed for motivation) that under these conditions we have  $H^1(G; M) = 0$ . In particular, if  $p$  is an odd prime, then this applies to the action of  $\mathrm{Sp}_{2g}(\mathbb{Z}_p)$  on  $H_1(\Sigma_g; \mathbb{Z}_p)$ , since we have  $-\mathbb{I}_{2g}$  in the center of  $\mathrm{Sp}_{2g}(\mathbb{Z}_p)$ .

For  $p = 2$ , this fails, and in fact Pollatsek [51] showed that  $H^1(\mathrm{Sp}_{2g}(\mathbb{Z}_2); H_1(\Sigma_g; \mathbb{Z}_2)) \cong \mathbb{Z}_2$  for  $g \geq 2$ . The following theorem slightly extends this.

**Theorem 4.1.** *For  $g \geq 3$  and  $L \geq 2$  such that  $2 \mid L$  and  $4 \nmid L$ , we have  $H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_L); H_1(\Sigma_g; \mathbb{Z}_2)) = \mathbb{Z}_2$ .*

*Remark.* Computer calculations indicate that the condition  $4 \nmid L$  is probably unnecessary.

*Proof of Theorem 4.1.* The algebraic intersection form is a nondegenerate pairing on  $H_1(\Sigma_g; \mathbb{Z}_2)$ , so we have an isomorphism  $H_1(\Sigma_g; \mathbb{Z}_2) \cong (H_1(\Sigma_g; \mathbb{Z}_2))^*$  of  $\mathrm{Sp}_{2g}(\mathbb{Z}_L)$ -modules. Using Lemma 2.1, this implies that it is enough to prove that  $H^1(\mathrm{Sp}_{2g}(\mathbb{Z}_L); H_1(\Sigma_g; \mathbb{Z}_2)) = \mathbb{Z}_2$ . By Lemma 3.8, we have  $\mathrm{Sp}_{2g}(\mathbb{Z}_L) \cong \mathrm{Sp}_{2g}(\mathbb{Z}_2) \oplus \mathrm{Sp}_{2g}(\mathbb{Z}_{L/2})$  (this is where we use the fact that  $4 \nmid L$ ). Clearly  $\mathrm{Sp}_{2g}(\mathbb{Z}_{L/2})$  acts trivially on  $H_1(\Sigma_g; \mathbb{Z}_2)$ , and Lemma 3.6 says that  $H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_{L/2}); \mathbb{Z}) = 0$ . We can thus apply Lemma 2.3 to deduce that  $H^1(\mathrm{Sp}_{2g}(\mathbb{Z}_L); H_1(\Sigma_g; \mathbb{Z}_2)) \cong H^1(\mathrm{Sp}_{2g}(\mathbb{Z}_2); H_1(\Sigma_g; \mathbb{Z}_2))$ . Pollatsek's theorem [51] says that this is 0, and the result follows.  $\square$

## 4.2 Some preliminaries for Theorem E

### 4.2.1 The transgression in the cohomology Hochschild-Serre spectral sequence

In [27], Huebschmann gives a recipe for computing the first transgression in the Hochschild-Serre spectral sequence. We will need a very special case of this theorem.

Consider a short exact sequence of groups

$$1 \longrightarrow A \longrightarrow G \longrightarrow Q \longrightarrow 1, \quad (6)$$

where  $A$  is abelian. Regarding  $A$  as a  $G$ -module, there is an associated Hochschild-Serre spectral sequence converging to  $H^*(G;A)$ . The first transgression of this spectral sequence is the differential

$$d : H^0(Q;H^1(A;A)) \longrightarrow H^2(Q;H^0(A;A)).$$

Now,  $H^0(Q;H^1(A;A)) \cong \text{Hom}_Q(A,A)$  contains a distinguished element, namely, the identity homomorphism  $i : A \rightarrow A$ . There is thus a canonical element  $d(i) \in H^2(Q;H^0(A;A)) \cong H^2(Q;A)$  associated to the extension (6). The Euler class of (6) provides another canonical element of  $H^2(Q;A)$ . Huebschmann proved that these two elements are equal.

**Theorem 4.2** (Huebschmann, [27, Theorem 6]). *Up to signs,  $d(i) \in H^2(Q;A)$  is the Euler class of the extension (6). In particular, if (6) is a nontrivial extension, then  $d(i)$  is nonzero.*

*Remark.* In fact, Huebschmann proves a much more general result than Theorem 4.2, but in our special case it is easy to see that Huebschmann's theorem reduces to Theorem 4.2.

#### 4.2.2 The symplectic Lie algebra

We will need two facts about  $\mathfrak{sp}_{2g}(\mathbb{Z}_p)$ . They are both analogues of familiar results in characteristic 0, but their proofs are somewhat different. The first is the following.

**Proposition 4.3.** *For  $g \geq 2$  and  $p$  an odd prime, the group  $\text{Hom}_{\text{Sp}_{2g}(\mathbb{Z}_p)}(\mathfrak{sp}_{2g}(\mathbb{Z}_p), \mathfrak{sp}_{2g}(\mathbb{Z}_p))$  is cyclic of order  $p$  and generated by the identity.*

For the proof of Proposition 4.3, we will need the following theorem. In it, we denote by  $\overline{\mathbb{F}}_p$  the algebraic closure of the field with  $p$  elements.

**Theorem 4.4.** *For  $g \geq 1$  and  $p$  an odd prime,  $\mathfrak{sp}_{2g}(\overline{\mathbb{F}}_p)$  is an irreducible  $\text{Sp}_{2g}(\mathbb{Z}_p)$ -module.*

Theorem 4.4 can be extracted from the literature as follows. A theorem of Jacobson [31, Lemma 7] says that  $\mathfrak{sp}_{2g}(\overline{\mathbb{F}}_p)$  is an irreducible Lie algebra (we remark that Jacobson showed that this fails if  $p = 2$ ). The standard theory of algebraic groups then says that  $\mathfrak{sp}_{2g}(\overline{\mathbb{F}}_p)$  is an irreducible representation of  $\text{Sp}_{2g}(\overline{\mathbb{F}}_p)$  (see [6, §6] for more details on this point). A theorem of Curtis [13] finally allows us to conclude that this representation remains irreducible upon being restricted to  $\text{Sp}_{2g}(\mathbb{Z}_p) < \text{Sp}_{2g}(\overline{\mathbb{F}}_p)$ , as desired.

*Remark.* Curtis's paper [13] makes the assumption that  $p > 7$ , but this assumption is not necessary; see [6, Corollary 7.3] for the details of the general case.

*Proof of Proposition 4.3.* Consider

$$f \in \text{Hom}_{\text{Sp}_{2g}(\mathbb{Z}_p)}(\mathfrak{sp}_{2g}(\mathbb{Z}_p), \mathfrak{sp}_{2g}(\mathbb{Z}_p)).$$

We get an induced  $\text{Sp}_{2g}(\mathbb{Z}_p)$ -module homomorphism  $\bar{f} : \mathfrak{sp}_{2g}(\overline{\mathbb{F}}_p) \rightarrow \mathfrak{sp}_{2g}(\overline{\mathbb{F}}_p)$ . Theorem 4.4 says that  $\mathfrak{sp}_{2g}(\overline{\mathbb{F}}_p)$  is an irreducible  $\text{Sp}_{2g}(\mathbb{Z}_p)$ -module, so Schur's lemma implies that  $\bar{f}$  is scalar multiplication by an element of  $\overline{\mathbb{F}}_p$ . The homomorphism  $f$  is then multiplication by the same element, so we deduce that in fact  $f$  is scalar multiplication by an element of  $\mathbb{Z}_p$ , and the proposition follows.  $\square$

The second result we will need is as follows.

**Lemma 4.5.** *For  $g \geq 1$  and  $p$  an odd prime, the  $\mathrm{Sp}_{2g}(\mathbb{Z}_p)$ -modules  $\mathfrak{sp}_{2g}(\mathbb{Z}_p)$  and  $(\mathfrak{sp}_{2g}(\mathbb{Z}_p))^*$  are isomorphic.*

*Remark.* Lemma 4.5 is false for  $p = 2$ . Indeed, if it were true, then by Lemma 2.1 we would have  $H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_2); \mathfrak{sp}_{2g}(\mathbb{Z}_2)) \cong H^1(\mathrm{Sp}_{2g}(\mathbb{Z}_2); \mathfrak{sp}_{2g}(\mathbb{Z}_2))$ . However, while we will prove below that  $H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_2); \mathfrak{sp}_{2g}(\mathbb{Z}_2)) = 0$ , a computer calculation similar to the one we will describe in Step 1 of the proof of Theorem E shows that  $H^1(\mathrm{Sp}_{2g}(\mathbb{Z}_2); \mathfrak{sp}_{2g}(\mathbb{Z}_2)) = \mathbb{Z}_2$ .

If we were working in characteristic 0, then Lemma 4.5 would follow from the nondegeneracy of the Killing form  $\mathrm{Tr}(\mathrm{ad}(x)\mathrm{ad}(y))$ . However, it turns out that the Killing form is degenerate if  $p \mid g + 1$  (on [57, p. 47], this observation is attributed to Dynkin [17]). Thankfully, in our situation something even more simple-minded works. Namely, elements of  $\mathfrak{sp}_{2g}(\mathbb{Z}_p)$  are *matrices*, so we can define

$$(x, y) = \mathrm{Tr}(xy).$$

This is clearly a symmetric bilinear form invariant under the action of  $\mathrm{Sp}_{2g}(\mathbb{Z}_p)$ , and the following lemma says that it is nondegenerate.

**Lemma 4.6.** *If  $g \geq 1$  and  $p$  is an odd prime, then  $(\cdot, \cdot)$  is a nondegenerate bilinear form on  $\mathfrak{sp}_{2g}(\mathbb{Z}_p)$ .*

This lemma, which has Lemma 4.5 as an immediate corollary, seems to be well-known. For instance, it is asserted without proof on [57, p. 47]. However, we have been unable to find a reference proving it, so we include a proof.

*Proof of Lemma 4.6.* We will write down a basis  $S$  for  $\mathfrak{sp}_{2g}(\mathbb{Z}_p)$  such that for  $x \in S$ , there is a unique  $x' \in S$  with  $\mathrm{Tr}(xx') \neq 0$ . Choosing an ordering of the elements of  $S$ , it will follow that the matrix  $M$  representing the bilinear form  $(\cdot, \cdot)$  has a unique non-zero element in each row. Since the form is symmetric,  $M$  will also have a unique non-zero element in each column. We will thus be able to conclude that up to signs, the determinant of  $M$  is the product of these non-zero elements, and the lemma will follow.

For  $1 \leq i, j \leq 2g$ , let  $E_{i,j}$  denote the matrix with a 1 at position  $(i, j)$  and 0's elsewhere. Define

$$S = \{A_{i,j} \mid 1 \leq i, j \leq g\} \cup \{B_i, B'_i \mid 1 \leq i \leq g\} \cup \{C_{i,j}, C'_{i,j} \mid 1 \leq i < j \leq g\} \subset \mathfrak{sp}_{2g}(\mathbb{Z}_p),$$

where

$$\begin{aligned} A_{i,j} &= E_{i,j} - E_{g+i,g+j}, & C_{i,j} &= E_{g+i,j} + E_{g+j,i}, & C'_{i,j} &= E_{i,g+j} + E_{j,g+i}, \\ B_i &= E_{g+i,i}, & B'_i &= E_{i,g+i}. \end{aligned}$$

It is clear that  $S$  is a basis for  $\mathfrak{sp}_{2g}(\mathbb{Z}_p)$ . The lemma is now a simple case by case check. For instance, the only  $x \in S$  with  $\mathrm{Tr}(A_{i,j}x) \neq 0$  is  $A_{j,i}$ , and

$$\mathrm{Tr}(A_{i,j}A_{j,i}) = \mathrm{Tr}((E_{i,j} - E_{g+i,g+j})(E_{j,i} - E_{g+j,g+i})) = \mathrm{Tr}(E_{i,i} + E_{g+j,g+j}) = 2.$$

The other cases are similar. □

### 4.2.3 Homological stability

Fix a ring  $R$ . Define a *coefficient system* for  $\mathrm{Sp}_{2g}(R)$  to be a sequence  $\{M_g, i_g\}_{g \geq 1}$ , where  $M_g$  is an  $\mathrm{Sp}_{2g}(R)$ -module and  $i_g : M_g \rightarrow M_{g+1}$  is a map that is equivariant with respect to the natural inclusion  $\mathrm{Sp}_{2g}(R) \hookrightarrow \mathrm{Sp}_{2(g+1)}(R)$ . One example of such a coefficient system is  $M_g = \mathfrak{sp}_{2g}(R)$ , where the maps  $i_g : M_g \rightarrow M_{g+1}$  are the obvious inclusions.

For  $R$  sufficiently nice, Charney [11] proved a homological stability result for  $\mathrm{Sp}_{2g}(R)$  with respect to coefficient systems that are *central of degree  $k$*  for some  $k \geq 0$ . We will not reproduce the definition of this here, but merely record the following fact, whose proof is similar to that of Example 3 on [11, p. 123].

**Lemma 4.7.** *The coefficient system  $\mathfrak{sp}_{2g}(R)$  is central of degree 2.*

The statement of Charney's theorem depends on the connectivity of a certain space  $HU(R^{2g})$ , the *poset of hyperbolic unimodular sequences* in  $R^{2g}$  (equipped with the standard symplectic form). The key property of  $HU(R^{2g})$  for us is the following lemma.

**Lemma 4.8.** *For  $g \geq 1$  and  $L \geq 2$ , the complex  $HU((\mathbb{Z}_L)^{2g})$  is  $\frac{g-4}{2}$ -connected.*

Lemma 4.8 can be extracted from the literature in the following way. In [45, Theorem 5.8], Mirzaii and van der Kallen proved a much more general result, namely, that if  $R$  has *stable rank*  $\mathrm{sr}(R)$ , then  $HU(R^{2g})$  is  $\frac{g-\mathrm{sr}(R)-3}{2}$ -connected. The lemma then follows from the fact that  $\mathbb{Z}_L$  has stable rank 1 (see [2, Proposition V.3.4.a]; a ring  $R$  has stable rank  $n$  in the sense of [45] if it satisfies Bass's condition  $SR_{n+1}(R)$ ).

We can now state Charney's theorem.

**Theorem 4.9** (Charney, [11, Theorem 4.3]). *Let  $R$  be a finitely generated ring and let  $a \geq 3$  be such that  $HU(R^{2g})$  is  $\frac{g-a-1}{2}$ -connected for all  $g \geq 1$ . Let  $\{M_g, i_g\}_{g \geq 1}$  be a coefficient system for  $\mathrm{Sp}_{2g}(R)$  that is central of degree  $k$ . Then the natural map*

$$H_i(\mathrm{Sp}_{2g}(R); M_g) \rightarrow H_i(\mathrm{Sp}_{2(g+1)}(R); M_{g+1})$$

*is surjective for  $g \geq 2i + k + a$  and bijective for  $g \geq 2i + k + a + 1$ .*

*Remark.* Charney proved a result similar to Lemma 4.8 for  $R$  a PID. This is enough to cover most of our uses of Theorem 4.9; however, in Step 2 of the proof of Theorem E below, we will need to apply Theorem 4.9 with  $R = \mathbb{Z}_9$ , which is not a domain.

## 4.3 The proof of Theorem E

Recall that we wish to prove that

$$H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_L); \mathfrak{sp}_{2g}(\mathbb{Z}_L)) = 0$$

for  $g \geq 3$  and  $L \geq 2$  such that  $4 \nmid L$ . The proof will have four steps. In all the steps, we fix  $g \geq 3$ .

**Step 1.** *Let  $p$  be prime. Then  $H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_p); \mathfrak{sp}_{2g}(\mathbb{Z}_p)) = 0$ .*

Lemma 2.1 says that

$$(\mathrm{H}_1(\mathrm{Sp}_{2g}(\mathbb{Z}_p); \mathfrak{sp}_{2g}(\mathbb{Z}_p)))^* = \mathrm{H}^1(\mathrm{Sp}_{2g}(\mathbb{Z}_p); (\mathfrak{sp}_{2g}(\mathbb{Z}_p))^*). \quad (7)$$

It is enough to prove that either side of (7) vanishes. For  $p$  odd, Lemma 4.5 says that  $(\mathfrak{sp}_{2g}(\mathbb{Z}_p))^* \cong \mathfrak{sp}_{2g}(\mathbb{Z}_p)$ . A theorem of Volklein [62] says that  $\mathrm{H}^1(\mathrm{Sp}_{2g}(\mathbb{Z}_p); \mathfrak{sp}_{2g}(\mathbb{Z}_p)) = 0$  for  $g \geq 3$  and primes  $p > 5$ . We thus only need to deal with  $p \in \{2, 3, 5\}$ . By the homological stability theorem for the symplectic group (Theorem 4.9) together with Lemmas 4.7 and 4.8, for a fixed  $p$  the desired result will hold for all  $g$  if it holds for  $g$  such that  $3 \leq g \leq 8$ . This reduces the result to checking a finite number of cases. These cases may be easily checked on a computer using the algorithm for calculating twisted first cohomology groups given in [26, §7.6.1]. Computer code for performing these calculations (written in C++) can be found on the author's webpage.

**Step 2.** *Let  $p$  be an odd prime and let  $k \geq 1$ . Then  $\mathrm{H}_1(\mathrm{Sp}_{2g}(\mathbb{Z}_{p^k}); \mathfrak{sp}_{2g}(\mathbb{Z}_p)) = 0$ .*

Since  $p$  is odd, Lemma 2.1 together with Lemma 4.5 says that the desired claim is equivalent to proving that  $\mathrm{H}^1(\mathrm{Sp}_{2g}(\mathbb{Z}_{p^k}); \mathfrak{sp}_{2g}(\mathbb{Z}_p)) = 0$ . The proof of this will be by induction on  $k$ . The case  $k = 1$  follows from Step 1. Our inductive argument also does not cover the case  $(p, k) = (3, 2)$ , but this case can be handled by a computer calculation just as in Step 1.

Assume now that  $k \geq 1$  and that  $\mathrm{H}^1(\mathrm{Sp}_{2g}(\mathbb{Z}_{p^k}); \mathfrak{sp}_{2g}(\mathbb{Z}_p)) = 0$ . If  $p = 3$ , then we will also assume that  $k \geq 2$ . We must prove that  $\mathrm{H}^1(\mathrm{Sp}_{2g}(\mathbb{Z}_{p^{k+1}}); \mathfrak{sp}_{2g}(\mathbb{Z}_p)) = 0$ . By Lemma 3.9, we have a nonsplit short exact exact sequence

$$1 \longrightarrow \mathfrak{sp}_{2g}(\mathbb{Z}_p) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}_{p^{k+1}}) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}_{p^k}) \longrightarrow 1. \quad (8)$$

Observe that the restriction of the action of  $\mathrm{Sp}_{2g}(\mathbb{Z}_{p^{k+1}})$  on  $\mathfrak{sp}_{2g}(\mathbb{Z}_p)$  to the subgroup  $\mathfrak{sp}_{2g}(\mathbb{Z}_p)$  is trivial. The  $E_2$  page of the associated Hochschild-Serre spectral sequence for cohomology with coefficients in  $\mathfrak{sp}_{2g}(\mathbb{Z}_p)$  is of the form

$$\begin{array}{|c} \mathrm{H}^0(\mathrm{Sp}_{2g}(\mathbb{Z}_{p^k}); \mathrm{H}^1(\mathfrak{sp}_{2g}(\mathbb{Z}_p); \mathfrak{sp}_{2g}(\mathbb{Z}_p))) \\ \hline * \qquad \qquad \qquad \mathrm{H}^1(\mathrm{Sp}_{2g}(\mathbb{Z}_p); \mathfrak{sp}_{2g}(\mathbb{Z}_p)) \quad \mathrm{H}^2(\mathrm{Sp}_{2g}(\mathbb{Z}_p); \mathfrak{sp}_{2g}(\mathbb{Z}_p)) \end{array}$$

By assumption, we have  $\mathrm{H}^1(\mathrm{Sp}_{2g}(\mathbb{Z}_p); \mathfrak{sp}_{2g}(\mathbb{Z}_p)) = 0$ . Let

$$d : \mathrm{H}^0(\mathrm{Sp}_{2g}(\mathbb{Z}_{p^k}); \mathrm{H}^1(\mathfrak{sp}_{2g}(\mathbb{Z}_p); \mathfrak{sp}_{2g}(\mathbb{Z}_p))) \longrightarrow \mathrm{H}^2(\mathrm{Sp}_{2g}(\mathbb{Z}_p); \mathfrak{sp}_{2g}(\mathbb{Z}_p))$$

be the differential. Since the action of  $\mathrm{Sp}_{2g}(\mathbb{Z}_{p^k})$  on  $\mathfrak{sp}_{2g}(\mathbb{Z}_p)$  factors through  $\mathrm{Sp}_{2g}(\mathbb{Z}_p)$ , we can use Proposition 4.3 to deduce that

$$\mathrm{H}^0(\mathrm{Sp}_{2g}(\mathbb{Z}_{p^k}); \mathrm{H}^1(\mathfrak{sp}_{2g}(\mathbb{Z}_p); \mathfrak{sp}_{2g}(\mathbb{Z}_p))) \cong \mathrm{Hom}_{\mathrm{Sp}_{2g}(\mathbb{Z}_{p^k})}(\mathfrak{sp}_{2g}(\mathbb{Z}_p), \mathfrak{sp}_{2g}(\mathbb{Z}_p)).$$

is an order  $p$  cyclic group generated by the identity homomorphism  $i : \mathfrak{sp}_{2g}(\mathbb{Z}_p) \rightarrow \mathfrak{sp}_{2g}(\mathbb{Z}_p)$ . Since (8) does not split, Theorem 4.2 implies that  $d(i) \neq 0$ . We conclude that the  $E_3$  page of the spectral sequence is of the form

$$\begin{array}{|c} 0 \\ \hline * \quad 0 \quad * \end{array}$$

and the desired result follows.

**Step 3.** Let  $L \geq 2$  satisfy  $4 \nmid L$  and let  $p$  be a prime dividing  $L$ . Then  $H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_L); \mathfrak{sp}_{2g}(\mathbb{Z}_p)) = 0$ .

Write  $L = p_1^{k_1} \cdots p_n^{k_n}$ , where the  $p_i$  are distinct primes and  $p_1 = p$ . By Lemma 3.8, we have

$$\mathrm{Sp}_{2g}(\mathbb{Z}_L) \cong \bigoplus_{i=1}^n \mathrm{Sp}_{2g}(\mathbb{Z}_{p_i^{k_i}}).$$

For  $i > 1$ , the action of  $\mathrm{Sp}_{2g}(\mathbb{Z}_{p_i^{k_i}})$  on  $\mathfrak{sp}_{2g}(\mathbb{Z}_p)$  is trivial. Additionally, Lemma 3.6 says that  $H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_{p_i^{k_i}}); \mathbb{Z}) = 0$  for all  $i$ . Lemma 2.3 thus implies that

$$H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_L); \mathfrak{sp}_{2g}(\mathbb{Z}_p)) \cong H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_{p_1^{k_1}}); \mathfrak{sp}_{2g}(\mathbb{Z}_p)).$$

If  $p_1$  is odd, then Step 2 says that  $H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_{p_1^{k_1}}); \mathfrak{sp}_{2g}(\mathbb{Z}_p)) = 0$ . If  $p_1 = 2$ , then the fact that  $4 \nmid L$  implies that  $k_1 = 1$ , so Step 1 says that  $H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_{p_1^{k_1}}); \mathfrak{sp}_{2g}(\mathbb{Z}_p)) = 0$ . The result follows.

**Step 4.** Let  $L \geq 2$  satisfy  $4 \nmid L$  and let  $L' > 1$  satisfy  $L' \mid L$ . Then  $H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_L); \mathfrak{sp}_{2g}(\mathbb{Z}_{L'})) = 0$ . In particular,  $H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_L); \mathfrak{sp}_{2g}(\mathbb{Z}_L)) = 0$ .

Write  $L = p_1^{k_1} \cdots p_n^{k_n}$  and  $L' = p_1^{k'_1} \cdots p_n^{k'_n}$ , where the  $p_i$  are distinct odd primes and  $k_i \geq 1$  for  $1 \leq i \leq n$ . Observe that  $0 \leq k'_i \leq k_i$  for  $1 \leq i \leq n$ . The proof will be by induction on  $\sum |k'_i|$ .

The base case is  $\sum |k'_i| = 1$ , in which case  $L' = p_j$  for some  $1 \leq j \leq n$  and the result follows from Step 3. Assume now that  $\sum |k'_i| > 1$  and that the desired result is true for all smaller sums. Pick  $1 \leq m \leq n$  such that  $k'_m \geq 1$ . Set

$$k''_i = \begin{cases} k'_i & \text{if } i \neq m \\ k'_i - 1 & \text{if } i = m \end{cases} \quad (1 \leq i \leq n)$$

and  $L'' = p_1^{k''_1} \cdots p_n^{k''_n}$ . There is a short exact sequence

$$0 \longrightarrow \mathbb{Z}_{L''} \longrightarrow \mathbb{Z}_{L'} \longrightarrow \mathbb{Z}_{p_m} \longrightarrow 0$$

of rings and a corresponding short exact sequence

$$0 \longrightarrow \mathfrak{sp}_{2g}(\mathbb{Z}_{L''}) \longrightarrow \mathfrak{sp}_{2g}(\mathbb{Z}_{L'}) \longrightarrow \mathfrak{sp}_{2g}(\mathbb{Z}_{p_m}) \longrightarrow 0$$

of  $\mathrm{Sp}_{2g}(\mathbb{Z}_L)$ -modules. This induces a long exact sequence in  $\mathrm{Sp}_{2g}(\mathbb{Z}_L)$ -homology which contains the segment

$$H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_L); \mathfrak{sp}_{2g}(\mathbb{Z}_{L''})) \rightarrow H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_L); \mathfrak{sp}_{2g}(\mathbb{Z}_{L'})) \rightarrow H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_L); \mathfrak{sp}_{2g}(\mathbb{Z}_{p_m})).$$

The inductive hypothesis implies that

$$H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_L); \mathfrak{sp}_{2g}(\mathbb{Z}_{L''})) = 0 \quad \text{and} \quad H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_L); \mathfrak{sp}_{2g}(\mathbb{Z}_{p_m})) = 0,$$

so  $H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_L); \mathfrak{sp}_{2g}(\mathbb{Z}_{L'})) = 0$ , as desired.

## 5 The moduli space of principally polarized abelian varieties

### 5.1 Definitions

We now introduce the moduli space of principally polarized abelian varieties. For more information, see [21, 29, 47].

The Siegel upper half plane  $\mathcal{H}_g$  is the space of  $g \times g$  symmetric complex matrices with positive definite imaginary parts. A principal polarization of an abelian variety  $A$  is an extra bit of data whose precise nature does not concern us here. All we will need to know is that it provides a symplectic form on  $H_1(A; \mathbb{Z})$ . The space  $\mathcal{H}_g$  parametrizes pairs  $(A, B)$ , where  $A$  is a principally polarized abelian variety and  $B$  is a symplectic basis for  $H_1(A; \mathbb{Z})$ . The abelian variety associated to  $\Omega \in \mathcal{H}_g$  is  $\mathbb{C}^g / (\mathbb{Z}^g + \Omega \mathbb{Z}^g)$ . The space  $\mathcal{H}_g$  is a contractible complex manifold and the group  $\mathrm{Sp}_{2g}(\mathbb{Z})$  acts on it holomorphically and properly discontinuously. If  $M \in \mathrm{Sp}_{2g}(\mathbb{Z})$  and  $\Omega \in \mathcal{H}_g$ , then

$$M(\Omega) = \frac{A\Omega + B}{C\Omega + D},$$

where  $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$  is the decomposition of  $M$  into  $g \times g$  blocks. One can view this action as changing the symplectic basis associated to an element of  $\mathcal{H}_g$ .

To simplify our statements, we will denote  $\mathrm{Sp}_{2g}(\mathbb{Z})$  and  $\mathcal{A}_g$  by  $\mathrm{Sp}_{2g}(\mathbb{Z}, 1)$  and  $\mathcal{A}_g(1)$ , respectively. For  $L \geq 1$ , the space  $\mathcal{A}_g(L)$  is the orbifold  $(\mathcal{H}_g, \mathrm{Sp}_{2g}(\mathbb{Z}, L))$ . Thus  $\mathcal{A}_g(L')$  is a covering orbifold for  $\mathcal{A}_g(L)$  whenever  $L|L'$ . For  $L \geq 3$ , the group  $\mathrm{Sp}_{2g}(\mathbb{Z}, L)$  acts freely on  $\mathcal{H}_g$ , so in these cases  $\mathcal{A}_g(L)$  is a trivial orbifold. It is known that  $\mathcal{H}_g / \mathrm{Sp}_{2g}(\mathbb{Z}, L)$  is a quasiprojective variety for all  $L$  (see [29] for a proof and history of this result). We conclude that  $\mathcal{A}_g(L)$  is a quasiprojective orbifold with quasiprojective finite cover  $\mathcal{A}_g(L')$  for any  $L' \geq 3$  such that  $L|L'$ .

### 5.2 Some classical results

To investigate  $\mathrm{Pic}(\mathcal{A}_g(L))$ , our main tool will be Lemma 2.7. To verify the conditions of that lemma, we will need a number of classical results, the first of which are the following group-cohomology calculations.

**Theorem 5.1.** *For  $g \geq 3$ , we have  $H_1(\mathrm{Sp}_{2g}(\mathbb{Z}); \mathbb{Z}) = 0$ . For  $g \geq 4$ , we have  $H_2(\mathrm{Sp}_{2g}(\mathbb{Z}); \mathbb{Z}) \cong \mathbb{Z}$ .*

*Remark.* Theorem 5.1 is classical, but I do not know a good reference for it. See the remark after the proof of Lemma 7.5 below for one way to extract it from the literature.

**Theorem 5.2** (Kazhdan, [38]). *Fix  $g \geq 2$  and  $L \geq 1$ . We then have  $H^1(\mathrm{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z}) = 0$ .*

**Theorem 5.3** (Borel, [7, 8]). *Fix  $g \geq 3$  and  $L \geq 1$ . We then have  $H_2(\mathrm{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Q}) = \mathbb{Q}$ .*

*Remark.* The range of  $g$  in Theorem 5.3 is better than that claimed in [7], and follows from a criterion that can be found in [8].

Recall that  $\mathrm{Pic}(\mathcal{A}_g)$  is the set of  $\mathrm{Sp}_{2g}(\mathbb{Z})$ -equivariant line bundles on  $\mathcal{H}_g$ . There is a standard line bundle  $\overline{\mathcal{L}}_g \in \mathrm{Pic}(\mathcal{A}_g)$  whose sections are the Siegel modular forms (of weight 1 and level 1; see [21] or [47]). The fiber of  $\overline{\mathcal{L}}_g$  over a point of  $\mathcal{H}_g$  can be identified with the space of holomorphic 1-forms on the associated abelian variety. The following theorem gives its first Chern class.

**Theorem 5.4** ([23, Theorem 17.4]). *For  $g \geq 3$ , the class  $c_1(\overline{\mathcal{L}}_g)$  generates  $H^2(\mathrm{Sp}_{2g}(\mathbb{Z}); \mathbb{Z}) \cong \mathbb{Z}$ .*

Define  $\overline{\mathcal{L}}_g(L) \in \text{Pic}(\mathcal{A}_g(L))$  to be the pullback of  $\overline{\mathcal{L}}_g$ . For even levels, the transformation formulas for the theta-nulls (see, e.g., [21, §I.5]) show that they provide a square root for  $\overline{\mathcal{L}}_g(L)$  when  $L$  is even. We thus have the following divisibility result.

**Lemma 5.5.** *For  $g \geq 3$  and  $L \geq 2$  such that  $L$  is even,  $\overline{\mathcal{L}}_g(L) \in \text{Pic}(\mathcal{A}_g(L))$  is divisible by 2.*

We now verify exact sequence (2) from the introduction. Fix  $g \geq 3$  and  $L \geq 2$  (we are *not* assuming that  $4 \nmid L$ ). By Theorem 5.3, the universal coefficients exact sequence for  $H^2(\text{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z})$  is of the form

$$0 \longrightarrow \text{Hom}(H_1(\text{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \longrightarrow H^2(\text{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z}) \longrightarrow \mathbb{Z} \longrightarrow 0. \quad (9)$$

Combining Theorem 5.2 with the first conclusion of Lemma 2.4, we can apply Theorem 2.6 to deduce that there is an injection  $\text{Pic}(\mathcal{A}_g(L)) \hookrightarrow \text{Pic}_{\text{top}}(\mathcal{A}_g(L))$ . Combining Lemma 2.5 with the first conclusion of Lemma 2.4, we get an injection  $\text{Pic}_{\text{top}}(\mathcal{A}_g(L)) \hookrightarrow H^2(\text{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z})$ . The image of  $\text{Pic}(\mathcal{A}_g(L))$  in  $H^2(\text{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z})$  contains the entire torsion subgroup, as every element of  $\text{Hom}(H_1(\text{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z}), \mathbb{Q}/\mathbb{Z})$  is the first Chern class of a flat line bundle. Moreover,  $\overline{\mathcal{L}}_g(L) \in \text{Pic}(\mathcal{A}_g(L))$  has infinite order, so the image of  $\text{Pic}(\mathcal{A}_g(L))$  in  $H^2(\text{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z})$  cannot simply be the torsion subgroup. Restricting (9) to the image of  $\text{Pic}(\mathcal{A}_g(L))$  in  $H^2(\text{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z})$ , we therefore obtain an exact sequence

$$0 \longrightarrow \text{Hom}(H_1(\text{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \longrightarrow \text{Pic}(\mathcal{A}_g(L)) \longrightarrow \mathbb{Z} \longrightarrow 0,$$

as desired.

### 5.3 The proofs of Theorems C and D

We wish to apply Lemma 2.7 with  $(X, G) = (\mathcal{H}_g, \text{Sp}_{2g}(\mathbb{Z}))$  and  $G' = \text{Sp}_{2g}(\mathbb{Z}, L)$  and  $\lambda = \overline{\mathcal{L}}_g$  and

$$n = \begin{cases} 1 & \text{if } L \text{ is odd,} \\ 2 & \text{if } L \text{ is even.} \end{cases}$$

The results of §5.1–5.2 show that all the conditions of Lemma 2.7 are satisfied except possibly the condition that the image  $A$  of  $H_2(\text{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z})$  in  $H_2(\text{Sp}_{2g}(\mathbb{Z}); \mathbb{Z}) \cong \mathbb{Z}$  is  $m\mathbb{Z}$  for some  $m$  with  $1 \leq m \leq n$ , which we now prove.

We will use the Hochschild-Serre spectral sequence in group homology arising from the short exact sequence

$$1 \longrightarrow \text{Sp}_{2g}(\mathbb{Z}, L) \longrightarrow \text{Sp}_{2g}(\mathbb{Z}) \longrightarrow \text{Sp}_{2g}(\mathbb{Z}_L) \longrightarrow 1.$$

Theorem 3.7 says that  $H_2(\text{Sp}_{2g}(\mathbb{Z}_L); \mathbb{Z}) = 0$ . Theorem 2.2 thus implies that the  $E^\infty$  page of our spectral sequence is of the form

$$\begin{array}{|ccc} A & & \\ * & B & \\ * & * & 0 \end{array}$$

with  $B$  a quotient of  $H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_L); H_1(\mathrm{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z}))$ . Our goal is to show that  $B = 0$  if  $L$  is odd and that  $B$  is either  $0$  or  $\mathbb{Z}_2$  if  $L$  is even (it will then follow that in fact  $B = \mathbb{Z}_2$ ). To do this, it is enough to prove a similar result for  $H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_L); H_1(\mathrm{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z}))$ .

If  $L$  is odd, then Corollary 3.3 says that  $H_1(\mathrm{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z}) \cong \mathfrak{sp}_{2g}(\mathbb{Z}_L)$ , so we can apply Theorem E to deduce that

$$H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_L); H_1(\mathrm{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z})) \cong H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_L); \mathfrak{sp}_{2g}(\mathbb{Z}_L)) \cong 0.$$

If  $L$  is even, then Corollary 3.5 gives a short exact sequence

$$0 \longrightarrow H_1(\Sigma_g; \mathbb{Z}_2) \longrightarrow H_1(\mathrm{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z}) \longrightarrow \mathfrak{sp}_{2g}(\mathbb{Z}_L) \longrightarrow 0$$

of  $\mathrm{Sp}_{2g}(\mathbb{Z}_L)$ -modules. The long exact sequence in  $\mathrm{Sp}_{2g}(\mathbb{Z}_L)$ -homology contains the exact sequence

$$H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_L); H_1(\Sigma_g; \mathbb{Z}_2)) \longrightarrow H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_L); H_1(\mathrm{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z})) \longrightarrow H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_L); \mathfrak{sp}_{2g}(\mathbb{Z}_L)).$$

Using Theorems E and 4.1, this reduces to

$$\mathbb{Z}_2 \longrightarrow H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_L); H_1(\mathrm{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z})) \longrightarrow 0,$$

so  $H_1(\mathrm{Sp}_{2g}(\mathbb{Z}_L); H_1(\mathrm{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z}))$  is either  $0$  or  $\mathbb{Z}_2$ , as desired.

## 6 The Torelli group

### 6.1 Definition and generators

To prove our theorems about the moduli space of curves, we will need some results about the Torelli group (see [35] for a survey). Fix some  $g \geq 1$  and  $b \leq 1$ . Recall that the Torelli group  $\mathcal{I}_{g,b}$  is the subgroup of  $\mathrm{Mod}_{g,b}$  consisting of mapping classes that act trivially on  $H_1(\Sigma_{g,b}; \mathbb{Z})$ . We thus have a short exact sequence

$$1 \longrightarrow \mathcal{I}_{g,b} \longrightarrow \mathrm{Mod}_{g,b} \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1.$$

For several proofs that the map  $\mathrm{Mod}_{g,b} \rightarrow \mathrm{Sp}_{2g}(\mathbb{Z})$  is surjective, see [19, §7.1].

Two key elements of  $\mathcal{I}_{g,b}$  are as follows. Let  $T_x$  denote the Dehn twist about a simple closed curve  $x$ .

1. If  $x$  is a non-nullhomotopic separating curve (see Figure 1.a), then  $x$  is nullhomologous, and it follows that  $T_x \in \mathcal{I}_{g,b}$ . We will call  $T_x$  a *separating twist*.
2. If  $\{y, z\}$  is a pair of disjoint non-homotopic simple closed nonseparating curves such that  $y \cup z$  separates  $\Sigma_{g,b}$  (see Figure 1.a), then  $y$  and  $z$  are homologous. Thus  $T_y$  and  $T_z$  act in the same way on  $H_1(\Sigma_{g,b}; \mathbb{Z})$ , so  $T_y T_z^{-1} \in \mathcal{I}_{g,b}$ . We will call  $\{y, z\}$  a *bounding pair* and  $T_y T_z^{-1}$  a *bounding pair map*.

Following work of Birman [3], it was proven by Powell [52] that  $\mathcal{I}_{g,b}$  is generated by the set of all bounding pair maps and separating twists. For  $g \geq 3$ , this was improved by Johnson.

**Theorem 6.1** (Johnson, [32]). *For  $g \geq 3$  and  $b \leq 1$ , the group  $\mathcal{I}_{g,b}$  is generated by bounding pair maps.*

*Remark.* Later, Johnson [36] proved that  $\mathcal{I}_{g,b}$  is generated by finitely many bounding pair maps for  $g \geq 3$ . This should be contrasted with work of McCullough and Miller [43], who showed that  $\mathcal{I}_2$  is *not* finitely generated. It is still unknown whether or not  $\mathcal{I}_{g,b}$  is finitely presentable for  $g \geq 3$ .

## 6.2 Abelianization

### 6.2.1 The Johnson homomorphism

We now turn to the abelianization of  $\mathcal{S}_{g,b}$ . There are two key abelian quotients of  $\mathcal{S}_{g,b}$ . The first is the *Johnson homomorphism*, which was constructed by Johnson in [33]. The precise definition of this homomorphism will not be needed in this paper (it can be viewed as measuring the “unipotent” part of the action of  $\mathcal{S}_{g,b}$  on the second nilpotent truncation of  $\pi_1(\Sigma_{g,b})$ ), so we only list the properties of it that we need. Letting  $H = H_1(\Sigma_{g,b}; \mathbb{Z})$ , it is a surjective  $\text{Mod}_{g,b}$ -equivariant homomorphism of the form

$$\tau : \mathcal{S}_{g,1} \longrightarrow \wedge^3 H \quad \text{or} \quad \tau : \mathcal{S}_g \longrightarrow (\wedge^3 H)/H,$$

depending on whether  $b$  is 1 or 0. Here  $H$  is embedded in  $\wedge^3 H$  as follows. Let  $\{a_1, b_1, \dots, a_g, b_g\}$  be a symplectic basis for  $H$ . The element  $\omega = a_1 \wedge b_1 + \dots + a_g \wedge b_g \in \wedge^2 H$  is then independent of the basis, and we have an injection  $H \hookrightarrow \wedge^3 H$  that takes  $h \in H$  to  $h \wedge \omega$ .

### 6.2.2 The Birman-Craggs-Johnson homomorphism

The second abelian quotient of  $\mathcal{S}_{g,b}$  is given by the Birman-Craggs-Johnson homomorphism. In [5], Birman and Craggs constructed a large family of  $\mathbb{Z}_2$  quotients of  $\mathcal{S}_{g,b}$  using the Rochlin invariant of homology 3-spheres. Later, in [34], Johnson packaged all these abelian quotients into one homomorphism. Its target is easiest to describe when  $b = 1$ . Let  $\Omega$  be the set of all  $\mathbb{Z}_2$ -quadratic forms on  $H_1(\Sigma_{g,b}; \mathbb{Z}_2)$  inducing the standard intersection form. In other words,  $\Omega$  is the set of all maps  $f : H_1(\Sigma_{g,b}; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$  satisfying  $f(x+y) = f(x) + f(y) + i(x,y)$  for  $x, y \in H_1(\Sigma_{g,b}; \mathbb{Z}_2)$ , where  $i(\cdot, \cdot)$  is the algebraic intersection form. The target of the Birman-Craggs-Johnson homomorphism is a certain additive subspace of the ring  $\text{Map}(\Omega, \mathbb{Z}_2)$  of  $\mathbb{Z}_2$ -valued functions on  $\Omega$ .

For each  $x \in H_1(\Sigma_{g,b}; \mathbb{Z}_2)$ , there is a function  $\bar{x} : \Omega \rightarrow \mathbb{Z}_2$  that takes  $f \in \Omega$  to  $f(x)$ . It is clear that  $\bar{x}^2 = \bar{x}$  and  $\overline{x+y} = \bar{x} + \bar{y} + i(x,y)$  for  $x, y \in H_1(\Sigma_{g,b}; \mathbb{Z}_2)$ ; here  $i(x,y) \in \mathbb{Z}_2$  denotes the constant function. Motivated by this, we make the following definitions. Let  $B'(2g)$  denote the ring of polynomials with  $\mathbb{Z}_2$  coefficients in the (formal) symbols  $\{\bar{x} \mid x \in H_1(\Sigma_{g,b}; \mathbb{Z}_2), x \neq 0\}$ . Next, let  $B(2g)$  denote the quotient of  $B'(2g)$  by the ideal generated by set of relations

$$\begin{aligned} \bar{x}^2 &= \bar{x} && \text{for } x \in H_1(\Sigma_{g,b}; \mathbb{Z}_2) \text{ with } x \neq 0, \\ \overline{x+y} &= \bar{x} + \bar{y} + i(x,y) && \text{for } x, y \in H_1(\Sigma_{g,b}; \mathbb{Z}_2) \text{ with } x, y \neq 0 \text{ and } x \neq y. \end{aligned}$$

One can view elements of  $B(2g)$  as “square-free”  $\mathbb{Z}_2$ -polynomials in a fixed basis for  $H_1(\Sigma_g; \mathbb{Z}_2)$ . As such, elements of  $B(2g)$  have a natural degree. Let  $B_n(2g)$  denote the elements of  $B(2g)$  of degree at most  $n$ . Observe that  $B_n(2g)$  is an abelian 2-group with a natural action of  $\text{Sp}_{2g}(\mathbb{Z})$ . Moreover, we have a natural ring homomorphism  $B(2g) \rightarrow \text{Map}(\Omega, \mathbb{Z}_2)$ .

Johnson proved that the map  $B_3(2g) \rightarrow \text{Map}(\Omega, \mathbb{Z}_2)$  is injective. Its image is the target of the Birman-Craggs-Johnson homomorphism on a surface with one boundary component. Summing up, the Birman-Craggs-Johnson homomorphism on a surface with one boundary component is a surjective homomorphism

$$\sigma : \mathcal{S}_{g,1} \longrightarrow B_3(2g)$$

that is  $\text{Mod}_{g,1}$ -equivariant.

For closed surfaces, we have to take a further quotient of  $B_3(2g)$ . Associated to every  $f \in \Omega$  is its *Arf invariant*, which can be computed as follows. Let  $\{a_1, b_1, \dots, a_g, b_g\}$  be a symplectic basis

for  $H_1(\Sigma_{g,b}; \mathbb{Z}_2)$ . Then the Arf invariant of  $f$  is  $f(a_1)f(b_1) + \cdots + f(a_g)f(b_g)$ . One can show that this is independent of the symplectic basis and that two elements of  $\Omega$  are in the same  $\mathrm{Sp}_{2g}(\mathbb{Z}_2)$  orbit if and only if they have the same Arf invariant. Let  $\Omega_0 \subset \Omega$  be the set of quadratic forms of Arf invariant 0. There is then a restriction map  $\pi : \mathrm{Map}(\Omega, \mathbb{Z}_2) \rightarrow \mathrm{Map}(\Omega_0, \mathbb{Z}_2)$ . Let  $\bar{B}_n(2g) = \pi(B_n(2g))$ . The Birman-Craggs-Johnson homomorphism on a closed surface is a surjective  $\mathrm{Mod}_g$ -equivariant homomorphism

$$\sigma : \mathcal{S}_g \longrightarrow \bar{B}_3(2g).$$

See [34] for a calculation of the dimension of  $\bar{B}_3(2g)$ .

### 6.2.3 Combining the abelian quotients

Both the Johnson homomorphisms and the Birman-Craggs-Johnson homomorphisms factor through  $H_1(\mathcal{S}_{g,b}; \mathbb{Z})$ , and we will frequently regard them as homomorphisms with domain  $H_1(\mathcal{S}_{g,b}; \mathbb{Z})$ . In [37], Johnson proved that the Johnson homomorphisms and the Birman-Craggs-Johnson homomorphisms combine to give the abelianization of  $\mathcal{S}_{g,b}$  when  $g \geq 3$ . However, they are not independent. Let  $H = H_1(\Sigma_{g,b}; \mathbb{Z})$  and  $H_2 = H_1(\Sigma_{g,b}; \mathbb{Z}_2)$ . Observe that  $B_3(2g)/B_2(2g) \cong \wedge^3 H_2$ . Johnson proved the following theorem.

**Theorem 6.2** (Johnson, [37]). *For  $g \geq 3$ , there is a commutative diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & B_2(2g) & \longrightarrow & H_1(\mathcal{S}_{g,1}; \mathbb{Z}) & \xrightarrow{\tau} & \wedge^3 H & \longrightarrow & 0 \\ & & \downarrow = & & \downarrow \sigma & & \downarrow & & \\ 0 & \longrightarrow & B_2(2g) & \longrightarrow & B_3(2g) & \longrightarrow & \wedge^3 H_2 & \longrightarrow & 0 \end{array} \quad (10)$$

In particular,  $H_1(\mathcal{S}_{g,1}; \mathbb{Z}) \cong B_2(2g) \oplus \wedge^3 H$ .

*Remark.* The top exact sequence in (10) does *not* split in a manner compatible with the action of  $\mathrm{Sp}_{2g}(\mathbb{Z})$  on  $H_1(\mathcal{S}_{g,1}; \mathbb{Z})$ . Thus the decomposition  $H_1(\mathcal{S}_{g,1}; \mathbb{Z}) \cong B_2(2g) \oplus \wedge^3 H$  is not an isomorphism of  $\mathrm{Sp}_{2g}(\mathbb{Z})$ -modules.

Similarly, one can show that  $\bar{B}_3(2g)/\bar{B}_2(2g) \cong (\wedge^3 H_2)/H_2$ , and Johnson proved the following.

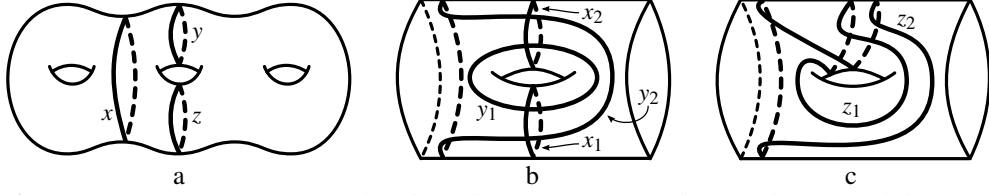
**Theorem 6.3** (Johnson, [37]). *For  $g \geq 3$ , there is a commutative diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bar{B}_2(2g) & \longrightarrow & H_1(\mathcal{S}_g; \mathbb{Z}) & \xrightarrow{\tau} & (\wedge^3 H)/H & \longrightarrow & 0 \\ & & \downarrow = & & \downarrow \sigma & & \downarrow & & \\ 0 & \longrightarrow & \bar{B}_2(2g) & \longrightarrow & \bar{B}_3(2g) & \longrightarrow & (\wedge^3 H_2)/H_2 & \longrightarrow & 0 \end{array} \quad (11)$$

In particular,  $H_1(\mathcal{S}_g; \mathbb{Z}) \cong \bar{B}_2(2g) \oplus (\wedge^3 H)/H$ .

## 6.3 Coinvariants

Recall that if a group  $G$  acts on a ring  $R$ , then the coinvariants of that action are denoted  $R_G$ . The goal of this section is to prove Proposition 6.6 below, which gives  $(H_1(\mathcal{S}_{g,b}; \mathbb{Z}))_{\mathrm{Sp}_{2g}(\mathbb{Z}, L)}$ . This is proceeded by two lemmas. In the statement of the first one, we will write  $\mathrm{Sp}_{2g}(\mathbb{Z}, 1)$  and  $\mathrm{Mod}_g(1)$  for  $\mathrm{Sp}_{2g}(\mathbb{Z})$  and  $\mathrm{Mod}_g$ , respectively.



**Figure 1:** *a.* A separating twist and a bounding pair map *b–c.* The crossed lantern relation  $(T_{y_1} T_{y_2}^{-1})(T_{x_1} T_{x_2}^{-1}) = (T_{z_1} T_{z_2}^{-1})$ .

**Lemma 6.4.** For  $g \geq 3$  and  $b \leq 1$  and  $L \geq 1$ , let  $v \in (\mathbf{H}_1(\mathcal{J}_{g,b}; \mathbb{Z}))_{\mathrm{Sp}_{2g}(\mathbb{Z}, L)}$ . Then  $L \cdot v = 0$ .

*Remark.* In [42, proof of Theorem 1.1], McCarthy proved that for  $v \in \mathbf{H}_1(\mathrm{Mod}_{g,b}(L); \mathbb{Z})$ , we have  $L \cdot v = 0$ . Though this is related to Lemma 6.4, it seems difficult to use McCarthy’s techniques to prove Lemma 6.4.

*Proof of Lemma 6.4.* For  $f \in \mathcal{J}_{g,b}$ , we will denote the associated element of  $(\mathbf{H}_1(\mathcal{J}_{g,b}; \mathbb{Z}))_{\mathrm{Sp}_{2g}(\mathbb{Z}, L)}$  by  $[f]_L$ . By Theorem 6.1, we can assume that  $v = [T_{x_1} T_{x_2}^{-1}]_L$  for some bounding pair  $\{x_1, x_2\}$ . Embed  $\{x_1, x_2\}$  in a 2-holed torus as in Figure 1.b. We will make use of the *crossed lantern relation* from [53]. Letting  $\{y_1, y_2\}$  and  $\{z_1, z_2\}$  be the other bounding pair maps depicted in Figures 1.b–c, this relation says that

$$(T_{y_1} T_{y_2}^{-1})(T_{x_1} T_{x_2}^{-1}) = (T_{z_1} T_{z_2}^{-1}).$$

Observe that for  $i = 1, 2$  we have  $z_i = T_{x_2}(y_i)$ . The key observation is that for all  $n \geq 0$  we have another crossed lantern relation

$$(T_{T_{x_2}^n(y_1)} T_{T_{x_2}^n(y_2)}^{-1})(T_{x_1} T_{x_2}^{-1}) = (T_{T_{x_2}^{n+1}(y_1)} T_{T_{x_2}^{n+1}(y_2)}^{-1}).$$

Since  $T_{x_2}^L \in \mathrm{Mod}_{g,n}(L)$ , we conclude that in  $(\mathbf{H}_1(\mathcal{J}_{g,n}; \mathbb{Z}))_{\mathrm{Sp}_{2g}(\mathbb{Z}, L)}$  we have  $[T_{y_1} T_{y_2}^{-1}]_L$  equal to

$$\begin{aligned} [T_{x_2}^L (T_{y_1} T_{y_2}^{-1}) T_{x_2}^{-L}]_L &= [(T_{T_{x_2}^L(y_1)} T_{T_{x_2}^L(y_2)}^{-1})]_L \\ &= [T_{x_1} T_{x_2}^{-1}]_L + [(T_{T_{x_2}^{L-1}(y_1)} T_{T_{x_2}^{L-1}(y_2)}^{-1})]_L \\ &= 2[T_{x_1} T_{x_2}^{-1}]_L + [(T_{T_{x_2}^{L-2}(y_1)} T_{T_{x_2}^{L-2}(y_2)}^{-1})]_L \\ &\quad \vdots \\ &= L[T_{x_1} T_{x_2}^{-1}]_L + [T_{y_1} T_{y_2}^{-1}]_L, \end{aligned}$$

so  $L[T_{x_1} T_{x_2}^{-1}]_L = 0$ , as desired.  $\square$

**Lemma 6.5.** For  $g \geq 3$  and  $L \geq 2$ , define  $H = \mathbf{H}_1(\Sigma_g; \mathbb{Z})$  and  $H_L = \mathbf{H}_1(\Sigma_g; \mathbb{Z}/L\mathbb{Z})$ . Then

$$(\wedge^3 H)_{\mathrm{Sp}_{2g}(\mathbb{Z}, L)} \cong \wedge^3 H_L \quad \text{and} \quad ((\wedge^3 H)/H)_{\mathrm{Sp}_{2g}(\mathbb{Z}, L)} \cong (\wedge^3 H_L)/H_L.$$

*Proof.* Letting  $S = \{a_1, b_1, \dots, a_g, b_g\}$  be a symplectic basis for  $H$ , the groups  $\wedge^3 H$  and  $(\wedge^3 H)/H$  are generated by  $T := \{x \wedge y \wedge z \mid x, y, z \in S \text{ distinct}\}$ . Consider  $x \wedge y \wedge z \in T$ . It is enough to show that in the indicated rings of coinvariants we have  $L(x \wedge y \wedge z) = 0$ . Now, one of  $x, y$ , and  $z$  must have algebraic intersection number 0 with the other two terms. Assume that  $x = a_1$  and  $y, z \in$

$\{a_2, b_2, \dots, a_g, b_g\}$  (the other cases are similar). There is then some  $\phi \in \mathrm{Sp}_{2g}(\mathbb{Z}, L)$  such that  $\phi(b_1) = b_1 + La_1 = b_1 + Lx$  and such that  $\phi(y) = y$  and  $\phi(z) = z$ . We conclude that in the indicated rings of coinvariants we have  $b_1 \wedge y \wedge z = (b_1 + Lx) \wedge y \wedge z$ , so  $L(x \wedge y \wedge z) = 0$ , as desired.  $\square$

**Proposition 6.6.** *Fix  $g \geq 3$  and  $L \geq 2$ . Set  $H_L = H_1(\Sigma_{g,b}; \mathbb{Z}_L)$ . If  $L$  is odd, then*

$$(H_1(\mathcal{J}_{g,1}; \mathbb{Z}))_{\mathrm{Sp}_{2g}(\mathbb{Z}, L)} = \wedge^3 H_L \quad \text{and} \quad (H_1(\mathcal{J}_g; \mathbb{Z}))_{\mathrm{Sp}_{2g}(\mathbb{Z}, L)} = (\wedge^3 H_L)/H_L,$$

while if  $L$  is even, then

$$(H_1(\mathcal{J}_{g,1}; \mathbb{Z}))_{\mathrm{Sp}_{2g}(\mathbb{Z}, L)} = B_2(2g) \oplus \wedge^3 H_L \quad \text{and} \quad (H_1(\mathcal{J}_g; \mathbb{Z}))_{\mathrm{Sp}_{2g}(\mathbb{Z}, L)} = \bar{B}_2(2g) \oplus (\wedge^3 H_L)/H_L.$$

*Proof.* We will do the case of  $\mathcal{J}_{g,1}$ ; the case of  $\mathcal{J}_g$  is similar. Let  $H = H_1(\Sigma_g; \mathbb{Z})$ , so  $H_1(\mathcal{J}_{g,1}; \mathbb{Z}) \cong B_2(2g) \oplus \wedge^3 H$ . If  $L$  is odd, then by Lemma 6.4 the 2-torsion  $B_2(2g)$  goes to 0 upon passing to the ring of coinvariants. Thus by Lemma 6.5 we have

$$(H_1(\mathcal{J}_{g,1}; \mathbb{Z}))_{\mathrm{Sp}_{2g}(\mathbb{Z}, L)} \cong (\wedge^3 H)_{\mathrm{Sp}_{2g}(\mathbb{Z})} \cong \wedge^3 H_L.$$

Now assume that  $L$  is even. Set

$$K = \{x - g(x) \mid x \in H_1(\mathcal{J}_{g,1}; \mathbb{Z}), g \in \mathrm{Sp}_{2g}(\mathbb{Z}, L)\},$$

so  $(H_1(\mathcal{J}_{g,1}; \mathbb{Z}))_{\mathrm{Sp}_{2g}(\mathbb{Z}, L)} = H_1(\mathcal{J}_{g,1}; \mathbb{Z})/K$ . Since  $\mathrm{Sp}_{2g}(\mathbb{Z}, L)$  acts trivially on  $H_1(\Sigma_{g,1}; \mathbb{Z}_2)$ , the image of  $K$  in  $B_3(2g)$  under the Birman-Craggs-Johnson homomorphism is 0. In particular, examining commutative diagram (10) from Theorem 6.2, we see that  $K \cap B_2(2g) = 0$ . Using Lemma 6.5, we thus conclude that

$$(H_1(\mathcal{J}_{g,1}; \mathbb{Z}))_{\mathrm{Sp}_{2g}(\mathbb{Z}, L)} \cong B_2(2g) \oplus (\wedge^3 H)_{\mathrm{Sp}_{2g}(\mathbb{Z})} \cong B_2(2g) \oplus \wedge^3 H_L \quad \square$$

## 6.4 Extending abelian quotients of $\mathcal{J}_{g,b}$ over $\mathrm{Mod}_{g,b}(L)$

The last 3 terms of the 5 term exact sequence associated to the exact sequence

$$1 \longrightarrow \mathcal{J}_{g,b} \longrightarrow \mathrm{Mod}_{g,b}(L) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}, L) \longrightarrow 1$$

are

$$(H_1(\mathcal{J}_{g,b}; \mathbb{Z}))_{\mathrm{Sp}_{2g}(\mathbb{Z}, L)} \xrightarrow{j} H_1(\mathrm{Mod}_{g,b}(L); \mathbb{Z}) \longrightarrow H_1(\mathrm{Sp}_{2g}(\mathbb{Z}, L)) \longrightarrow 0. \quad (12)$$

In this section, we discuss two known results about the image of  $j$ .

As we proved in Proposition 6.6, the group  $(H_1(\mathcal{J}_{g,b}; \mathbb{Z}))_{\mathrm{Sp}_{2g}(\mathbb{Z}, L)}$  has two pieces, one coming from the Johnson homomorphism and the other from the Birman-Craggs-Johnson homomorphism (the latter only existing if  $L$  is even). We begin by discussing the Johnson homomorphism  $\tau$ . Analogues of the Johnson homomorphism “mod  $L$ ” were constructed independently by Broaddus-Farb-Putman, Perron, and Sato. The following theorem gives their main properties.

**Theorem 6.7** (Broaddus-Farb-Putman, [9, Example 5.3 and Theorem 5.8], Perron [49], Sato [56]). *Fix  $g \geq 3$  and  $L \geq 2$ . Set  $H = H_1(\Sigma_g; \mathbb{Z})$  and  $H_L = H_1(\Sigma_g; \mathbb{Z}_L)$ . There then exist homomorphisms  $\mathrm{Mod}_{g,1}(L) \rightarrow \wedge^3 H_L$  and  $\mathrm{Mod}_g(L) \rightarrow (\wedge^3 H_L)/H_L$  that fit into commutative diagrams of the form*

$$\begin{array}{ccc} \mathcal{J}_{g,1} & \xrightarrow{\tau} & \wedge^3 H \\ \downarrow & & \downarrow \\ \mathrm{Mod}_{g,1}(L) & \longrightarrow & \wedge^3 H_L \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{J}_g & \xrightarrow{\tau} & (\wedge^3 H)/H \\ \downarrow & & \downarrow \\ \mathrm{Mod}_g(L) & \longrightarrow & (\wedge^3 H_L)/H_L \end{array}$$

Here the right hand vertical arrows are reduction mod  $L$ .

An immediate consequence of this theorem is that the terms of  $(\mathbf{H}_1(\mathcal{J}_{g,b}; \mathbb{Z}))_{\mathrm{Sp}_{2g}(\mathbb{Z}, L)}$  given by the quotients of the Johnson homomorphisms inject into  $\mathbf{H}_1(\mathrm{Mod}_{g,b}(L); \mathbb{Z})$ .

We now turn to the Birman-Craggs-Johnson homomorphism  $\sigma$ . A version of this was constructed on  $\mathrm{Mod}_{g,b}(2)$  by Sato. However, he did not manage to get the entire image of  $\sigma$ . Observe that  $B_3(2g)$  and  $\overline{B}_3(2g)$  contain the ‘‘constant’’ subgroups  $B_0(2g)$  and  $\overline{B}_0(2g)$ , both of which are isomorphic to  $\mathbb{Z}_2$ . Define

$$\sigma' : \mathcal{J}_{g,1} \longrightarrow B_3(2g)/B_0(2g) \quad \text{and} \quad \sigma' : \mathcal{J}_g \longrightarrow \overline{B}_3(2g)/\overline{B}_0(2g)$$

to be the compositions of the Birman-Craggs-Johnson homomorphisms with the quotients by  $B_0(2g)$  and  $\overline{B}_0(2g)$ . Sato proved the following theorem.

**Theorem 6.8** (Sato [56]). *Fix  $g \geq 3$ . There then exist abelian groups  $A$  and  $\overline{A}$  together with homomorphisms  $\mathrm{Mod}_{g,1}(2) \rightarrow A$  and  $\mathrm{Mod}_g \rightarrow \overline{A}$  and injections  $i : B_3(2g)/B_0(2g) \hookrightarrow A$  and  $\bar{i} : \overline{B}_3(2g)/\overline{B}_0(2g) \rightarrow \overline{A}$  that fit into commutative diagrams of the form*

$$\begin{array}{ccc} \mathcal{J}_{g,1} & \xrightarrow{\sigma'} & B_3(2g)/B_0(2g) \\ \downarrow & & \downarrow i \\ \mathrm{Mod}_{g,1}(2) & \longrightarrow & A \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{J}_g & \xrightarrow{\sigma'} & \overline{B}_3(2g)/\overline{B}_0(2g) \\ \downarrow & & \downarrow \bar{i} \\ \mathrm{Mod}_g(2) & \longrightarrow & \overline{A} \end{array}$$

*Remark.* In [56], it is also proven that there does *not* exist an abelian group  $B$  together with a homomorphism  $\mathrm{Mod}_{g,1}(2) \rightarrow B$  and an injection  $j : B_3(2g) \hookrightarrow B$  such that the diagram

$$\begin{array}{ccc} \mathcal{J}_{g,1} & \xrightarrow{\sigma} & B_3(2g) \\ \downarrow & & \downarrow j \\ \mathrm{Mod}_{g,1}(2) & \longrightarrow & B \end{array}$$

commutes, and similarly for  $\mathrm{Mod}_g(2)$ . For  $L > 2$ , Theorem 7.8 below implies that a similar statement is true for  $\mathrm{Mod}_{g,1}(L)$  and  $\mathrm{Mod}_g(L)$ .

For  $L$  even, we can restrict the homomorphisms given by Theorem 6.8 to  $\mathrm{Mod}_{g,b}(L)$  and obtain that the only possible subgroups of the pieces of  $(\mathbf{H}_1(\mathcal{J}_{g,b}; \mathbb{Z}))_{\mathrm{Sp}_{2g}(\mathbb{Z}, L)}$  given by the Birman-Craggs-Johnson homomorphisms that can go to 0 in  $\mathbf{H}_1(\mathrm{Mod}_{g,b}(L); \mathbb{Z})$  are  $B_0(2g)$  and  $\overline{B}_0(2g)$ .

Summing up, we have obtained the following lemma.

**Lemma 6.9.** *Fix  $g \geq 3$  and  $L \geq 2$  and  $0 \leq b \leq 1$ . If  $L$  is odd, then the map*

$$(\mathbf{H}_1(\mathcal{J}_{g,b}; \mathbb{Z}))_{\mathrm{Sp}_{2g}(\mathbb{Z}, L)} \rightarrow \mathbf{H}_1(\mathrm{Mod}_{g,b}(L); \mathbb{Z})$$

*from (12) is injective. If  $L$  is even, then the kernel of this map is isomorphic to either 0 or  $\mathbb{Z}_2$ . If the kernel is  $\mathbb{Z}_2$ , then it is equal to  $B_0(2g)$  if  $b = 1$  and  $\overline{B}_0(2g)$  if  $b = 0$ .*

## 7 The moduli space of curves

### 7.1 Definitions

We now introduce the moduli space of curves. For more information, see [23, 25].

Let  $\mathcal{T}_g$  be *Teichmüller space*, i.e. the space of pairs  $(S, B)$  where  $S$  is a genus  $g$  Riemann surface and  $B$  is a standard basis for  $\pi_1(S)$  (up to conjugacy). The space  $\mathcal{T}_g$  is a contractible complex manifold and the group  $\text{Mod}_g$  acts on it holomorphically and properly discontinuously in the following way. If  $\phi \in \text{Mod}_g$  and  $(S, B) \in \mathcal{T}_g$ , then  $\phi(B)$  is well-defined up to conjugacy and  $\phi(S, B) = (S, \phi(B))$ .

To simplify our statements, we will denote  $\text{Mod}_g$  and  $\mathcal{M}_g$  by  $\text{Mod}_g(1)$  and  $\mathcal{M}_g(1)$ , respectively. For  $L \geq 1$ , the space  $\mathcal{M}_g(L)$  is the orbifold  $(\mathcal{T}_g, \text{Mod}_g(L))$ . Thus  $\mathcal{M}_g(L')$  is a covering orbifold for  $\mathcal{M}_g(L)$  whenever  $L|L'$ . For  $L \geq 3$ , the group  $\text{Mod}_g(L)$  acts freely on  $\mathcal{T}_g$ , so in these cases  $\mathcal{M}_g(L)$  is a trivial orbifold. It is known that  $\mathcal{T}_g / \text{Mod}_g(L)$  is a quasiprojective variety for all  $L$  (this was originally proven by Baily [4]; see also [15]). We conclude that  $\mathcal{M}_g(L)$  is a quasiprojective orbifold with quasiprojective finite cover  $\mathcal{M}_g(L')$  for any  $L' \geq 3$  such that  $L|L'$ . The representation  $\text{Mod}_g \rightarrow \text{Sp}_{2g}(\mathbb{Z})$  is induced by the orbifold map  $\mathcal{M}_g \rightarrow \mathcal{A}_g$  that takes a Riemann surface to its Jacobian.

## 7.2 Some known results

Just like in the case of  $\text{Pic}(\mathcal{A}_g(L))$ , our main tool for investigating  $\text{Pic}(\mathcal{M}_g(L))$  will be Lemma 2.7. To verify the conditions of that lemma, we will need a number of results, the first of which are the following group-cohomology calculations.

**Theorem 7.1** (Powell, [52]). *For  $g \geq 3$ , we have  $H_1(\text{Mod}_g; \mathbb{Z}) = 0$ .*

**Theorem 7.2** (Harer, [24]). *For  $g \geq 4$ , we have  $H_2(\text{Mod}_g; \mathbb{Z}) \cong \mathbb{Z}$ .*

*Remark.* Harer's paper [24] only considers the case  $g \geq 5$  and contains an erroneous torsion term. The case  $g = 4$  is essentially due to Pitsch [50], though his paper only considers surfaces with boundary. For more information, see the survey [39], which also gives a very short proof (due to Harer) of Theorem 7.1.

**Theorem 7.3** (Hain, [22]). *For  $g \geq 3$  and  $L \geq 2$ , we have  $H^1(\text{Mod}_g(L); \mathbb{Z}) = 0$ .*

**Theorem 7.4** (Putman, [55]). *For  $g \geq 5$  and  $L \geq 2$ , we have  $H_2(\text{Mod}_g(L); \mathbb{Q}) \cong \mathbb{Q}$ .*

To relate  $\mathcal{M}_g$  and  $\mathcal{A}_g$ , we will need the following folklore result. I do not know a reference for it, so I include a proof.

**Lemma 7.5.** *For  $g \geq 4$ , the surjection  $\text{Mod}_g \rightarrow \text{Sp}_{2g}(\mathbb{Z})$  induces an isomorphism  $H_2(\text{Mod}_g; \mathbb{Z}) \cong H_2(\text{Sp}_{2g}(\mathbb{Z}))$ .*

*Proof.* Theorem 7.2 says that  $H_2(\text{Mod}_g; \mathbb{Z}) \cong \mathbb{Z}$  and Theorem 5.3 says that  $H_2(\text{Sp}_{2g}(\mathbb{Z}); \mathbb{Z})$  has rank 1, so it is enough to show that the map  $H_2(\text{Mod}_g; \mathbb{Z}) \rightarrow H_2(\text{Sp}_{2g}(\mathbb{Z}); \mathbb{Z})$  is a surjection. The 5 term exact sequence coming from the short exact sequence

$$1 \longrightarrow \mathcal{T}_g \longrightarrow \text{Mod}_g \longrightarrow \text{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1$$

contains the segment

$$H_2(\text{Mod}_g; \mathbb{Z}) \longrightarrow H_2(\text{Sp}_{2g}(\mathbb{Z}); \mathbb{Z}) \longrightarrow (H_1(\mathcal{T}_g; \mathbb{Z}))_{\text{Sp}_{2g}(\mathbb{Z})}.$$

By Lemma 6.4, we have  $(H_1(\mathcal{T}_g; \mathbb{Z}))_{\text{Sp}_{2g}(\mathbb{Z})} = 0$ , and the lemma follows.  $\square$

*Remark.* One can obtain Theorem 5.1 from the above results as follows. First, the map  $\text{Mod}_g \rightarrow \text{Sp}_{2g}(\mathbb{Z})$  is a surjection, so Theorem 7.1 implies that  $H_1(\text{Sp}_{2g}(\mathbb{Z}); \mathbb{Z}) = 0$  for  $g \geq 3$ . Second, one can combine Lemma 7.5 and Theorem 7.2 to deduce that  $H_2(\text{Sp}_{2g}(\mathbb{Z}); \mathbb{Z}) = \mathbb{Z}$  for  $g \geq 4$ .

Next, let  $\mathcal{L}_g \in \text{Pic}(\mathcal{M}_g)$  be the determinantal bundle of the Hodge bundle, as in the introduction. As is shown in [23, Corollary 17.4], the line bundle  $\mathcal{L}_g$  is the pullback of  $\overline{\mathcal{L}}_g \in \text{Pic}(\mathcal{A}_g)$ . We will need two properties of  $\mathcal{L}_g$ . First, it generates  $\text{Pic}(\mathcal{M}_g)$ .

**Theorem 7.6** (Arbarello-Cornalba [1], see also [23, Theorem 17.3]). *For  $g \geq 4$ , the class  $c_1(\mathcal{L}_g)$  generates  $H^2(\text{Mod}_g; \mathbb{Z}) \cong \mathbb{Z}$ .*

For the second, let  $\mathcal{L}_g(L) \in \text{Pic}(\mathcal{M}_g(L))$  be the pullback of  $\mathcal{L}_g$ . We will need the following recent theorem of G. Farkas.

**Theorem 7.7** (Farkas, [20, Theorem 0.2]). *For  $g \geq 3$  and  $L \geq 2$  such that  $L$  is even,  $\mathcal{L}_g(L) \in \text{Pic}(\mathcal{M}_g(L))$  is divisible by 4.*

*Remark.* The statement of [20, Theorem 0.2] differs from Theorem 7.7 in two ways. First, the line bundle  $x$  given there actually lives in Picard group of the moduli space  $\mathcal{S}_g^+$  of curves with even spin structures. For  $L$  even, the orbifold  $\mathcal{M}_g(L)$  is a finite cover of  $\mathcal{S}_g^+$ , so the line bundle whose existence is asserted by Theorem 7.7 can be obtained by pulling back  $x$ . Second, the statement of [20, Theorem 0.2] actually refers to a compactification of  $\mathcal{S}_g^+$  and contains a number of extraneous boundary terms that vanish when restricted to the actual moduli space.

Exactly like in §5.2, the above results immediately yield (1) from the introduction, which we recall is of the form

$$0 \longrightarrow \text{Hom}(H_1(\text{Mod}_g(L); \mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \longrightarrow \text{Pic}(\mathcal{M}_g(L)) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

### 7.3 The proofs of Theorems A and B

We wish to apply Lemma 2.7 with  $(X, G) = (\mathcal{S}_g, \text{Mod}_g)$  and  $G' = \text{Mod}_g(L)$  and  $\lambda = \mathcal{L}_g$  and

$$n = \begin{cases} 1 & \text{if } L \text{ is odd,} \\ 4 & \text{if } L \text{ is even.} \end{cases}$$

The results of §7.1–7.2 show that all the conditions of Lemma 2.7 are satisfied except possibly the condition that the image of  $H_2(\text{Mod}_g(L); \mathbb{Z})$  in  $H_2(\text{Mod}_g; \mathbb{Z}) \cong \mathbb{Z}$  is  $m\mathbb{Z}$  for some  $m$  with  $1 \leq m \leq n$ , which we now prove.

By Lemma 7.5, it is enough to show that the image of  $H_2(\text{Mod}_g(L); \mathbb{Z})$  in  $H_2(\text{Sp}_{2g}(\mathbb{Z}); \mathbb{Z}) \cong \mathbb{Z}$  is  $m\mathbb{Z}$  for some  $m$  with  $1 \leq m \leq n$ . The 5-term exact sequence associated with the short exact sequence

$$1 \longrightarrow \mathcal{S}_g \longrightarrow \text{Mod}_g(L) \longrightarrow \text{Sp}_{2g}(\mathbb{Z}, L) \longrightarrow 1$$

contains the segment

$$H_2(\text{Mod}_g(L); \mathbb{Z}) \longrightarrow H_2(\text{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z}) \longrightarrow (H_1(\mathcal{S}_g; \mathbb{Z}))_{\text{Sp}_{2g}(\mathbb{Z}, L)} \xrightarrow{i} H_1(\text{Mod}_g(L); \mathbb{Z}).$$

Assume first that  $L$  is odd. Lemma 6.9 says that  $\ker(i) = 0$ , so the map  $H_2(\text{Mod}_g(L); \mathbb{Z}) \rightarrow H_2(\text{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z})$  is surjective. Theorem D says that the map  $H_2(\text{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z}) \rightarrow H_2(\text{Sp}_{2g}(\mathbb{Z}); \mathbb{Z})$

is surjective, so we conclude that the map  $H_2(\text{Mod}_g(L); \mathbb{Z}) \rightarrow H_2(\text{Sp}_{2g}(\mathbb{Z}); \mathbb{Z})$  is surjective, as desired.

Assume now that  $L$  is even. Lemma 6.9 says that  $\ker(i)$  is either 0 or  $\mathbb{Z}_2$ , so the cokernel of the map  $H_2(\text{Mod}_g(L); \mathbb{Z}) \rightarrow H_2(\text{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z})$  is either 0 or  $\mathbb{Z}_2$ . Theorem D says that the image of  $H_2(\text{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z})$  in  $H_2(\text{Sp}_{2g}(\mathbb{Z}); \mathbb{Z}) \cong \mathbb{Z}$  is  $2\mathbb{Z}$ , so we conclude that the image of the map  $H_2(\text{Mod}_g(L); \mathbb{Z}) \rightarrow H_2(\text{Sp}_{2g}(\mathbb{Z}); \mathbb{Z})$  is either  $2\mathbb{Z}$  or  $4\mathbb{Z}$ , as desired.

## 7.4 The abelianization of $\text{Mod}_g(L)$

The following theorem subsumes Theorem F from the introduction.

**Theorem 7.8.** *Fix  $g \geq 5$  and  $0 \leq b \leq 1$  and  $L \geq 2$  such that  $4 \nmid L$ . Set  $H_L = H_1(\Sigma_g; \mathbb{Z}_L)$ . There is then a short exact sequence*

$$0 \longrightarrow K_{g,b} \longrightarrow H_1(\text{Mod}_{g,b}(L); \mathbb{Z}) \longrightarrow H_1(\text{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z}) \longrightarrow 0,$$

where

$$K_{g,1} = \wedge^3 H_L \quad \text{and} \quad K_g \cong (\wedge^3 H_L) / H_L$$

if  $L$  is odd and

$$K_{g,1} = (B_2(2g)/B_0(2g)) \oplus \wedge^3 H_L \quad \text{and} \quad K_g \cong (\overline{B}_2(2g)/\overline{B}_0(2g)) \oplus (\wedge^3 H_L) / H_L$$

if  $L$  is even.

*Proof.* The 5-term exact sequence coming from the short exact sequence

$$1 \longrightarrow \mathcal{I}_{g,b} \longrightarrow \text{Mod}_{g,b}(L) \longrightarrow \text{Sp}_{2g}(\mathbb{Z}, L) \longrightarrow 1$$

contains the subsequence

$$(H_1(\mathcal{I}_{g,b}; \mathbb{Z}))_{\text{Sp}_{2g}(\mathbb{Z}, L)} \xrightarrow{i_{g,b}} H_1(\text{Mod}_{g,b}(L); \mathbb{Z}) \longrightarrow H_1(\text{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z}) \longrightarrow 0.$$

Setting

$$K_{g,b} = (H_1(\mathcal{I}_{g,b}; \mathbb{Z}))_{\text{Sp}_{2g}(\mathbb{Z}, L)} / \ker(i_{g,b}),$$

it is enough to prove the indicated description of  $K_{g,b}$ . If  $L$  is odd, then we showed in Lemma 6.9 that  $i_{g,b}$  is injective, so the theorem follows from Proposition 6.6. If  $L$  is even and  $b = 0$ , then in the proofs of Theorems A and B in §7.3, we showed that  $\ker(i_{g,b}) \cong \mathbb{Z}_2$ . The theorem thus follows in this case from Proposition 6.6 and Lemma 6.9. If  $L$  is even and  $b = 1$ , then we can use the case  $b = 0$  and the commutative diagram

$$\begin{array}{ccccc} H_2(\text{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z}) & \longrightarrow & (H_1(\mathcal{I}_{g,1}; \mathbb{Z}))_{\text{Sp}_{2g}(\mathbb{Z}, L)} & \xrightarrow{i_{g,1}} & H_1(\text{Mod}_{g,1}(L); \mathbb{Z}) \\ \downarrow = & & \downarrow & & \downarrow \\ H_2(\text{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z}) & \longrightarrow & (H_1(\mathcal{I}_g; \mathbb{Z}))_{\text{Sp}_{2g}(\mathbb{Z}, L)} & \xrightarrow{i_{g,0}} & H_1(\text{Mod}_g(L); \mathbb{Z}) \end{array}$$

arising from the naturality of the 5-term exact sequence to deduce that  $\ker(i_{g,1})$  must be nonzero. Just like before, the theorem now follows from Proposition 6.6 and Lemma 6.9.  $\square$

## References

- [1] E. Arbarello and M. Cornalba, The Picard groups of the moduli spaces of curves, *Topology* **26** (1987), no. 2, 153–171.
- [2] H. Bass, *Algebraic K-theory*, W. A. Benjamin, Inc., New York, 1968.
- [3] J. S. Birman, On Siegel’s modular group, *Math. Ann.* **191** (1971), 59–68.
- [4] W. L. Baily, Jr., On the moduli of Jacobian varieties, *Ann. of Math. (2)* **71** (1960), 303–314.
- [5] J. S. Birman and R. Craggs, The  $\mu$ -invariant of 3-manifolds and certain structural properties of the group of homeomorphisms of a closed, oriented 2-manifold, *Trans. Amer. Math. Soc.* **237** (1978), 283–309.
- [6] A. Borel, Properties and linear representations of Chevalley groups, in *Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69)*, 1–55, Lecture Notes in Math., 131, Springer, Berlin.
- [7] A. Borel, Stable real cohomology of arithmetic groups, *Ann. Sci. École Norm. Sup. (4)* **7** (1974), 235–272 (1975).
- [8] A. Borel, Stable real cohomology of arithmetic groups. II, in *Manifolds and Lie groups (Notre Dame, Ind., 1980)*, 21–55, *Progr. Math.*, 14, Birkhäuser, Boston, Mass.
- [9] N. Broaddus, B. Farb and A. Putman, Irreducible  $\mathrm{Sp}$ -representations and subgroup distortion in the mapping class group, to appear in *Comment. Math. Helv.*
- [10] K. S. Brown, *Cohomology of groups*, Corrected reprint of the 1982 original, Springer, New York, 1994.
- [11] R. Charney, A generalization of a theorem of Vogtmann, *J. Pure Appl. Algebra* **44** (1987), no. 1-3, 107–125.
- [12] P. E. Conner, *Lectures on the action of a finite group*, Lecture Notes in Math., 73, Springer, Berlin, 1968.
- [13] C. W. Curtis, On projective representations of certain finite groups, *Proc. Amer. Math. Soc.* **11** (1960), 852–860.
- [14] P. Deligne, Extensions centrales non résiduellement finies de groupes arithmétiques, *C. R. Acad. Sci. Paris Sér. A-B* **287** (1978), no. 4, A203–A208.
- [15] P. Deligne and D. Mumford, The irreducibility of the space of curves of given genus, *Inst. Hautes Études Sci. Publ. Math. No. 36* (1969), 75–109.
- [16] R. K. Dennis and M. R. Stein, The functor  $K_2$ : a survey of computations and problems, in *Algebraic K-theory, II: “Classical” algebraic K-theory and connections with arithmetic (Proc. Conf., Seattle Res. Center, Battelle Memorial Inst., 1972)*, 243–280, Lecture Notes in Math., 342, Springer, Berlin.
- [17] E. B. Dynkin, The structure of semi-simple algebras, *Amer. Math. Soc. Translation* **1950** (1950), no. 17, 143 pp.
- [18] B. Farb, Some problems on mapping class groups and moduli space, in *Problems on mapping class groups and related topics*, 11–55, *Proc. Sympos. Pure Math.*, 74, Amer. Math. Soc., Providence, RI.
- [19] B. Farb and D. Margalit, A primer on mapping class groups, preprint 2009.
- [20] G. Farkas, The birational type of the moduli space of spin curves, preprint 2008.
- [21] E. Freitag, *Singular modular forms and theta relations*, Lecture Notes in Math., 1487, Springer, Berlin, 1991.
- [22] R. M. Hain, Torelli groups and geometry of moduli spaces of curves, in *Current topics in complex algebraic geometry (Berkeley, CA, 1992/93)*, 97–143, Cambridge Univ. Press, Cambridge.

- [23] R. Hain, Moduli of Riemann surfaces, transcendental aspects, in *School on Algebraic Geometry (Trieste, 1999)*, 293–353, Abdus Salam Int. Cent. Theoret. Phys., Trieste.
- [24] J. Harer, The second homology group of the mapping class group of an orientable surface, *Invent. Math.* **72** (1983), no. 2, 221–239.
- [25] J. Harris and I. Morrison, *Moduli of curves*, Springer, New York, 1998.
- [26] D. F. Holt, B. Eick and E. A. O’Brien, *Handbook of computational group theory*, Chapman & Hall/CRC, Boca Raton, FL, 2005.
- [27] J. Huebschmann, Group extensions, crossed pairs and an eight term exact sequence, *J. Reine Angew. Math.* **321** (1981), 150–172.
- [28] J. Igusa, On the graded ring of theta-constants, *Amer. J. Math.* **86** (1964), 219–246.
- [29] J. Igusa, *Theta functions*, Springer, New York, 1972.
- [30] S. Illman, Existence and uniqueness of equivariant triangulations of smooth proper  $G$ -manifolds with some applications to equivariant Whitehead torsion, *J. Reine Angew. Math.* **524** (2000), 129–183.
- [31] N. Jacobson, Classes of restricted Lie algebras of characteristic  $p$ . I, *Amer. J. Math.* **63** (1941), 481–515.
- [32] D. L. Johnson, Homeomorphisms of a surface which act trivially on homology, *Proc. Amer. Math. Soc.* **75** (1979), no. 1, 119–125.
- [33] D. Johnson, An abelian quotient of the mapping class group  $\mathcal{S}_g$ , *Math. Ann.* **249** (1980), no. 3, 225–242.
- [34] D. Johnson, Quadratic forms and the Birman-Craggs homomorphisms, *Trans. Amer. Math. Soc.* **261** (1980), no. 1, 235–254.
- [35] D. Johnson, A survey of the Torelli group, in *Low-dimensional topology (San Francisco, Calif., 1981)*, 165–179, *Contemp. Math.*, 20, Amer. Math. Soc., Providence, RI.
- [36] D. Johnson, The structure of the Torelli group. I. A finite set of generators for  $\mathcal{S}$ , *Ann. of Math. (2)* **118** (1983), no. 3, 423–442.
- [37] D. Johnson, The structure of the Torelli group. III. The abelianization of  $\mathcal{S}$ , *Topology* **24** (1985), no. 2, 127–144.
- [38] D. A. Každan, On the connection of the dual space of a group with the structure of its closed subgroups, *Funkcional. Anal. i Priložen.* **1** (1967), 71–74.
- [39] M. Korkmaz, Low-dimensional homology groups of mapping class groups: a survey, *Turkish J. Math.* **26** (2002), no. 1, 101–114.
- [40] R. Lee and R. H. Szczarba, On the homology and cohomology of congruence subgroups, *Invent. Math.* **33** (1976), no. 1, 15–53.
- [41] S. Mac Lane, *Homology*, Reprint of the 1975 edition, Springer, Berlin, 1995.
- [42] J. D. McCarthy, On the first cohomology group of cofinite subgroups in surface mapping class groups, *Topology* **40** (2001), no. 2, 401–418.
- [43] D. McCullough and A. Miller, The genus 2 Torelli group is not finitely generated, *Topology Appl.* **22** (1986), no. 1, 43–49.
- [44] J. Milnor, *Introduction to algebraic K-theory*, *Ann. of Math. Stud.*, 72, Princeton Univ. Press, Princeton, N.J., 1971.
- [45] B. Mirzaii and W. van der Kallen, Homology stability for unitary groups, *Doc. Math.* **7** (2002), 143–166 (electronic).

- [46] D. Mumford, Abelian quotients of the Teichmüller modular group, *J. Analyse Math.* **18** (1967), 227–244.
- [47] D. Mumford, *Tata lectures on theta. I*, Birkhäuser Boston, Boston, MA, 1983.
- [48] M. Newman and J. R. Smart, Symplectic modular groups, *Acta Arith* **9** (1964), 83–89.
- [49] B. Perron, Filtration de Johnson et groupe de Torelli modulo  $p$ ,  $p$  premier, *C. R. Math. Acad. Sci. Paris* **346** (2008), no. 11-12, 667–670.
- [50] W. Pitsch, Un calcul élémentaire de  $H_2(\mathcal{M}_{g,1}, \mathbf{Z})$  pour  $g \geq 4$ , *C. R. Acad. Sci. Paris Sér. I Math.* **329** (1999), no. 8, 667–670.
- [51] H. Pollatsek, First cohomology groups of some linear groups over fields of characteristic two, *Illinois J. Math.* **15** (1971), 393–417.
- [52] J. Powell, Two theorems on the mapping class group of a surface, *Proc. Amer. Math. Soc.* **68** (1978), no. 3, 347–350.
- [53] A. Putman, An infinite presentation for the Torelli group, to appear in GAFA.
- [54] A. Putman The abelianization of the level  $L$  mapping class group, preprint 2008.
- [55] A. Putman, The second rational homology group of the moduli space of curves with level structures, preprint 2008.
- [56] M. Sato, The abelianization of the level 2 mapping class group, preprint 2008.
- [57] G. B. Seligman, *Modular Lie algebras*, Springer-Verlag New York, Inc., New York, 1967.
- [58] M. R. Stein, Surjective stability in dimension 0 for  $K_2$  and related functors, *Trans. Amer. Math. Soc.* **178** (1973), 165–191.
- [59] R. Steinberg, Générateurs, relations et revêtements de groupes algébriques, in *Colloq. Théorie des Groupes Algébriques (Bruxelles, 1962)*, 113–127, Librairie Universitaire, Louvain.
- [60] R. Steinberg, *Lectures on Chevalley groups*, Yale University, New Haven, Conn., 1968.
- [61] W. Thurston, The geometry and topology of three-manifolds, unpublished notes.
- [62] H. Völklein, The 1-cohomology of the adjoint module of a Chevalley group, *Forum Math.* **1** (1989), no. 1, 1–13.

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