# The Johnson homomorphism and its kernel

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**Abstract**

We give a new proof of a celebrated theorem of Dennis Johnson that asserts that the kernel of the Johnson homomorphism on the Torelli subgroup of the mapping class group is generated by separating twists. In fact, we prove a more general result that also applies to "subsurface Torelli groups". Using this, we extend Johnson's calculation of the rational abelianization of the Torelli group not only to the subsurface Torelli groups, but also to finite-index subgroups of the Torelli group that contain the kernel of the Johnson homomorphism.

## 1 Introduction

Let $\Sigma_{g,n}$ be an oriented genus $g$ surface with $n$ boundary components (we will often omit the $n$ if it vanishes). The *mapping class group* of $\Sigma_{g,n}$, denoted $\text{Mod}(\Sigma_{g,n})$, is the group of isotopy classes of orientation preserving homeomorphisms of $\Sigma_{g,n}$ that act as the identity on $\partial\Sigma_{g,n}$ (see [9]). The group $\text{Mod}(\Sigma_{g,n})$ acts on $\text{H}_1(\Sigma_{g,n};\mathbb{Z})$ and preserves the algebraic intersection form. If $n \leq 1$, then this is a nondegenerate alternating form, so we obtain a representation $\text{Mod}(\Sigma_{g,n}) \rightarrow \text{Sp}_{2g}(\mathbb{Z})$. This representation is well-known to be surjective. Its kernel is known as the *Torelli group* and will be denoted $\mathscr{I}(\Sigma_{g,n})$. This is all summarized in the exact sequence

$$1 \longrightarrow \mathscr{I}(\Sigma_{g,n}) \longrightarrow \text{Mod}(\Sigma_{g,n}) \longrightarrow \text{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1.$$

In the early 1980's, Dennis Johnson published a sequence of remarkable papers on $\mathscr{I}(\Sigma_{g,n})$ (see [17] for a survey). Many of his results center on the so-called *Johnson homomorphisms*, which were defined in [15]. Letting $H = \text{H}_1(\Sigma_{g,n};\mathbb{Z})$, these are homomorphisms

$$\tau_{\Sigma_{g,1}} : \mathscr{I}(\Sigma_{g,1}) \longrightarrow \wedge^3 H \quad \text{and} \quad \tau_{\Sigma_g} : \mathscr{I}(\Sigma_g) \longrightarrow (\wedge^3 H)/H$$

which measure the "unipotent" part of the action of $\mathscr{I}(\Sigma_{g,n})$ on the second nilpotent truncation of $\pi_1(\Sigma_{g,n})$. They have since been found to play a role in the study of invariants of 3-manifolds and in the geometry/topology of the moduli space of curves.

The goal of this paper is to generalize two of Johnson's theorems.

1. Define $\mathscr{K}(\Sigma_{g,n}) = \ker(\tau_{\Sigma_{g,n}})$. In [19], Johnson proved that $\mathscr{K}(\Sigma_{g,n})$ is generated by right Dehn twists $T_x$ about separating curves $x$. We will generalize this to the "subsurface Torelli groups" defined by the author in [25]. Our proof is independent of Johnson's work, so in particular it provides a new proof of of his theorem. Our new proof is much less computationally intensive than Johnson's proof.

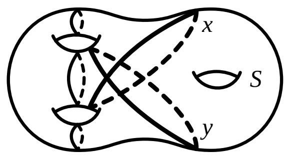


**Figure 1:** $S \cong \Sigma_{1,3}$ *is embedded in* $\Sigma_3$. *We conjecture that* $[T_x, T_y]$ *cannot be written as a product of separating twists in* $\mathscr{I}(\Sigma_3, S)$

2. In [20], Johnson proved that if $g \geq 3$, then modulo torsion the Johnson homomorphism gives the abelianization of $\mathscr{I}(\Sigma_{g,n})$. We will generalize this not only to subsurface Torelli groups, but also to certain finite-index subgroups of $\mathscr{I}(\Sigma_{g,n})$.

**Statements of theorems.** Fix a compact connected oriented surface $\Sigma$ with at most 1 boundary component and let $S \hookrightarrow \Sigma$ be a compact connected embedded subsurface. We say that $f \in \mathrm{Mod}(\Sigma)$ is *supported on S* if it can be represented by a homeomorphism of $\Sigma$ that acts as the identity on $\Sigma \setminus S$. Denote by $\mathrm{Mod}(\Sigma, S)$ the subgroup of $\mathrm{Mod}(\Sigma)$ consisting of mapping classes supported on $S$. Also, define $\mathscr{I}(\Sigma, S) = \mathscr{I}(\Sigma) \cap \mathrm{Mod}(\Sigma, S)$ and $\mathscr{K}(\Sigma, S) = \mathscr{K}(\Sigma) \cap \mathrm{Mod}(\Sigma, S)$. Our first theorem is as follows. It reduces to Johnson's theorem when $S = \Sigma$.

**Theorem 1.1** (Generators for kernel of Johnson homomorphism)**.** *Let* $\Sigma$ *be a compact connected orientable surface with at most* 1 *boundary component and let* $S \hookrightarrow \Sigma$ *be an embedded compact connected subsurface whose genus is at least* 2*. Then the group* $\mathscr{K}(\Sigma, S)$ *is generated by the set*

$$\{T_\gamma \mid \gamma \text{ is a separating curve on } \Sigma \text{ and } \gamma \subset S\}.$$

*Remark.* We believe that the restriction on the genus of $S$ is necessary. Indeed, let $S \hookrightarrow \Sigma_3$ be the subsurface depicted in Figure 1 and let $x, y \subset S$ be the curves shown there. Then it is not hard to see that $[T_x, T_y] \in \mathscr{K}(\Sigma_3, S)$, but we conjecture that $[T_x, T_y]$ cannot be written as a product of separating twists lying in $S$.

*Remark.* As will become clear from the proof sketch below, the first step of our proof reduces Theorem 1.1 to Johnson's result. However, the extra generality of Theorem 1.1 is needed for the second step, which reduces Johnson's result to understanding subsurfaces. In other words, from our point of view the more general statement is necessary even if you only care about the classical case.

*Remark.* We recently learned that Church [6] has independently proven Theorem 1.1. He deduces it from Johnson's work; in the proof sketch below, his argument is a version of our Step 1.

Our second theorem is as follows. It reduces to Johnson's theorem when $S = \Sigma$ and $\Gamma = \mathscr{I}(\Sigma, S)$.

**Theorem 1.2** (Rational $\mathrm{H}_1$ of finite index subgroups of Torelli)**.** *Let* $\Sigma$ *be a compact connected orientable surface with at most* 1 *boundary component and let* $S \hookrightarrow \Sigma$ *be an embedded compact connected subsurface whose genus is at least* 3*. Let* $\Gamma$ *be a finite-index subgroup of* $\mathscr{I}(\Sigma, S)$ *with* $\mathscr{K}(\Sigma, S) < \Gamma$*. Then* $\mathrm{H}_1(\Gamma; \mathbb{Q}) \cong \tau_\Sigma(\mathscr{I}(\Sigma, S)) \otimes \mathbb{Q}$.

*Remark.* Mess [22] showed that $\mathscr{I}(\Sigma_2)$ is an infinitely generated free group. Using this, it is not hard to show that if $S \hookrightarrow \Sigma$ is a subsurface of genus 2, then $\mathrm{H}_1(\mathscr{I}(\Sigma, S); \mathbb{Q})$ has infinite rank. In particular, the conclusion of Theorem 1.2 does not hold.

*Remark.* Theorem 1.2 is inspired by a well-known conjecture of Ivanov that asserts that if $g \geq 3$ and $\Gamma$ is a finite-index subgroup of $\mathrm{Mod}(\Sigma_g)$, then $\mathrm{H}_1(\Gamma;\mathbb{Q}) \cong \mathrm{H}_1(\mathrm{Mod}(\Sigma_g);\mathbb{Q}) = 0$ (see [13] for a recent discussion). In analogy with this, one might guess that the condition that $\mathscr{K}(\Sigma,S) < \Gamma$ is unnecessary. We remark that Ivanov's conjecture is only known in a few special cases – Hain [10] proved it if $\mathscr{I}(\Sigma_g) < \Gamma$, while Boggi [4] and the author [28] later independently proved that it holds if $\mathscr{K}(\Sigma_g) < \Gamma$.

**Proof ideas for Theorem 1.1.** Let $S$ and $\Sigma$ be as in Theorem 1.1. One can view $\mathscr{I}(\Sigma,S)$ as a sort of "Torelli group on a surface with boundary". In [25], the author studied these groups. One of the main results of that paper (Theorem 2.6 below) gave generators for the groups $\mathscr{I}(\Sigma,S)$, generalizing and reproving a classical result of Birman [3] and Powell [24], who did this in the case $S = \Sigma$. Our proof of Theorem 1.1 follows the same basic outline as the proof of this result in [25].

The proof is by induction on the genus and number of boundary components of $S$. The base case is $S = \Sigma = \Sigma_2$, where $\tau_\Sigma = 0$ and the result is trivial. The induction has 2 basic steps.

**Step 1** (Reduction to the closed case)**.** *Assume that $S$ is a surface with boundary and let $\Sigma_g$ be the closed surface that results from gluing discs to all of its boundary components. Then $\mathscr{K}(\Sigma,S)$ is generated by separating twists if $\mathscr{K}(\Sigma_g)$ is generated by separating twists.*

The argument in this step proceeds by investigating the effect on $\mathscr{K}(\Sigma,S)$ of gluing a disc to a boundary component of $S$. For the whole mapping class group, we have the fundamental Birman exact sequence ([2]; see §2.1 below), which is a short exact sequence of the form

$$1 \longrightarrow \pi_1(U\Sigma_{g,n}) \longrightarrow \mathrm{Mod}(\Sigma_{g,n+1}) \longrightarrow \mathrm{Mod}(\Sigma_{g,n}) \longrightarrow 1.$$

Here $U\Sigma_{g,n}$ is the unit tangent bundle of $\Sigma_{g,n}$ and the map $\mathrm{Mod}(\Sigma_{g,n+1}) \to \mathrm{Mod}(\Sigma_{g,n})$ is induced by gluing a disc to a boundary component $\beta$ of $\Sigma_{g,n+1}$. The kernel $\pi_1(U\Sigma_{g,n}) \lhd \mathrm{Mod}(\Sigma_{g,n+1})$ consists of mapping classes that "drag" $\beta$ around $\Sigma_{g,n}$ while allowing it to rotate. We prove an appropriate analogue of this for $\mathscr{K}(\Sigma,S)$ (see §4).

**Step 2** (Reducing the genus)**.** *Assume that for all subsurfaces $S$ of $\Sigma_g$ whose genus is $g-1$, the group $\mathscr{K}(\Sigma_g,S)$ is generated by separating twists. Then $\mathscr{K}(\Sigma_g)$ is generated by separating twists.*

Let $\gamma$ be a nonseparating simple closed curve on $\Sigma_g$. If $S = \Sigma_g \setminus N(\gamma)$, where $N(\gamma)$ is a regular neighborhood of $\gamma$, then the assumption implies that $\mathscr{K}(\Sigma_g,S)$ is generated by separating twists. Observe that $\mathscr{K}(\Sigma_g,S)$ is the stabilizer in $\mathscr{K}(\Sigma_g)$ of the isotopy class of $\gamma$. Now, the set of isotopy classes of nonseparating simple closed curves forms the vertices of the *nonseparating complex of curves*, which was defined by Harer in [11]. This complex, which we will denote $\mathscr{C}_g^{\mathrm{ns}}$, is the simplicial complex whose simplices are sets $\{\gamma_1,\ldots,\gamma_k\}$ of isotopy classes of nonseparating simple closed curves on $\Sigma_g$ that can be realized disjointly with $\Sigma_g \setminus (\gamma_1 \cup \cdots \gamma_k)$ connected.

For technical reasons (it simplifies the proof a certain connectivity result), we will actually make use of a larger complex, which is defined as follows. Denote by $i(\gamma_1,\gamma_2)$ the *geometric intersection number* of 2 simple closed curves $\gamma_1$ and $\gamma_2$, i.e. the minimum over all curves $\gamma_1'$ and $\gamma_2'$ with $\gamma_i'$ isotopic to $\gamma_i$ for $1 \leq i \leq 2$ of the number of points of $\gamma_1' \cap \gamma_2'$.

**Definition.** For $k \geq 0$, denote by $\mathscr{C}_g^{\mathrm{ns},k}$ the simplicial complex whose simplices are sets $\{\gamma_1,\ldots,\gamma_k\}$ of isotopy classes of nonseparating simple closed curves on $\Sigma_g$ such that $i(\gamma_i,\gamma_j) \leq k$ for $1 \leq i,j \leq k$.

We will use this in the case $k = 2$. The group $\mathscr{K}(\Sigma_g)$ acts on $\mathscr{C}_g^{\text{ns},2}$, and to prove the desired result, it is enough to show that $\mathscr{K}(\Sigma_g)$ is generated by the vertex stabilizers of this action. A basic result of Armstrong (see Theorem 5.1 below) says that this will hold if both $\mathscr{C}_g^{\text{ns},2}$ and $\mathscr{C}_g^{\text{ns},2}/\mathscr{K}(\Sigma_g)$ are simply connected. These connectivity results are proven in §6. A key role is played by a theorem of Harer [11] that says that $\mathscr{C}_g^{\text{ns}}$ is $(g-2)$-connected.

**Proof ideas for Theorem 1.2.** Johnson deduced that $\text{H}_1(\mathscr{I}(\Sigma_{g,n});\mathbb{Q}) = \tau_{\Sigma_{g,n}}(\mathscr{I}(\Sigma_{g,n}))\otimes\mathbb{Q}$ when $g \geq 3$ and $n \leq 1$ by proving that separating twists go to torsion in $\text{H}_1(\mathscr{I}(\Sigma_{g,n});\mathbb{Z})$. The theorem then follows from the fact that the kernel of $\tau_{\Sigma_{g,n}}$ is generated by separating twists. For Theorem 1.2, we prove that separating twists go to torsion in the abelianizations of finite-index subgroups of $\mathscr{I}(\Sigma,S)$ that contain $\mathscr{K}(\Sigma,S)$, where $\Sigma$ and $S$ are as in that theorem. The proof combines a small generalization of Johnson's argument with the trick used in [28] to show that powers of Dehn twists go to torsion in the abelianizations of finite-index subgroups of the mapping class group.

**Comments on Johnson's work.** To put this paper in perspective, we now give a brief sketch of Johnson's proof [19] that $\mathscr{K}(\Sigma)$ is generated by separating twists. We will focus on the case $\Sigma = \Sigma_{g,1}$ and ignore some low-genus issues. Set $H = \text{H}_1(\Sigma_{g,1};\mathbb{Z})$.

Johnson's proof proceeds by a sequence of ingenious and difficult calculations. Let $\mathscr{S} \lhd \mathscr{I}(\Sigma_{g,1})$ be the subgroup generated by separating twists. Recall that the Johnson homomorphism takes the form

$$\tau_{\Sigma_{g,1}} : \mathscr{I}(\Sigma_{g,1}) \longrightarrow \wedge^3 H.$$

The free abelian group $\wedge^3 H$ has rank $\binom{g}{3}$. Johnson begins by writing down an explicit set $X$ of $\binom{g}{3}$ elements of $\mathscr{I}(\Sigma_{g,1})$ such that $\tau(X)$ is a basis for $\wedge^3 H$. He then does the following.

1. By examining the conjugates of $X$ under a standard generating set for $\text{Mod}(\Sigma_{g,1})$, he shows that $X$ generates a normal subgroup of $\text{Mod}(\Sigma_{g,1})/\mathscr{S}$. The set $X$ contains a normal generating set for $\mathscr{I}(\Sigma_{g,1})$ identified by Powell ([24]; see §2.2 for the details of Powell's generating set). Thus Johnson can conclude that $X$ generates $\mathscr{I}(\Sigma_{g,1})/\mathscr{S}$.

2. Via a brute-force calculation, Johnson next shows that the elements of $X$ commute modulo $\mathscr{S}$, so $\mathscr{I}(\Sigma_{g,1})/\mathscr{S}$ is an abelian group generated by $\binom{g}{3}$ elements.

3. The Johnson homomorphism induces a surjection $\mathscr{I}(\Sigma_{g,1})/\mathscr{S} \to \wedge^3 H$. By the previous step, this map has to be an isomorphism, and the result follows.

**Outline of paper.** In §2, we review some preliminaries about the mapping class group, the Birman exact sequence, the Torelli group, and the Johnson homomorphisms. Next, in §3 we prove a linear-algebraic lemma that will be used several times. A version of the Birman exact sequence for the Johnson kernel is then constructed in §4, and the proof of Theorem 1.1 is contained in §5. That proof depends on the simple connectivity of $\mathscr{C}_g^{\text{ns},2}$ and $\mathscr{C}_g^{\text{ns},2}/\mathscr{K}(\Sigma_g)$, which we establish in §6. Finally, in §7 we prove Theorem 1.2.

**Notation and conventions.** Throughout this paper, by a *surface* we will mean a compact oriented connected surface with boundary. All homology groups will have $\mathbb{Z}$ coefficients unless otherwise specified. We will identify $\text{H}_1(\Sigma_g)$ and $\text{H}_1(\Sigma_{g,1})$ without comment. We will denote

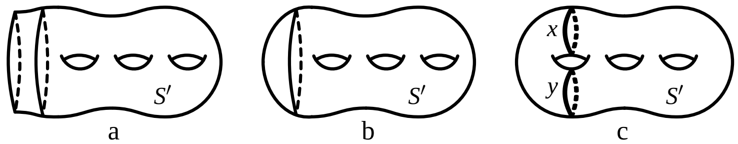


**Figure 2:** *a,b. The subsurface $S'$ is not cleanly embedded in the ambient surface $S$. c. The subsurface $S'$ is cleanly embedded in $S$, but the map $\mathrm{Mod}(S') \to \mathrm{Mod}(S)$ has a nontrivial kernel (generated by $T_x T_y^{-1}$)*

the algebraic intersection pairing on $\mathrm{H}_1(\Sigma_g)$ by $i_{\mathrm{alg}}(\cdot,\cdot)$. If $V$ is a subgroup of $\mathrm{H}_1(\Sigma_g)$, then $V^{\perp}$ will denote the subgroup $\{x \mid i_{\mathrm{alg}}(x,v) = 0 \text{ for all } v \in V\}$. A *symplectic basis* for $\mathrm{H}_1(\Sigma_g)$ is a basis $\{a_1, b_1, \ldots, a_g, b_g\}$ for $\mathrm{H}_1(\Sigma_g)$ such that $i_{\mathrm{alg}}(a_i, b_i) = 1$ and $i_{\mathrm{alg}}(a_i, a_j) = i_{\mathrm{alg}}(a_i, b_j) = i_{\mathrm{alg}}(b_i, b_j) = 0$ for distinct $1 \leq i, j \leq g$. If $c$ is an oriented 1-submanifold of the surface, then $[c]$ will denote the homology class of $c$. We will confuse a curve on a surface with its isotopy class. Several times we will have formulas involving $[\gamma]$, where $\gamma$ is an unoriented simple closed curve. By this we will mean that the formula holds for an appropriate choice of orientation.

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# 2 Preliminaries and notation

## 2.1 The mapping class group and the Birman exact sequence

If $S'$ is an embedded subsurface of $S$, then there is an induced map $\mathrm{Mod}(S') \to \mathrm{Mod}(S)$ ("extend by the identity") whose image is $\mathrm{Mod}(S,S')$. This map is often injective; in fact, we have the following theorem of Paris–Rolfsen.

**Theorem 2.1** ([23])**.** *Let $S_1$ and $S_2$ be surfaces and let $S_1 \hookrightarrow S_2$ be an embedding. Assume that no components of $\overline{S_2 \setminus S_1}$ are homeomorphic to discs or annuli. Then the map $\mathrm{Mod}(S_1) \to \mathrm{Mod}(S_2)$ is injective.*

If $S'$ is an embedded subsurface of $S$ and if some component $C$ of $\overline{S \setminus S'}$ is either a disc or an annulus with $C \cap S'$ a single loop, then $\mathrm{Mod}(S,S') = \mathrm{Mod}(S, S' \cup C)$. This motivates the following definition.

**Definition.** If $S'$ is an embedded subsurface of $S$, then $S'$ is *cleanly embedded* if no components $C$ of $\overline{S \setminus S'}$ are either discs or annuli with $C \cap S'$ a single loop (see Figure 2).

*Remark.* It is *not* true that if $S'$ is a cleanly embedded subsurface of $S$, then the map $\mathrm{Mod}(S') \to \mathrm{Mod}(S)$ is injective. See Figure 2.c for an example.

We now turn to the Birman exact sequence. In its original form (due to Birman [2]), it dealt with the effect on mapping class groups of punctured surfaces of "forgetting the punctures". Later, Johnson [18] gave the following version for surfaces with boundary. Call a surface *exceptional* if it is homeomorphic to $\Sigma_{1,1}$ or $\Sigma_{0,n}$ for $n \leq 2$.

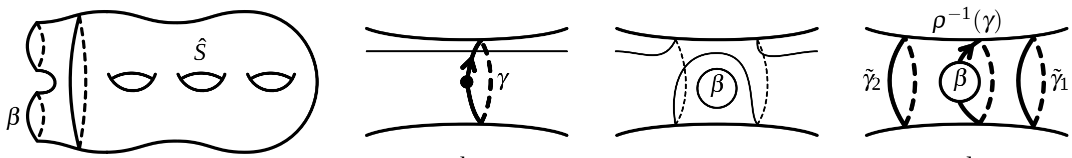



**Figure 3:** *a. $\hat{S}$ is embedded in S and is homeomorphic to S with a disc glued to $\beta$. It provides a splitting of the Birman exact sequence. b. $\gamma \in \pi_1(\hat{S})$ together with a transverse arc to illustrate effect of Push$(\gamma)$ c. The effect of Push$(\gamma)$ on the surface d. Push$(\gamma) = T_{\tilde{\gamma}_1} T_{\tilde{\gamma}_2}^{-1}$*

**Theorem 2.2** ([18, p. 430])**.** *Let S be a nonexceptional surface and let $\beta$ be a boundary component of S. Denote by $\hat{S}$ the result of gluing a disc to $\beta$. There is then a short exact sequence*

$$1 \longrightarrow \pi_1(U\hat{S}) \longrightarrow \mathrm{Mod}(S) \longrightarrow \mathrm{Mod}(\hat{S}) \longrightarrow 1,$$

*where $U\hat{S}$ is the unit tangent bundle of $\hat{S}$. If $\hat{S}$ is not closed, then this exact sequence splits via an embedding $\hat{S} \hookrightarrow S$ with $\overline{S \setminus \hat{S}}$ a 3-holed sphere containing $\beta$ and sharing exactly 2 boundary components with S (see Figure 3.a).*

For later use, we will make the following definition.

**Definition.** Let $S$ be a nonexceptional surface and let $\beta$ be a boundary component of $S$. We will denote by $\mathscr{P}(S;\beta)$ the kernel of the Birman exact sequence given by Theorem 2.2.

Let $S$, $\beta$, and $\hat{S}$ be as in Theorem 2.2. The mapping classes in the kernel $\mathscr{P}(S;\beta) \cong \pi_1(U\hat{S})$ "push" $\beta$ around the surface while allowing it to rotate. The loop around the fiber (with an appropriate orientation) corresponds to $T_\beta$. As for the other elements of $\mathscr{P}(S;\beta)$, it is easiest to understand them via the projection $\mathscr{P}(S;\beta) \cong \pi_1(U\hat{S}) \to \pi_1(\hat{S})$. Consider $\gamma \in \pi_1(\hat{S})$. Assume that $\gamma$ can be realized by a simple closed curve, which we can assume is smoothly embedded in $\hat{S}$. Taking the derivative of such an embedding, we get a well-defined lift $\mathrm{Push}(\gamma) \in \pi_1(U\hat{S}) \cong \mathscr{P}(S;\beta) < \mathrm{Mod}(S)$.

*Remark.* This is well-defined since any 2 realizations of $\gamma$ as a smoothly embedded simple closed curve are smoothly isotopic through an isotopy fixing the basepoint.

As shown in Figure 3.b–d, for such a $\gamma$ we have $\mathrm{Push}(\gamma) = T_{\tilde{\gamma}_1} T_{\tilde{\gamma}_2}^{-1}$ for simple closed curves $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ in $S$ constructed as follows. Observe that we can obtain $S$ from $\hat{S}$ by blowing up the basepoint to a boundary component. Let $\rho : S \to \hat{S}$ be the projection. Then $\rho^{-1}(\gamma)$ consists of $\beta$ together with an oriented arc joining 2 points of $\beta$. The boundary of a regular neighborhood of $\rho^{-1}(\gamma)$ has 2 components, and the curves $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are the components lying to the right and left of the arc, respectively.

We will need the following lemma about the Birman exact sequence.

**Lemma 2.3.** *Let S be a nonexceptional cleanly embedded subsurface of $\Sigma$ and let $\beta$ be a boundary component of $\Sigma$ that is not contained in S. The following then holds holds.*

- *If the component T of $\overline{\Sigma \setminus S}$ containing $\beta$ is homeomorphic to $\Sigma_{0,3}$ and $\partial T$ has components $\beta$, x, and y with $x, y \subset S$, then $\mathscr{P}(\Sigma;\beta) \cap Mod(\Sigma, S) \cong \langle T_x T_y^{-1} \rangle \cong \mathbb{Z}$.*
- *Otherwise, $\mathscr{P}(\Sigma;\beta) \cap Mod(\Sigma, S) = 1$.*

*Proof.* Since $S$ is cleanly embedded, the results of [23] imply that the kernel of the map $\text{Mod}(S) \to \text{Mod}(\Sigma)$ is generated by the set

$$K_1 = \{T_{b_1} T_{b_2}^{-1} \mid C \text{ is a component of } \overline{\Sigma \setminus S} \text{ with } C \cong \Sigma_{0,2} \text{ and } \partial C = b_1 \cup b_2\}.$$

Similarly, the kernel of the map $\text{Mod}(S) \to \text{Mod}(\hat{\Sigma})$ is generated by the set

$$K_2 = \{T_{b_1} T_{b_2}^{-1} \mid C \text{ is a component of } \overline{\hat{\Sigma} \setminus S} \text{ with } C \cong \Sigma_{0,2} \text{ and } \partial C = b_1 \cup b_2\}.$$

Let $T$ be the component of $\Sigma \setminus S$ containing $\beta$ and let $\hat{T}$ be the result of gluing a disc to the boundary component $\beta$ of $T$. If $\hat{T}$ is not an annulus, then $K_1 = K_2$, so $\text{Mod}(\Sigma, S) \cong \text{Mod}(\hat{\Sigma}, S)$ and the result follows. Otherwise, letting $x$ and $y$ be the components of $\partial \hat{T}$ we have $\text{Mod}(\Sigma, S)/\langle T_x T_y^{-1} \rangle \cong \text{Mod}(\hat{\Sigma}, S)$, and the result follows. □

## 2.2 Basic facts about Torelli group

In [25], the author investigated the subgroups of the Torelli group which are supported on subsurfaces. We begin with the following definition.

**Definition.** Let $\Sigma$ be a surface with at most 1 boundary component and let $S \hookrightarrow \Sigma$ be a cleanly embedded subsurface. Let $i : \text{Mod}(S) \to \text{Mod}(\Sigma)$ be the induced map. We define

$$\mathscr{I}(\Sigma, S) = \text{Mod}(\Sigma, S) \cap \mathscr{I}(\Sigma),$$
$$\tilde{\mathscr{I}}(\Sigma, S) = i^{-1}(\mathscr{I}(\Sigma)).$$

*Remark.* The group $\tilde{\mathscr{I}}(\Sigma, S) < \text{Mod}(S)$ is *not* the kernel of the action of $\text{Mod}(S)$ on $\text{H}_1(S)$.

*Remark.* Our primary interest is in $\mathscr{I}(\Sigma, S)$; however, we need to use $\tilde{\mathscr{I}}(\Sigma, S)$ so that the exact sequence in Theorem 2.8 below will split in all cases.

The groups $\tilde{\mathscr{I}}(\Sigma, S)$ depend strongly on the embedding $S \hookrightarrow \Sigma$, not merely on the homeomorphism type of $S$. Associated to every embedding $S \hookrightarrow \Sigma$ is a partition $P$ of the boundary components of $S$, namely

$$P = \{\partial T \mid T \text{ a component of } \overline{\Sigma \setminus S}\} \cup \{\partial S \cap \partial \Sigma\}.$$

We will call $P$ the *partition associated with the embedding* $S \hookrightarrow \Sigma$. For instance, if $S$ and $\Sigma$ are as in Figure 4.a, then the partition associated with $S \hookrightarrow \Sigma$ is $\{\{b_1\}, \{b_2, b_3\}, \{b_4\}, \{b_5, b_6, b_7\}\}$. We have the following theorem.

**Theorem 2.4** ([25, Theorem 1.1])**.** *Let $\Sigma_1$ and $\Sigma_2$ be 2 surfaces with at most 1 boundary component. Let $S \hookrightarrow \Sigma_1$ and $S \hookrightarrow \Sigma_2$ be clean embeddings. Then $\tilde{\mathscr{I}}(\Sigma_1, S) = \tilde{\mathscr{I}}(\Sigma_2, S)$ if and only if the partitions of $\partial S$ associated to $S \hookrightarrow \Sigma_1$ and $S \hookrightarrow \Sigma_2$ are the same.*

We now introduce several important elements of the Torelli group.

**Definition.** Let $\Sigma$ be a surface with at most 1 boundary component.

- Let $\gamma$ be a simple closed curve that separates $\Sigma$ (for instance, the curve $x_1$ in Figure 4.b). Then it is not hard to see that $T_\gamma \in \mathscr{I}(\Sigma)$. These are known as *separating twists*.

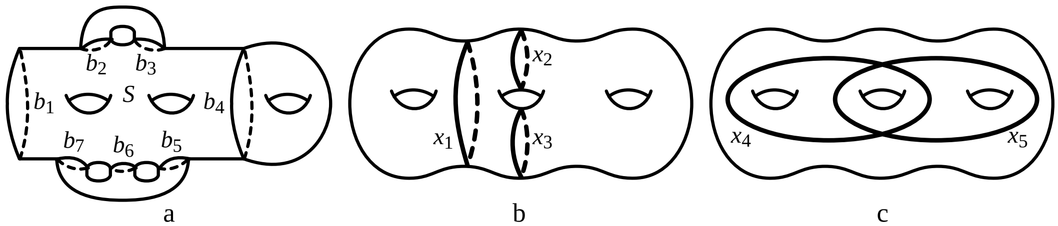



**Figure 4:** *a. The partition associated with $S \hookrightarrow \Sigma_{6,1}$ is $\{\{b_1\},\{b_2,b_3\},\{b_4\},\{b_5,b_6,b_7\}\}$ b. A separating twist $T_{x_1}$ and a bounding pair map $T_{x_2}T_{x_3}^{-1}$ c. A simply intersecting pair map $[T_{x_4},T_{x_5}]$*

- Let $\{\gamma_1,\gamma_2\}$ be a pair of non-isotopic disjoint homologous curves on $\Sigma$ (for instance, the pair of curves $\{x_2,x_3\}$ from Figure 4.b). Then $T_{\gamma_1}$ and $T_{\gamma_2}$ map to the same element of $\mathrm{Sp}_{2g}(\mathbb{Z})$, so $T_{\gamma_1}T_{\gamma_2}^{-1} \in \mathscr{I}(\Sigma)$. These are known as *bounding pair maps* and the pair $\{\gamma_1,\gamma_2\}$ is known as a *bounding pair*.

- Let $\{\gamma_1,\gamma_2\}$ be a pair of curves on $\Sigma$ with $i_{\mathrm{alg}}([\gamma_1],[\gamma_2]) = 0$ and $i(\gamma_1,\gamma_2) = 2$ (for instance, the pair of curves $\{x_4,x_5\}$ from Figure 4.c). Then the images of $T_{\gamma_1}$ and $T_{\gamma_2}$ in $\mathrm{Sp}_{2g}(\mathbb{Z})$ commute, so $[T_{\gamma_1},T_{\gamma_2}] \in \mathscr{I}(\Sigma)$. These are known as *simply intersecting pair maps* and the pair $\{\gamma_1,\gamma_2\}$ is known as a *simply intersecting pair*.

- Let $S$ be a cleanly embedded subsurface of $\Sigma$. An element of either $\mathscr{I}(\Sigma,S)$ or $\tilde{\mathscr{I}}(\Sigma,S)$ will be called a bounding pair map, etc. if its image in $\mathscr{I}(\Sigma)$ is a bounding pair map, etc.

Following up on work of Birman [3], it was proven by Powell [24] that $\mathscr{I}(\Sigma)$ is generated by separating twists and bounding pair maps. Johnson [14] later proved that for $g \geq 3$ no separating twists are needed.

**Theorem 2.5** ([14])**.** *Let $\Sigma$ be a surface with at most 1 boundary component. Assume that the genus of $\Sigma$ is at least 3. Then $\mathscr{I}(\Sigma_g)$ is generated by the set of bounding pair maps $T_xT_y^{-1}$ such that $x \cup y$ cuts off a subsurface homeomorphic to $\Sigma_{1,2}$.*

*Remark.* All the $T_xT_y^{-1}$ in Theorem 2.5 are conjugate in $\mathrm{Mod}(\Sigma_g)$.

In [25], the author gave a new proof of Powell's theorem and extended it to the relative case.

**Theorem 2.6** ([25, Theorem 1.3])**.** *Let $\Sigma$ be a surface with at most 1 boundary component and let $S \hookrightarrow \Sigma_g$ be a cleanly embedded subsurface. Assume that the genus of $S$ is at least 1. Then both $\mathscr{I}(\Sigma,S)$ and $\tilde{\mathscr{I}}(\Sigma,S)$ are generated by separating twists and bounding pair maps.*

## 2.3 The Birman exact sequence for the Torelli group

The first version of the Birman exact sequence for the Torelli group is due to Johnson.

**Theorem 2.7** ([18])**.** *Fix $g \geq 2$. The Birman exact sequence for $\mathrm{Mod}(\Sigma_{g,1})$ restricts to a short exact sequence*

$$1 \longrightarrow \pi_1(U\Sigma_g) \longrightarrow \mathscr{I}(\Sigma_{g,1}) \longrightarrow \mathscr{I}(\Sigma_g) \longrightarrow 1.$$

In [25], the author proved a version of the Birman exact sequence for subsurface Torelli groups. Let $\Sigma$ be a surface with at most 1 boundary component, let $S \hookrightarrow \Sigma$ be a cleanly embedded nonexceptional subsurface, and let $\beta$ be a boundary component of $S$. The kernel of the Birman exact sequence for the subsurface Torelli group will be as follows. We will give a more explicit description of it below in Theorems 2.9 and 2.10. Recall that $\mathscr{P}(S;\beta)$ is the kernel of the Birman exact sequence for the mapping class group.

**Definition.** Define $\mathscr{P}_{\mathscr{I}}(\Sigma,S;\beta) = \mathscr{P}(S;\beta) \cap \tilde{\mathscr{I}}(\Sigma,S)$.

The cokernel is a bit more subtle. We will need the following definition.

**Definition.** We will call the component of $\overline{\Sigma \setminus S}$ containing $\beta$ the *complementary component* to $\beta$.

Let $P$ be the partition of the boundary components of $S$ induced by $S \hookrightarrow \Sigma$. Define $\hat{S}$ to be the result of gluing a disc to $\beta$ in $S$. There is an induced partition $\hat{P}$ of the boundary components of $\hat{S}$. In [25], the author showed that the cokernel should be the Torelli group on a surface with the induced partition $\hat{P}$, which motivates the following definition.

**Definition.** Call an embedding $\hat{S} \hookrightarrow \Sigma$ a *splitting surface* for $\beta$ in $S$ if it satisfies the following two conditions.

- $\hat{S} \subset S$ and $\overline{S \setminus \hat{S}}$ is a 3-holed sphere containing $\beta$ and sharing exactly 2 boundary components with $S$.
- If $\beta$ does not separate $S$, then the complementary component to $\beta$ intersects $\overline{S \setminus \hat{S}}$ in two components (see Figure 5.a).

*Remark.* Observe that if $\hat{S} \hookrightarrow \Sigma$ is a splitting surface for $\beta$ in $S$, then the induced partition of the boundary components of $\hat{S}$ is $\hat{P}$.

With these definitions, we have the following theorem.

**Theorem 2.8** (Birman exact sequence for Torelli, [25, Theorem 1.2])**.** *Let $\Sigma$ be a surface with at most 1 boundary component and let $S \hookrightarrow \Sigma$ be a cleanly embedded nonexceptional subsurface which has at least 2 boundary components. Let $\beta$ be a boundary component of $S$ and $\hat{S} \hookrightarrow \Sigma$ be a splitting surface for $\beta$ in $S$. We then have a split short exact sequence*

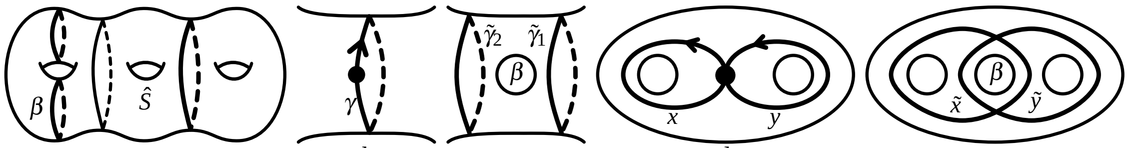
$$1 \longrightarrow \mathscr{P}_{\mathscr{I}}(\Sigma,S;\beta) \longrightarrow \tilde{\mathscr{I}}(\Sigma,S) \longrightarrow \tilde{\mathscr{I}}(\Sigma,\hat{S}) \longrightarrow 1.$$

We now give a more explicit description of the kernel $\mathscr{P}_{\mathscr{I}}(\Sigma,S;\beta)$. There are two cases, depending on whether $\beta$ separates $\Sigma$ or not. The easiest case to describe is the case that $\beta$ is separating.

**Theorem 2.9** ([25, Theorem 1.2])**.** *Let $\Sigma$ be a surface with at most 1 boundary component and let $S \hookrightarrow \Sigma$ be a cleanly embedded nonexceptional subsurface. Let $\beta$ be a boundary component of $S$ such that $\beta$ separates $\Sigma$. Let $\hat{S}$ be the result of gluing a disc to the boundary component $\beta$ of $S$. Then $\mathscr{P}_{\mathscr{I}}(\Sigma,S;\beta) = \mathscr{P}(S;\beta) = \pi_1(U\hat{S})$.*

We now deal with the case that $\beta$ is nonseparating, in which case $\mathscr{P}_{\mathscr{I}}(\Sigma,S;\beta)$ is almost (but not quite) isomorphic to $[\pi_1(\hat{S}),\pi_1(\hat{S})]$. We make the following definition.

**Definition.** Let $S_1$ be a subsurface of $S_2$. Define $[\pi_1(S_1),\pi_1(S_1)]_{S_2}$ to be the kernel of the composition $\pi_1(S_1) \to \pi_1(S_2) \to \mathrm{H}_1(S_2)$.



a b c d e

**Figure 5:** *a. Let $S \cong \Sigma_{1,3}$ be the subsurface bounded by the bold curves and let $\hat{S} \cong \Sigma_{1,2}$ be the indicated subsurface of S. Then $\hat{S}$ is a splitting surface for $\beta$ in S.* *b. $\gamma \in \pi_1(\hat{S})$ separates $\Sigma$.* *c. Push$(\gamma) = T_{\tilde{\gamma}_1} T_{\tilde{\gamma}_2}^{-1}$. Since $\beta$ does not separate $\Sigma$, exactly 1 of $T_{\tilde{\gamma}_1}$ and $T_{\tilde{\gamma}_2}$ is a separating twist.* *d,e. The other configuration*

*Remark.* If $S_1$ is a subsurface of $S_2$ and $K$ is the kernel of the map $\mathrm{H}_1(S_1) \to \mathrm{H}_1(S_2)$, then we have a short exact sequence

$$1 \longrightarrow [\pi_1(S_1), \pi_1(S_1)] \longrightarrow [\pi_1(S_1), \pi_1(S_1)]_{S_2} \longrightarrow K \longrightarrow 1. \tag{1}$$

**Definition.** Let $S_c$ be the complementary component to $\beta$. We will call $S_e = \overline{\Sigma \setminus S_c}$ the *expanded subsurface* of $S$ containing $\beta$. Observe that $S \subset S_e$.

We then have the following theorem.

**Theorem 2.10** ([25, Theorem 1.2])**.** *Let $\Sigma$ be a surface with at most 1 boundary component and let $S \hookrightarrow \Sigma$ be a cleanly embedded nonexceptional subsurface. Let $\beta$ be a boundary component of S that does not separate $\Sigma$ and let $S_e$ be the expanded subsurface of S containing $\beta$. Define $\hat{S}$ and $\hat{S}_e$ to be the result of gluing discs to the boundary components $\beta$ of S and $S_e$, respectively. We then have $\mathscr{P}_{\mathscr{I}}(\Sigma, S; \beta) \cong [\pi_1(\hat{S}), \pi_1(\hat{S})]_{\hat{S}_e}$.*

Let the notation be as in Theorem 2.10. The surface $\hat{S}$ has boundary, so $\mathscr{P}(S;\beta) \cong \pi_1(U\hat{S}) \cong \pi_1(\hat{S}) \oplus \mathbb{Z}$ (though this splitting is not natural), but $\mathscr{P}_{\mathscr{I}}(\Sigma, S; \beta) \cong [\pi_1(\hat{S}), \pi_1(\hat{S})]_{\hat{S}_e} < \mathscr{P}(S;\beta)$ does *not* lie in the first factor. Rather, there is a homomorphism $\psi : [\pi_1(\hat{S}), \pi_1(\hat{S})]_{\hat{S}_e} \to \mathbb{Z}$ such that $\mathscr{P}_{\mathscr{I}}(\Sigma, S; \beta)$ is the graph of $\psi$, i.e. the set of all pairs $(x, \psi(x)) \in \pi_1(U\hat{S})$ for $x \in [\pi_1(\hat{S}), \pi_1(\hat{S})]_{\hat{S}_e}$.

The homomorphism $\psi$ depends on the (unnatural) splitting $\pi_1(U\hat{S}) \cong \pi_1(\hat{S}) \oplus \mathbb{Z}$. We will not need to know an explicit splitting; rather, we will only need to know 2 examples of how elements of $[\pi_1(\hat{S}), \pi_1(\hat{S})]_{\hat{S}_e}$ lie in $\tilde{\mathscr{I}}(\Sigma, S)$. For $\gamma \in [\pi_1(\hat{S}), \pi_1(\hat{S})]_{\hat{S}_e}$, denote by $[\![\gamma]\!]$ the associated element of $\tilde{\mathscr{I}}(\Sigma, S)$. Also, for $x, y \in \pi_1(\hat{S})$, denote by $[\![x, y]\!] \in \tilde{\mathscr{I}}(\Sigma, S)$ the element associated to $[x, y] \in [\pi_1(\hat{S}), \pi_1(\hat{S})]_{\hat{S}_e}$. The proofs of the following descriptions can be found in [27, §3.2.1].

- Consider $\gamma \in [\pi_1(\hat{S}), \pi_1(\hat{S})]_{\hat{S}_e}$ that can be realized by a simple closed curve (see Figure 5.b). Let $\mathrm{Push}(\gamma) = T_{\tilde{\gamma}_1} T_{\tilde{\gamma}_2}^{-1}$. As shown in figure 5.c, exactly 1 of $T_{\tilde{\gamma}_1}$ or $T_{\tilde{\gamma}_2}$ is a separating twist (the point is that $\beta$ is a *nonseparating* curve in $\Sigma$). For concreteness we will assume that $T_{\tilde{\gamma}_1}$ is a separating twist, so $T_\beta T_{\tilde{\gamma}_2}^{-1}$ is a bounding pair map. Then $[\![\gamma]\!] = T_{\tilde{\gamma}_1}(T_\beta T_{\tilde{\gamma}_2}^{-1})$.

- Consider $x, y \in \pi_1(\hat{S})$ that can be realized by simple closed curves that only intersect at the basepoint. Assume that a regular neighborhood of $x \cup y$ is homeomorphic to $\Sigma_{0,3}$ (see Figure 5.d). Let $\tilde{x}$ and $\tilde{y}$ be as shown in Figure 5.e. Then for some $e_1, e_2 = \pm 1$ we have $[\![x, y]\!] = [T_{\tilde{x}}^{e_1}, T_{\tilde{y}}^{e_2}]$.

Finally, we will need the following result.

**Lemma 2.11.** *Let $S_1$ be an embedded subsurface of $S_2$. Assume that the genus of $S_1$ is at least* 1 *and that $S_1$ contains a basepoint in* $\mathrm{Int}(S_1)$. *Then $[\pi_1(S_1), \pi_1(S_1)]_{S_2}$ is generated by the set of all $\gamma \in \pi_1(S_1)$ that can be realized by simple closed curves which separate $S_1$ into* 2 *components, one of which is homeomorphic to either of the following.*

- *A torus with* 1 *boundary component.*
- *A genus* 0 *surface whose boundary consists of $\gamma \cup (\partial T \cap S_1)$, where $T$ is a connected component of $\overline{S_2 \setminus S_1}$.*

This is an immediate consequence of the following lemma of the author together with exact sequence (1).

**Lemma 2.12** ([25, Theorem A.1])**.** *Let $X$ be a surface whose genus is at least* 1. *Pick $* \in \mathrm{Int}(X)$. Then $[\pi_1(X,*), \pi_1(X,*)]$ is generated by elements that can be realized by simple closed curves that separate $X$ into* 2 *components,* 1 *of which is homeomorphic to $\Sigma_{1,1}$.*

## 2.4 The Johnson homomorphism

Fix some $g \geq 2$ and let $H = \mathrm{H}_1(\Sigma_g)$. The *Johnson homomorphisms* were constructed by Johnson in [15]. They can be defined in numerous ways (see [17] for an incomplete list), but we will not actually need the details of any construction. For surfaces with boundary, the Johnson homomorphisms are surjective homomorphisms

$$\tau_{\Sigma_{g,1}} : \mathscr{I}(\Sigma_{g,1}) \longrightarrow \wedge^3 H.$$

For closed surfaces, one has to modify the range a bit. The *canonical element* $\omega$ of $\wedge^2 H$ is the element $a_1 \wedge b_1 + \cdots + a_g \wedge b_g$, where $\{a_1, b_1, \ldots, a_g, b_g\}$ is a symplectic basis for $H$. It is independent of the choice of symplectic basis. Using the canonical element, we obtain the *standard embedding* $H \hookrightarrow \wedge^3 H$, namely $h \mapsto h \wedge \omega$. We will frequently confuse $H$ with its image in $\wedge^3 H$ under this embedding. The Johnson homomorphisms on closed surfaces are then surjective homomorphisms

$$\tau_{\Sigma_g} : \mathscr{I}(\Sigma_g) \longrightarrow (\wedge^3 H)/H.$$

The Johnson homomorphisms vanish on separating twists.

We now briefly discuss 2 naturality properties of the Johnson homomorphisms. The first is that they interact well with the Birman exact sequence. Namely, we have the following result, which is distilled from [15].

**Lemma 2.13.** *Fix $g \geq 2$ and set $H = \mathrm{H}_1(\Sigma_g)$. We then have a commutative diagram of short exact sequences*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(U\Sigma_g) & \longrightarrow & \mathscr{I}(\Sigma_{g,1}) & \longrightarrow & \mathscr{I}(\Sigma_g) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \tau_{\Sigma_{g,1}} & & \downarrow \tau_{\Sigma_g} & & \\ 0 & \longrightarrow & H & \longrightarrow & \wedge^3 H & \longrightarrow & (\wedge^3 H)/H & \longrightarrow & 0 \end{array}$$

*where the map $\pi_1(U\Sigma_g) \to H$ is the natural projection.*

For the second, let $S_1$ and $S_2$ be 2 surfaces with at most 1 boundary component and let $S_1 \hookrightarrow S_2$ be an embedding. Denote by $V_1$ and $V_2$ the targets of the Johnson homomorphisms on $\mathscr{I}(S_1)$ and

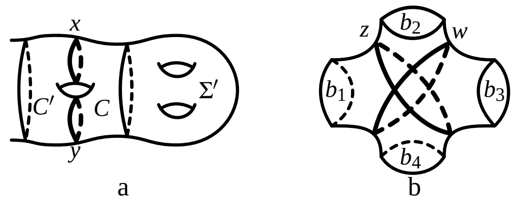



**Figure 6:** *a. Subsurfaces needed to calculate $\tau_\Sigma(T_x T_y^{-1})$. The whole subsurface pictured is $\Sigma''$. b. Curves needed to calculate $\tau_\Sigma([T_z, T_w])$*

$\mathscr{I}(S_2)$. There are then induced maps $\mathscr{I}(S_1) \to \mathscr{I}(S_2)$ and $V_1 \to V_2$ which fit into a commutative diagram

$$\begin{array}{ccc} \mathscr{I}(S_1) & \longrightarrow & \mathscr{I}(S_2) \\ \downarrow & & \downarrow \\ V_1 & \longrightarrow & V_2 \end{array} \tag{2}$$

We will use the commutativity of this diagram several times without comment.

Next, we will need some formulas for the Johnson homomorphisms. Let $\Sigma$ be a surface with at most 1 boundary component and let $\{x,y\}$ be a bounding pair on $\Sigma$ such that both $x$ and $y$ are nonseparating curves. Let $\Sigma'$ be a subsurface of $\Sigma$ satisfying the following 2 properties (see Figure 6.a).

- $\Sigma'$ has exactly 1 boundary component and is disjoint from $x \cup y$.
- There is a subsurface $C$ of $\Sigma$ such that $C \cong \Sigma_{0,3}$ and $\partial C = \partial \Sigma' \cup x \cup y$.

Denote by $\omega_{\Sigma'}$ the canonical element of $\wedge^2 \mathrm{H}_1(\Sigma')$. It was proven in [15] that

$$\tau_\Sigma(T_x T_y^{-1}) = [x] \wedge \omega_{\Sigma'};$$

here $x$ is oriented such that $C$ lies to the left of $x$. A useful equivalent formula is as follows. Let $\Sigma''$ be a subsurface of $\Sigma$ satisfying the following 2 properties (see Figure 6.a).

- $\Sigma''$ has exactly 1 boundary component and contains $x \cup y$.
- There is a subsurface $C'$ of $\Sigma''$ such that $C' \cong \Sigma_{0,3}$ and $\partial C = \partial \Sigma'' \cup x \cup y$.

Let $\omega_{\Sigma''}$ be the canonical element of $\wedge^2 \mathrm{H}_1(\Sigma'')$. We then have

$$\tau_\Sigma(T_x T_y^{-1}) = [x] \wedge \omega_{\Sigma''};$$

here $x$ is oriented such that $C'$ lies to the right of $x$.

*Remark.* This is implicitly contained in Lemma 2.13 above. The key point is that when considered as an element of $\mathscr{I}(\Sigma'')$, the mapping class $T_x T_y^{-1}$ is in the kernel of the Birman exact sequence obtained by gluing a disc to $\partial \Sigma''$.

Now let $\{z,w\}$ be a simply intersecting pair on $\Sigma$. Observe that a closed regular neighborhood $N$ of $z \cup w$ satisfies $N \cong \Sigma_{0,4}$ (see Figure 6.b). Let $\partial N = b_1 \cup \cdots \cup b_4$. We then have that

$$\tau_\Sigma([T_z, T_w]) = \pm [b_1] \wedge [b_2] \wedge [b_3],$$

where arbitrary orientations are chosen on the $b_i$. This can be proven directly, using the recipe given in [15]. It can also be deduced from Lemma 4.7 below in the same way that the calculation for bounding pair maps can be deduced from Lemma 2.13 above. We will leave the details of this calculation to the reader.

*Remark.* The above formula might seem strange, since $b_4$ played no role. However, by construction $[b_4] = \pm[b_1] \pm [b_2] \pm [b_3]$, so $[b_1] \wedge [b_2] \wedge [b_3] = \pm[b_1] \wedge [b_2] \wedge [b_4]$.

We conclude by defining the Johnson homomorphism on subsurface Torelli groups.

**Definition.** Let $\Sigma$ be a surface with at most 1 boundary component and let $S \hookrightarrow \Sigma$ be a cleanly embedded subsurface. Let $V$ be the target of $\tau_\Sigma$. Define $\tau_{\Sigma,S} : \mathscr{I}(\Sigma,S) \to V$ to be the restriction of $\tau_\Sigma$ to $\mathscr{I}(\Sigma,S)$ and define $\tilde{\tau}_{\Sigma,S} : \tilde{\mathscr{I}}(\Sigma,S) \to V$ to be the composition of the natural map $\tilde{\mathscr{I}}(\Sigma,S) \to \mathscr{I}(\Sigma,S)$ with $\tau_{\Sigma,S}$.

## 3 A bit of linear algebra

In this section, we prove the following linear-algebraic fact about $(\wedge^3 H)/H$.

**Lemma 3.1.** *For some $g \geq 1$, set $H = \mathrm{H}_1(\Sigma_g)$ and let $i : H \to \wedge^3 H$ be the standard embedding.*

1. *If $x$ is a primitive vector in $H$, then $(\wedge^3 \langle x \rangle^\perp) \cap i(H) = \langle i(x) \rangle \cong \mathbb{Z}$.*
2. *If $x$ and $y$ are 2 primitive vectors in $H$ with $0 \leq i_{\mathrm{alg}}(x,y) \leq 1$ that span a 2-dimensional summand, then $(\wedge^3 \langle x,y \rangle^\perp) \cap i(H) = 0$.*

*Proof.* The proofs of the 2 conclusions are similar; we will give the details for the first conclusion and leave the second to the reader. Choose a symplectic basis $\{a_1, b_1, \ldots, a_g, b_g\}$ for $H$ with $a_1 = x$. Letting $\omega = a_1 \wedge b_1 + \cdots + a_g \wedge b_g$, we have $i(h) = h \wedge \omega$ for $h \in H$. Observe that we have a direct sum decomposition

$$\wedge^3 H = (\wedge^2 \langle a_1 \rangle^\perp) \oplus (\wedge^3 \langle a_1 \rangle^\perp), \tag{3}$$

where the inclusion of the first factor is induced by the map $\wedge^2 \langle a_1 \rangle^\perp \to \wedge^3 H$ that takes $\theta \in \wedge^2 \langle a_1 \rangle^\perp$ to $\theta \wedge b_1$.

Consider $h \in H$. Write $h = ca_1 + db_1 + h'$ with $c, d \in \mathbb{Z}$ and $h' \in \langle a_2, b_2, \ldots, a_g, b_g \rangle$. Letting $\overline{\omega} = a_2 \wedge b_2 + \cdots + a_g \wedge b_g$, an easy calculation shows that the expression for $i(h)$ in terms of the decomposition (3) is

$$(d\overline{\omega} + h' \wedge a_1, (ca_1 + h') \wedge \overline{\omega}).$$

The first coordinate of this equals 0 if and only if $d = 0$ and $h' = 0$, and the lemma follows. □

The following special case of Lemma 3.1 will be used several times.

**Lemma 3.2.** *Fix $g' < g$, and let $H = \mathrm{H}_1(\Sigma_g)$ and $H' = \mathrm{H}_1(\Sigma_{g'})$. Choosing any symplectic embedding of $H'$ into $H$, the natural map $\wedge^3 H' \to (\wedge^3 H)/H$ is injective.*

# 4 The Johnson homomorphism and the Birman exact sequence

The goal of this section is to investigate the effect of the Johnson homomorphism on the terms of the Birman exact sequence for the Torelli groups. To that end, we make the following definition.

**Definition.** Let $\Sigma$ be a surface with at most 1 boundary component. Let $S \hookrightarrow \Sigma$ be a cleanly embedded subsurface and let $\beta$ be a boundary component of $S$. We define

$$\mathscr{K}(\Sigma,S) = \ker(\tau_{\Sigma,S}) < \mathscr{I}(\Sigma,S),$$
$$\tilde{\mathscr{K}}(\Sigma,S) = \ker(\tilde{\tau}_{\Sigma,S}) < \tilde{\mathscr{I}}(\Sigma,S),$$
$$\mathscr{P}_{\mathscr{K}}(\Sigma,S;\beta) = \mathscr{P}_{\mathscr{I}}(\Sigma,S;\beta) \cap \tilde{\mathscr{K}}(\Sigma,S).$$

The following 3 propositions will be our main results.

**Proposition 4.1** (Birman exact sequence for Johnson kernel)**.** *Let $\Sigma$ be a closed surface. Let $S \hookrightarrow \Sigma$ be a cleanly embedded nonexceptional subsurface with at least 2 boundary components and let $\beta$ be a boundary component of $S$. Let $S'$ be a splitting surface for $\beta$ in $S$. We then have a split short exact sequence*

$$1 \longrightarrow \mathscr{P}_{\mathscr{K}}(\Sigma,S;\beta) \longrightarrow \tilde{\mathscr{K}}(\Sigma,S) \longrightarrow \tilde{\mathscr{K}}(\Sigma,S') \longrightarrow 1.$$

**Proposition 4.2** (Kernel of exact sequence generated by separating twists I)**.** *Let $\Sigma$ be a surface with at most 1 boundary component. Let $S$ be a cleanly embedded nonexceptional subsurface of $\Sigma$ and let $\beta$ be a boundary component of $S$ such that $\beta$ is a separating curve in $\Sigma$. Assume that the genus of $S$ is at least 1 and that $\beta \subset \partial\Sigma$ if $\partial\Sigma \neq \emptyset$. Then $\mathscr{P}_{\mathscr{K}}(\Sigma,S;\beta)$ is contained in the subgroup of $\tilde{\mathscr{K}}(\Sigma,S)$ generated by separating twists.*

**Proposition 4.3** (Kernel of exact sequence generated by separating twists II)**.** *Let $\Sigma$ be a closed surface. Let $S$ be a cleanly embedded nonexceptional subsurface of $\Sigma$ and let $\beta$ be a boundary component of $S$ such that $\beta$ is a nonseparating curve in $\Sigma$. Assume that the genus of $S$ is at least 2. Then the image of $\mathscr{P}_{\mathscr{K}}(\Sigma,S;\beta) < \tilde{\mathscr{K}}(\Sigma,S)$ in $\mathscr{K}(\Sigma,S)$ is contained in the subgroup of $\mathscr{K}(\Sigma,S)$ generated by separating twists.*

Proposition 4.1 is proven in §4.3, Proposition 4.2 in §4.1, and Proposition 4.3 in §4.2.

*Remark.* Propositions 4.1 and 4.3 are also true when $\Sigma$ has a boundary component. Additionally, Proposition 4.2 is true when $\partial\Sigma \neq \emptyset$ but $\beta \cap \partial\Sigma = \emptyset$. However, the above results are all we need and their proofs are slightly easier.

## 4.1 Capping off a separating curve

In this section, $\Sigma$ is a fixed surface with at most 1 boundary component. Our goal is to prove Proposition 4.2. This is proceeded by 2 results concerning the effect of $\tilde{\tau}_{\Sigma,S}$ on $\mathscr{P}_{\mathscr{K}}(\Sigma,S;\beta)$.

**Lemma 4.4.** *Let $\beta$ be a non-nullhomotopic separating simple closed curve on $\Sigma$ and let $\Sigma'$ be a component of $\overline{\Sigma \setminus \beta}$ whose genus is at least 2. Assume that $\Sigma'$ has 1 boundary component. Define $\hat{\Sigma}'$ to be $\Sigma$ with a disc glued to $\beta$, so $\mathscr{P}_{\mathscr{I}}(\Sigma,\Sigma';\beta) \cong \pi_1(U\hat{\Sigma}')$. Let $V$ be the target of $\tilde{\tau}_{\Sigma,\Sigma'}$. The composition*

$$\pi_1(U\hat{\Sigma}') \cong \mathscr{P}_{\mathscr{I}}(\Sigma,\Sigma';\beta) \hookrightarrow \tilde{\mathscr{I}}(\Sigma,\Sigma') \xrightarrow{\tilde{\tau}_{\Sigma,\Sigma'}} V \qquad (4)$$

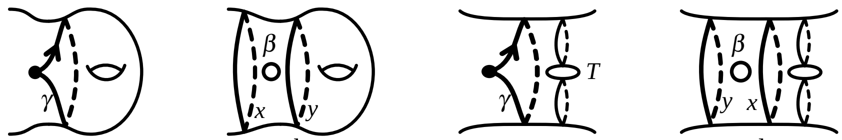



**Figure 7:** *a.* $\gamma \in \pi_1(\hat{S})$ *cuts off* $\Sigma_{1,1}$ *b. Push*$(\gamma) = T_x T_y^{-1}$ *c.* $\gamma \in \pi_1(\hat{S})$ *cuts off the boundary of* $T$*, a component of* $\hat{\Sigma}' \setminus \hat{S}$ *d. Push*$(\gamma) = T_x T_y^{-1}$

*then factors as*

$$\pi_1(U\hat{\Sigma}') \longrightarrow \pi_1(\hat{\Sigma}') \longrightarrow \mathrm{H}_1(\hat{\Sigma}') \longrightarrow \wedge^2 \mathrm{H}_1(\hat{\Sigma}') \longrightarrow V,$$

*where the map* $\mathrm{H}_1(\hat{\Sigma}') \to \wedge^2 \mathrm{H}_1(\hat{\Sigma}')$ *is the standard embedding and the map* $\wedge^2 \mathrm{H}_1(\hat{\Sigma}') \to V$ *is induced by the evident injection* $\mathrm{H}_1(\hat{\Sigma}') \to \mathrm{H}_1(\Sigma)$.

*Proof.* An immediate consequence of Lemma 2.13 together with the commutative diagram (2). □

**Corollary 4.5.** *Let* $S$ *be a cleanly embedded nonexceptional subsurface of* $\Sigma$*. Also, let* $\beta$ *be a boundary component of* $S$ *such that* $\beta$ *is a separating curve in* $\Sigma$ *and such that* $\beta \subset \partial\Sigma$ *if* $\partial\Sigma \neq \emptyset$*. Define* $\hat{S}$ *to be* $S$ *with a disc glued to* $\beta$*. Finally, let* $S_e$ *be the expanded subsurface of* $S$ *containing* $\beta$ *and let* $\hat{S}_e$ *be* $S_e$ *with a disc glued to* $\beta$*. Then* $\mathscr{P}_{\mathscr{K}}(\Sigma, S; \beta) < \mathscr{P}_{\mathscr{I}}(\Sigma, S; \beta) \cong \pi_1(U\hat{S})$ *is the pullback to* $\pi_1(U\hat{S})$ *of the subgroup* $[\pi_1(\hat{S}), \pi_1(\hat{S})]_{\hat{S}_e} < \pi_1(\hat{S})$.

*Proof.* Let $V$ be the target of $\tilde{\tau}_{\Sigma,S}$. By Lemma 4.4, the group $\mathscr{P}_{\mathscr{K}}(\Sigma, S; \beta)$ is the kernel of the composition

$$\pi_1(U\hat{S}) \longrightarrow \pi_1(\hat{S}) \longrightarrow \mathrm{H}_1(\hat{S}) \longrightarrow \mathrm{H}_1(\hat{S}_e) \longrightarrow \wedge^2 \mathrm{H}_1(\hat{S}_e) \longrightarrow V.$$

The map $\mathrm{H}_1(\hat{S}_e) \to \wedge^2 \mathrm{H}_1(\hat{S}_e)$ is injective. By Lemma 3.2, the map $\wedge^2 \mathrm{H}_1(\hat{S}_e) \longrightarrow V$ is injective. Finally, $[\pi_1(\hat{S}), \pi_1(\hat{S})]_{\hat{S}_e}$ is the kernel of the map $\pi_1(\hat{S}) \to \mathrm{H}_1(\hat{S}_e)$. The result follows. □

*Proof of Proposition 4.2.* Let $\hat{S}$ be $S$ with a disc glued to $\beta$. Also, let $S_e$ be the expanded subsurface of $\Sigma$ containing $\beta$ and let $\hat{S}_e$ be $S_e$ with a disc glued to $\beta$. By Corollary 4.5, the group $\mathscr{P}_{\mathscr{K}}(\Sigma, S; \beta) < \mathscr{P}(S; \beta) \cong \pi_1(U\hat{S})$ is the pullback to $\pi_1(U\hat{S})$ of the subgroup $[\pi_1(\hat{S}), \pi_1(\hat{S})]_{\hat{S}_e} < \pi_1(\hat{S})$. By Lemma 2.11, this pullback is generated by the following 3 types of elements.

- The loop around the fiber in $U\hat{S}$. With an appropriate choice of orientation, this corresponds to $T_\beta$, a separating twist.
- Push$(\gamma)$, where $\gamma \in \pi_1(\hat{S})$ can be realized by a simple closed separating curve that cuts off a subsurface homeomorphic to $\Sigma_{1,1}$ (see Figure 7.a). As depicted in Figure 7.b, the mapping class Push$(\gamma)$ equals $T_x T_y^{-1}$, where $T_x$ and $T_y$ are separating twists.
- Push$(\gamma)$, where $\gamma \in \pi_1(\hat{S})$ can be realized by a simple closed separating curve that cuts off a genus 0 subsurface whose boundary consists of $\gamma \cup (\partial T \cap \hat{S})$ for some connected component $T$ of $\overline{\hat{S}_e \setminus \hat{S}}$ (see Figure 7.c). As depicted in Figure 7.d, the mapping class Push$(\gamma)$ equals $T_x T_y^{-1}$, where $T_x$ and $T_y$ are separating twists.

The result follows. □

### 4.2 Capping off a nonseparating curve

#### 4.2.1 A bit of surface topology

Before proving Proposition 4.3, we will need a group-theoretic result. Let $S$ be a cleanly embedded subsurface of a surface $S'$. For simplicity, assume that $S'$ is not closed and that $\partial S' \subset S$ (this is all that is needed later on).

Recall that $[\pi_1(S), \pi_1(S)]_{S'}$ is the kernel of the induced map $\pi_1(S) \to \mathrm{H}_1(S')$. There is a natural map $[\pi_1(S'), \pi_1(S')] \to \wedge^2 \mathrm{H}_1(S')$ that takes $[x,y]$ to $[x] \wedge [y]$. It has kernel $[\pi_1(S'), [\pi_1(S'), \pi_1(S')]]$ and plays an important role in the definition of the Johnson homomorphism (see [15] for more details). There is thus an induced map $[\pi_1(S), \pi_1(S)]_{S'} \to \wedge^2 \mathrm{H}_1(S')$, and the following lemma identifies its kernel.

**Lemma 4.6.** *The kernel of the map* $\psi : [\pi_1(S), \pi_1(S)]_{S'} \to \wedge^2 \mathrm{H}_1(S')$ *is* $[\pi_1(S), [\pi_1(S), \pi_1(S)]_{S'}]$.

*Proof.* As notation, set $H_S = \mathrm{H}_1(S)$ and $H_{S'} = \mathrm{H}_1(S')$. Also, let $K$ and $Q$ be the kernel and image of the map $H_S \to H_{S'}$.

It is clear that $[\pi_1(S), [\pi_1(S), \pi_1(S)]_{S'}] < \ker(\psi)$. Setting

$$\Gamma = [\pi_1(S), \pi_1(S)]_{S'} / [\pi_1(S), [\pi_1(S), \pi_1(S)]_{S'}],$$

there is an induced homomorphism $\overline{\psi} : \Gamma \longrightarrow \wedge^2 H_{S'}$. Our goal is to show that $\overline{\psi}$ is an injection.

Let $T_1, \ldots, T_k$ be the components of $\overline{S' \setminus S}$. Observe that $K \cong \mathbb{Z}^k$ with generators $[\partial T_i]$ for $1 \leq i \leq k$ (see Figure 8.a; here we are using the fact that $\partial S' \subset S$). In particular, we have a short exact sequence

$$1 \longrightarrow [\pi_1(S), \pi_1(S)] \longrightarrow [\pi_1(S), \pi_1(S)]_{S'} \longrightarrow \mathbb{Z}^k \longrightarrow 1.$$

We have an isomorphism $[\pi_1(S), \pi_1(S)] / [\pi_1(S), [\pi_1(S), \pi_1(S)]] \cong \wedge^2 H_S$. Moreover, the kernel of the map $\wedge^2 H_S \to \wedge^2 H_{S'}$ is the subgroup $H_S \wedge K$. The upshot is that we have an isomorphism

$$[\pi_1(S), \pi_1(S)] / [\pi_1(S), [\pi_1(S), \pi_1(S)]_{S'}] \cong (\wedge^2 H_S)/(H_S \wedge K) \cong \wedge^2 Q.$$

As a consequence of this, the map $\overline{\psi}$ fits into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \wedge^2 Q & \longrightarrow & \Gamma & \longrightarrow & \mathbb{Z}^k & \longrightarrow & 0 \\ & & \downarrow = & & \downarrow \overline{\psi} & & \downarrow & & \\ 0 & \longrightarrow & \wedge^2 Q & \longrightarrow & \wedge^2 H_{S'} & \longrightarrow & (\wedge^2 H_{S'})/(\wedge^2 Q) & \longrightarrow & 0 \end{array}$$

To prove the lemma, it is enough to show that the map $\mathbb{Z}^k \to (\wedge^2 H_{S'})/(\wedge^2 Q)$ is injective.

For $1 \leq i \leq k$, let $\gamma_i$ be a simple closed curve on $S$ separates $S$ into 2 components, 1 of which is a genus zero surface with boundary $\gamma_i \cup \partial T_i$. Choose the $\gamma_i$ such that $\gamma_i \cap \gamma_j = \emptyset$ if $i \neq j$ (see Figure 8.a). Each $\gamma_i$ is a separating curve on $S'$ which divides $S'$ into 2 components, one of which contains $T_i$ and the other of which contains the $T_j$ for $j \neq i$. For $1 \leq i \leq k$, let $T_i'$ be the component of $S'$ cut along $\gamma_i$ that contains $T_i$. Observe that each $T_i'$ is a surface with 1 boundary component and that there is an injection

$$\bigoplus_{i=1}^{k} \mathrm{H}_1(T_i') \hookrightarrow H_{S'};$$

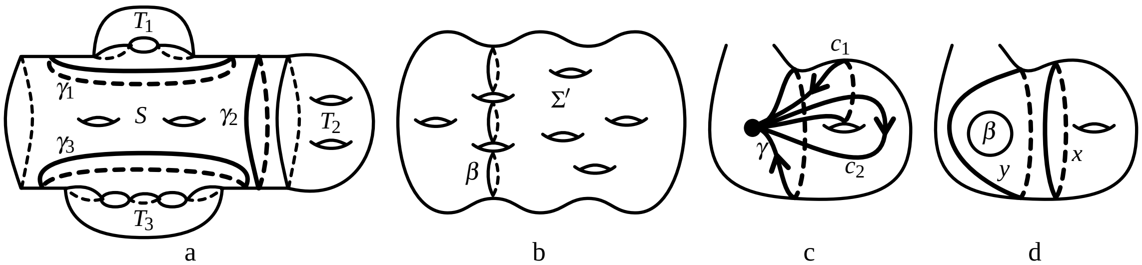



**Figure 8:** *a. The diagram for the proof of Lemma 4.6* *b.* $\Sigma \setminus \Sigma'$ *is connected* *c.* $\gamma = [c_1, c_2]$ *d.* $[\![\gamma]\!] = T_x(T_\beta T_y^{-1})$

we will identify this sum with its image. For $1 \leq i \leq k$, let $\omega_i \in \wedge^2 H_{S'}$ be the canonical element of $\wedge^2 \mathrm{H}_1(T_i')$. Tracing through the definitions, it is easy to see that $\omega_i$ is the image in $\wedge^2 H_{S'}$ of a lift of the appropriate generator of $\mathbb{Z}^k$ to $\Gamma$. Our goal is thus to show that the $\omega_i$ are linearly independent moduli $\wedge^2 Q$.

We can choose a basis $B$ for $H_{S'}$ with the following properties.

- For $1 \leq i \leq k$, there is some subset of $B$ forming a symplectic basis for $\mathrm{H}_1(T_i')$.
- There is a subset of $B$ that forms a basis for $Q$.

Choose a total ordering $\prec$ on $B$, so $\wedge^2 H_{S'}$ has the basis $B_\wedge = \{x \wedge y \mid x, y \in B, \, x \prec y\}$. A subset of $B_\wedge$ forms a basis for $\wedge^2 Q$. Moreover, for each $1 \leq i \leq k$, there exists some $v_i \in B_\wedge$ with the following properties.

- In the expansion of $\omega_i$ in the basis $B_\wedge$, the coordinate of $v_i$ is nonzero.
- $v_i \notin \wedge^2 Q$.
- $v_i \neq v_j$ for $i \neq j$.

It follows that the $\omega_i$ are linearly independent modulo $\wedge^2 Q$, and we are done. □

### 4.2.2 The proof of Proposition 4.3

In this section, we will assume that $\Sigma$ is a closed surface. We will prove Proposition 4.3 after proving the following 2 results.

**Lemma 4.7.** *Let* $\Sigma' \hookrightarrow \Sigma$ *be a cleanly embedded nonexceptional subsurface and let* $\beta$ *be a boundary component of* $\Sigma'$*. Assume that* $\beta$ *does not separate* $\Sigma$ *and that* $\Sigma \setminus \Sigma'$ *is connected (see Figure 8.b). Let* $\hat{\Sigma}'$ *be the result of gluing a disc to the boundary component* $\beta$ *of* $\Sigma'$*, so* $\mathscr{P}_{\mathscr{I}}(\Sigma, \Sigma'; \beta) \cong [\pi_1(\hat{\Sigma}'), \pi_1(\hat{\Sigma}')]$*. Setting* $H = \mathrm{H}_1(\Sigma)$*, the composition*

$$\mathscr{P}_{\mathscr{I}}(\Sigma, \Sigma'; \beta) \cong [\pi_1(\hat{\Sigma}'), \pi_1(\hat{\Sigma}')] \xrightarrow{\tilde{\tau}_{\Sigma,\Sigma'}} (\wedge^3 H)/H$$

*factors as*

$$[\pi_1(\hat{\Sigma}'), \pi_1(\hat{\Sigma}')] \longrightarrow \wedge^2 \mathrm{H}_1(\hat{\Sigma}') \longrightarrow (\wedge^3 H)/H.$$

*Here the map* $[\pi_1(\hat{\Sigma}'), \pi_1(\hat{\Sigma}')] \to \wedge^2 \mathrm{H}_1(\hat{\Sigma}')$ *is the standard map and the map* $\wedge^2 \mathrm{H}_1(\hat{\Sigma}') \to (\wedge^3 H)/H$ *takes* $\theta \in \wedge^2 \mathrm{H}_1(\hat{\Sigma}')$ *to the image of* $\tilde{\theta} \wedge [\beta]$ *in* $(\wedge^3 H)/H$*, where* $\tilde{\theta} \in \wedge^2 \mathrm{H}_1(\Sigma')$ *is any lift of* $\theta$*.*

*Remark.* This last map is well defined; indeed, if $x, y \in \mathrm{H}_1(\hat{\Sigma}')$, then we can lift $x$ and $y$ to elements $\tilde{x}, \tilde{y} \in \mathrm{H}_1(\Sigma')$ which are well-defined up to the addition of multiples of $[\beta]$, and thus the element $\tilde{x} \wedge \tilde{y} \wedge [\beta]$ is independent of all of our choices.

*Proof of Lemma 4.7.* It is enough to check this on a generating set for $[\pi_1(\hat{\Sigma}'), \pi_1(\hat{\Sigma}')]$. By Lemma 2.12, the group $[\pi_1(\hat{\Sigma}'), \pi_1(\hat{\Sigma}')]$ is generated by the set of all $\gamma \in \pi_1(\hat{\Sigma}')$ that can be realized by simple closed separating curves which cut off a subsurface homeomorphic to $\Sigma_{1,1}$ (see Figure 8.c). Consider such a $\gamma$. Inverting $\gamma$ if necessary, we can assume that the subsurface homeomorphic to $\Sigma_{1,1}$ lies to the right of $\gamma$. Write $\gamma$ as $[c_1, c_2]$, where $c_1$ and $c_2$ are as in 8.c. From Figure 8.d, we see that $[\![c_1, c_2]\!] = T_x(T_\beta T_y^{-1})$, where $T_x$ is a separating twist and $T_\beta T_y^{-1}$ is a bounding pair map. We then have

$$\tilde{\tau}_{\Sigma,\Sigma'}([\![c_1, c_2]\!]) = [c_1] \wedge [c_2] \wedge [\beta];$$

here the orientation of $\beta$ is chosen such that $\Sigma'$ lies to the right of $\beta$. The lemma follows. □

**Corollary 4.8.** *Let $S$ be a cleanly embedded subsurface of $\Sigma$. Also, let $\beta$ be a boundary component of $S$ such that $\beta$ is a nonseparating curve in $\Sigma$, let $S_c$ be the complementary component of $\beta$ in $S$, and let $S_e$ be the expanded subsurface of $S$ containing $\beta$. Let $\hat{S}$ and $\hat{S}_e$ be the results of gluing discs to the boundary component $\beta$ of $S$ and $S_e$, respectively. Then the following hold.*

- *If $S_c$ is not an annulus, then $\mathscr{P}_{\mathscr{K}}(\Sigma, S; \beta) < [\pi_1(\hat{S}), \pi_1(\hat{S})]_{\hat{S}_e}$ equals $[\pi_1(\hat{S}), [\pi_1(\hat{S}), \pi_1(\hat{S})]_{\hat{S}_e}]$.*
- *If $S_c$ is an annulus with boundary components $\beta$ and $\beta'$, then $\mathscr{P}_{\mathscr{K}}(\Sigma, S; \beta) < [\pi_1(\hat{S}), \pi_1(\hat{S})]_{\hat{S}_e}$ equals the group generated by $[\pi_1(\hat{S}), [\pi_1(\hat{S}), \pi_1(\hat{S})]_{\hat{S}_e}]$ and $\delta$, where $\delta \in \pi_1(\hat{S})$ is any element that can be realized by a simple closed curve that is freely homotopic to the boundary component of $\hat{S}$ corresponding to $\beta' \subset S$ (see Figure 9.e).*

*Proof.* Set $H_\Sigma = \mathrm{H}_1(\Sigma)$ and $H_{\hat{S}_e} = \mathrm{H}_1(\hat{S}_e)$ and $H_{S_e} = \mathrm{H}_1(S_e)$. Using Lemma 4.7, we can factor the composition

$$\mathscr{P}_{\mathscr{I}}(\Sigma, S; \beta) \cong [\pi_1(\hat{S}), \pi_1(\hat{S})]_{\hat{S}_e} \hookrightarrow \tilde{\mathscr{I}}(\Sigma, S) \longrightarrow \mathscr{I}(\Sigma, S) \hookrightarrow \mathscr{I}(\Sigma) \longrightarrow (\wedge^3 H_\Sigma)/H_\Sigma$$

as

$$[\pi_1(\hat{S}), \pi_1(\hat{S})]_{\hat{S}_e} \hookrightarrow [\pi_1(\hat{S}_e), \pi_1(\hat{S}_e)] \longrightarrow \wedge^2 H_{\hat{S}_e} \hookrightarrow \wedge^3 H_{S_e} \longrightarrow (\wedge^3 H_\Sigma)/H_\Sigma.$$

By Lemma 4.6, the kernel of the map $[\pi_1(\hat{S}), \pi_1(\hat{S})]_{\hat{S}_e} \to \wedge^3 H_{\hat{S}_e}$ is $[\pi_1(\hat{S}), [\pi_1(\hat{S}), \pi_1(\hat{S})]_{\hat{S}_e}]$. If $S_c$ is not an annulus, then we can find 2 linearly independent primitive vectors in $H_\Sigma$ with algebraic intersection number 0 that are orthogonal to $H_{S_e}$ (one of them can be $[\beta]$). Thus by Lemma 3.1 we can conclude that the map $\wedge^3 H_{S_e} \to (\wedge^3 H_\Sigma)/H_\Sigma$ is injective, and the corollary follows. Otherwise, we have $H_{S_e} = [\beta]^\perp$, and Lemma 3.1 implies that the map $\wedge^3 H_{S_e} \to (\wedge^3 H_\Sigma)/H_\Sigma$ has kernel $\mathbb{Z}$ generated by $[\beta] \wedge \omega$; here $\omega \in \wedge^2 H_\Sigma$ is the canonical element. This lifts to the simple closed curve $\delta$ described in the conclusion of the corollary, and we are done. □

*Proof of Proposition 4.3.* Our notation is as in Corollary 4.8. We investigate the generating set for $\mathscr{P}_{\mathscr{K}}(\Sigma, S; \beta)$ given by that Corollary. First consider $[\pi_1(\hat{S}), [\pi_1(\hat{S}), \pi_1(\hat{S})]_{\hat{S}_e}]$. By Lemma 2.11, the group $[\pi_1(\hat{S}), \pi_1(\hat{S})]_{\hat{S}_e}$ is generated by the set $X$ consisting of $\gamma \in \pi_1(\hat{S})$ that can be realized by simple closed separating curves that cut off either of the following.

- A torus with 1 boundary component.

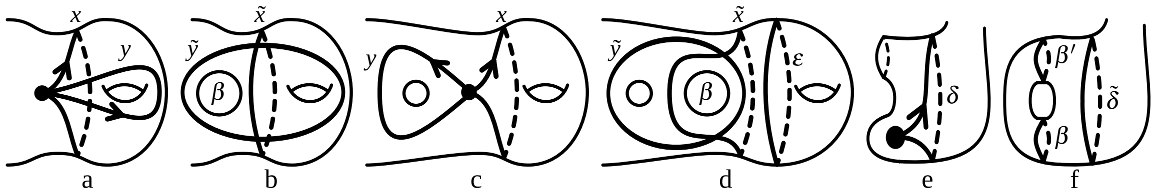



**Figure 9:** *a. x and y with y contained in the* 1*-holed torus* *b.* $[\![x,y]\!] = [T_{\tilde{x}}, T_{\tilde{y}}]$ *with* $T_{\tilde{x}}$ *a separating twist c. x and y with y no contained in the* 1*-holed torus* *d.* $[\![x,y]\!] = [T_{\tilde{x}}^{-1}, T_{\tilde{y}}]$ *with* $T_{\tilde{x}}$ not *a separating twist e,f. The curve* $\delta$ *has* $[\![\delta]\!] = T_{\tilde{\delta}}(T_{\beta} T_{\beta'}^{-1})$

- A genus 0 surface whose boundary consists of $\gamma \cup (\partial T \cap S_1)$, where $T$ is a connected component of $\overline{\hat{S}_e \setminus \hat{S}}$.

For $x \in X$, define

$$Y_x = \{\gamma \in \pi_1(\hat{S}) \mid \gamma \text{ can be realized by a simple closed nonseparating curve} \\ \text{that only intersects } x \text{ at the basepoint}\}.$$

Observe that $Y_x$ generates $\pi_1(\hat{S})$ for all $x \in X$, so the set $Z = \{[y,x] \mid x \in X, y \in Y_x\}$ normally generates $[\pi_1(\hat{S}), [\pi_1(\hat{S}), \pi_1(\hat{S})]_{\hat{S}_e}]$. Moreover, $Z$ is closed under conjugation (here we are using the fact that inner automorphisms of $\pi_1(\hat{S})$ can be realized by mapping classes fixing the basepoint). We conclude that $Z$ in fact generates $[\pi_1(\hat{S}), [\pi_1(\hat{S}), \pi_1(\hat{S})]_{\hat{S}_e}]$.

Consider $[y,x] \in Z$. The mapping class associated to $[y,x]$ is $[\![y,x]\!]$. We must express $[\![y,x]\!]$ as a product of separating twists. We will do the case that $x$ can be realized by a simple closed curve that cuts off a 1-holed torus; the other case is similar. There are 2 cases.

- The curve $y$ lies inside the 1-holed torus (see Figure 9.a). In this case, as shown in Figure 9.b we have $[\![y,x]\!] = [T_{\tilde{y}}^{e_1}, T_{\tilde{x}}^{e_2}]$ for curves $\tilde{x}$ and $\tilde{y}$ as depicted there and some signs $e_1, e_2 = \pm 1$ (the signs depend on the orientations of $x$ and $y$). Observe that $T_{\tilde{x}}$ is a separating twist, so we have
$$[\![y,x]\!] = T_{\tilde{y}}^{-e_1} T_{\tilde{x}}^{-e_2} T_{\tilde{y}}^{e_1} T_{\tilde{x}}^{e_2} = T_{T_{\tilde{y}}^{e_1}(\tilde{x})}^{-e_2} T_{\tilde{x}}^{e_2},$$
a product of separating twists.

- The curve $y$ lies outside the 1-holed torus (see Figure 9.c). In this case, as shown in Figure 9.d we have $[\![y,x]\!] = [T_{\tilde{y}}^{e_1}, T_{\tilde{x}}^{e_2}]$ for curves $\tilde{x}$ and $\tilde{y}$ as depicted there and some signs $e_1, e_2 = \pm 1$ (the signs depend on the orientations of $x$ and $y$). The mapping class $T_{\tilde{x}}$ is *not* a separating twist, so the previous trick does not apply. However, let $\varepsilon$ be the separating curve depicted in Figure 9.d and let $S'$ be the result of removing the 1 holed torus from $S$ by cutting along $\varepsilon$. We then have $[\![y,x]\!] \in \mathscr{P}_{\mathscr{K}}(\Sigma, S'; \varepsilon)$, so we can apply Proposition 4.2 to write $[\![y,x]\!]$ as a product of separating twists.

This completes the proof unless the complementary component to $\beta$ in $S$ is an annulus, say with boundary components $\beta$ and $\beta'$. In this case, we also need the curve $\delta$ depicted in Figure 8.d. As shown in Figure 8.d, the corresponding mapping class $[\![\delta]\!] \in \tilde{\mathscr{K}}(\Sigma, S)$ equals $T_{\tilde{\delta}}(T_{\beta} T_{\beta'}^{-1})$, where $T_{\tilde{\delta}}$ is a separating twist. However, when mapped to $\mathscr{K}(\Sigma, S)$ the bounding pair map $T_{\beta} T_{\beta'}^{-1}$ goes to 1, and we are done. □

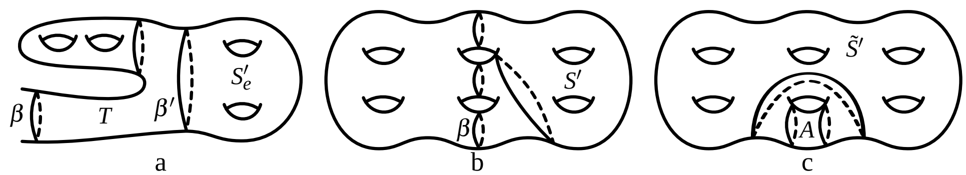



**Figure 10:** *a. The subsurface $S_e$ of $\Sigma$ with the boundary component $\beta$ (which separates $\Sigma$) together with the subsurfaces $T$ and $S'_e$ embedded in it b. An example of a splitting surface $S'$ for a nonseparating boundary component c. We can expand $S'$ to a splitting surface $\tilde{S}'$ for $\beta$ in $\tilde{S} = \overline{\Sigma \setminus A}$*

## 4.3 The exact sequence

In this section, we prove Proposition 4.1. For the proof, we will need the following lemma.

**Lemma 4.9.** *Let*

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

*be a split exact sequence of groups, let $Q$ be another group, and let $f : \Gamma \to Q$ be a homomorphism. Regarding $G$ as a subgroup of $\Gamma$ via the splitting, assume that $f(K) \cap f(G) = 1$. Then there is short exact sequence*

$$1 \longrightarrow \ker(f|_K) \longrightarrow \ker(f) \longrightarrow \ker(f|_G) \longrightarrow 1.$$

The proof of Lemma 4.9 is trivial and left to the reader.

*Proof of Proposition 4.1.* We begin by recalling the setup of the proposition. Let $\Sigma$ be a closed surface and let $S \hookrightarrow \Sigma$ be a cleanly embedded nonexceptional subsurface with at least 2 boundary components. Let $\beta$ be a boundary component of $S$ and let $S' \hookrightarrow S$ be a splitting surface for $\beta$. Theorem 2.8 gives a split exact sequence

$$1 \longrightarrow \mathscr{P}_{\mathscr{I}}(\Sigma, S; \beta) \longrightarrow \tilde{\mathscr{I}}(\Sigma, S) \longrightarrow \tilde{\mathscr{I}}(\Sigma, S') \longrightarrow 1. \tag{5}$$

We must prove that this restricts to a split exact sequence

$$1 \longrightarrow \mathscr{P}_{\mathscr{K}}(\Sigma, S; \beta) \longrightarrow \tilde{\mathscr{K}}(\Sigma, S) \longrightarrow \tilde{\mathscr{K}}(\Sigma, S') \longrightarrow 1, \tag{6}$$

where the terms are the kernels of the restrictions of $\tilde{\tau}_{\Sigma,S}$ to the terms in (5).

We will verify the hypothesis of Lemma 4.9 for the restriction of $\tilde{\tau}_{\Sigma,S}$ to $\tilde{\mathscr{I}}(\Sigma, S)$. This hypothesis asserts that

$$\tilde{\tau}_{\Sigma,S}(\mathscr{P}_{\mathscr{I}}(\Sigma, S; \beta)) \cap \tilde{\tau}_{\Sigma,S}(\tilde{\mathscr{I}}(\Sigma, S')) = 0. \tag{7}$$

There are 2 cases. Set $H_\Sigma = \mathrm{H}_1(\Sigma)$.

**Case 1.** *The curve $\beta$ separates $\Sigma$.*

Set $T = \overline{S \setminus S'}$, so $T \cong \Sigma_{0,3}$. Let $\beta' = \partial T \cap S'$, let $S_e$ be the expanded subsurface of $S$ containing $\beta$, and let $S'_e$ be the expanded subsurface of $S'$ containing $\beta'$ (see Figure 10.a). Observe that

$$\tilde{\tau}_{\Sigma,S}(\mathscr{P}_{\mathscr{I}}(\Sigma, S; \beta)) < \tilde{\tau}_{\Sigma,S_e}(\mathscr{P}_{\mathscr{I}}(\Sigma, S_e; \beta)) \quad \text{and} \quad \tilde{\tau}_{\Sigma,S}(\tilde{\mathscr{I}}(\Sigma, S')) < \tilde{\tau}_{\Sigma,S'_e}(\tilde{\mathscr{I}}(\Sigma, S'_e)).$$

Thus to prove (7), it is enough to prove that

$$\tilde{\tau}_{\Sigma,S_e}(\mathscr{P}_{\mathscr{I}}(\Sigma, S_e; \beta)) \cap \tilde{\tau}_{\Sigma,S'_e}(\tilde{\mathscr{I}}(\Sigma, S'_e)) = 0. \tag{8}$$

Set $H_{S_e} = \mathrm{H}_1(S_e)$ and $H_{S'_e} = \mathrm{H}_1(S'_e)$. Since the genus of $\overline{\Sigma \setminus S_e}$ is positive, Lemma 3.2 implies that the natural map $\wedge^3 H_{S_e} \to (\wedge^3 H_\Sigma)/H_\Sigma$ is injective. Both terms of (8) lie in the image of this injective map. It suffices, therefore, to check that

$$\tau_{S_e}(\mathscr{P}_{\mathscr{I}}(\Sigma, S_e; \beta)) \cap \tau_{S_e}(\mathscr{I}(S_e, S'_e)) = 0. \tag{9}$$

By Lemma 4.4, the image $\tau_{S_e}(\mathscr{P}_{\mathscr{I}}(\Sigma, S_e; \beta))$ is exactly the image of the standard embedding $H_{S_e} \hookrightarrow \wedge^3 H_{S_e}$. Also, it is clear that $\tau_{S_e}(\mathscr{I}(S_e, S'_e)$ is the image of $\wedge^3 H_{S'_e}$ in $\wedge^3 H_{S_e}$. Thus (9) is asserting that $H_{S_e} \cap \wedge^3 H_{S'_e} < \wedge^3 H_{S_e}$ is 0. Since $\overline{S_e \setminus S'_e}$ has positive genus, Lemma 3.2 implies that the map $\wedge^3 H_{S'_e} \to (\wedge^3 H_{S_e})/H_{S_e}$ is injective, and the result follows.

**Case 2.** *The curve $\beta$ does not separate $\Sigma$.*

Let $A$ be a closed annulus embedded in $\overline{\Sigma \setminus S}$ such that 1 component of $\partial A$ is $\beta$. Define $\tilde{S} = \overline{\Sigma \setminus A}$. We can find a splitting surface $\tilde{S}'$ for $\beta$ in $\tilde{S}$ such that $S' \subset \tilde{S}'$ (see Figures 10.b–c). Observe that

$$\tilde{\tau}_{\Sigma,S}(\mathscr{P}_{\mathscr{I}}(\Sigma, S; \beta)) < \tilde{\tau}_{\Sigma,\tilde{S}}(\mathscr{P}_{\mathscr{I}}(\Sigma, \tilde{S}; \beta)) \quad \text{and} \quad \tilde{\tau}_{\Sigma,S}(\tilde{\mathscr{I}}(\Sigma, S')) < \tilde{\tau}_{\Sigma,\tilde{S}}(\tilde{\mathscr{I}}(\Sigma, \tilde{S}')).$$

Thus to prove (7), it is enough to prove that

$$\tilde{\tau}_{\Sigma,\tilde{S}}(\mathscr{P}_{\mathscr{I}}(\Sigma, \tilde{S}; \beta)) \cap \tilde{\tau}_{\Sigma,\tilde{S}}(\tilde{\mathscr{I}}(\Sigma, \tilde{S}')) = 0. \tag{10}$$

Set $H_{\tilde{S}'} = \mathrm{H}_1(\tilde{S}')$. We have $\langle[\beta]\rangle^\perp = \langle[\beta]\rangle \oplus H_{\tilde{S}'}$. Associated to this is the decomposition

$$\wedge^3 \langle[\beta]\rangle^\perp = (\wedge^2 H_{\tilde{S}'}) \oplus (\wedge^3 H_{\tilde{S}'}),$$

where $\wedge^2 H_{\tilde{S}'}$ is embedded in $\wedge^3 \langle[\beta]\rangle^\perp$ via the map $\theta \mapsto \theta \wedge [\beta]$. Let $\phi : \wedge^3 \langle[\beta]\rangle^\perp \to (\wedge^3 H_\Sigma)/H_\Sigma$ be the obvious map. It is clear that $\tilde{\tau}_{\Sigma,\tilde{S}}(\tilde{\mathscr{I}}(\Sigma, \tilde{S}')) = \phi(\wedge^3 H_{\tilde{S}'})$, and by Lemma 4.7, we have $\tilde{\tau}_{\Sigma,\tilde{S}}(\mathscr{P}_{\mathscr{I}}(\Sigma, \tilde{S}; \beta)) = \phi(\wedge^2 H_{\tilde{S}'})$. We conclude that to prove (10), it is enough to prove

$$\phi(\wedge^3 H_{\tilde{S}'}) \cap \phi(\wedge^2 H_{\tilde{S}'}) = 0. \tag{11}$$

By Lemma 3.1, the kernel of $\phi$ is isomorphic to $\mathbb{Z}$ and is entirely contained in contained in the factor $(\wedge^2 H_{\tilde{S}'})$ (it is spanned by the canonical element). The condition (11) is an immediate consequence, and the proposition follows. □

## 5 The proof of Theorem 1.1

We now turn to the proof of Theorem 1.1. A key tool will be the following theorem, which is a formulation due to the author of a theorem of Armstrong [1]. A group $G$ acts on a simplicial complex $X$ *without rotations* if for all simplices $\sigma$ of $X$, the stabilizer $G_\sigma$ fixes $\sigma$ pointwise.

**Theorem 5.1** ([25, Theorem 2.1])**.** *Consider a group $G$ acting without rotations on a simply connected simplicial complex $X$. The group $G$ is generated by the set*

$$\bigcup_{v \in X^{(0)}} G_v$$

*if and only if $X/G$ is simply connected.*

We will apply this to the action of $\mathcal{K}(\Sigma_g)$ on the simplicial complex $\mathcal{C}_g^{\text{ns},2}$ defined in the introduction. To do this, we will need the following proposition.

**Proposition 5.2.** *For $g \geq 3$, the complexes $\mathcal{C}_g^{ns,2}$ and $\mathcal{C}_g^{ns,2}/\mathcal{K}(\Sigma_g)$ are simply connected.*

The proof of Proposition 5.2 is contained in §6. We will also need the following theorem of Farb-Leininger-Margalit.

**Theorem 5.3** ([8, Proposition 3.2])**.** *Let $\gamma$ be the isotopy class of a nontrivial simple closed curve on $\Sigma_g$ and let $f \in \mathcal{K}(\Sigma_g)$. Assume that $f(\gamma) \neq \gamma$. Then $i(\gamma, f(\gamma)) \geq 4$.*

*Proof of Theorem 1.1.* Let $\Sigma$ be a surface with at most 1 boundary component and let $S \hookrightarrow \Sigma$ be an embedded subsurface whose genus is at least 2. Our goal is to prove that $\mathcal{K}(\Sigma, S) < \mathcal{I}(\Sigma)$ is generated by separating twists. Without loss of generality, we can assume that $S$ is cleanly embedded in $\Sigma$. The proof will be by induction on the genus and number of boundary components of $S$.

Our induction will be carried out in several steps. An outline of the argument is as follows. The base case (where $S = \Sigma = \Sigma_2$) is in Step 1. In Steps 2–4, we show how to go from $\mathcal{K}(\Sigma_g)$ being generated by separating twists to $\mathcal{K}(\Sigma, S)$ being generated by separating twists when $S \cong \Sigma_{g,n}$. Finally, in Step 5 we show that if $\mathcal{K}(\Sigma_{g+1}, S)$ is generated by separating twists whenever $S \cong \Sigma_{g,2}$, then $\mathcal{K}(\Sigma_{g+1})$ is generated by separating twists.

**Step 1.** *The group $\mathcal{K}(\Sigma_2)$ is generated by separating twists.*

In fact, $\Sigma_2$ contains no bounding pair maps, so $\mathcal{I}(\Sigma_2)$ is generated by separating twists. This implies that $\tau_{\Sigma_2} = 0$, and the result follows.

**Step 2.** *Assume that $\Sigma$ has 1 boundary component $\beta$. Define $\hat{\Sigma}$ to be the result of gluing a disc to $\beta$. If $\beta \subset S$, then let $\hat{S}$ be $S$ with a disc glued to $\beta$; otherwise let $\hat{S} = S$. Regard $\hat{S}$ as a cleanly embedded subsurface of $\hat{\Sigma}$ in the obvious way. Assume that $\mathcal{K}(\hat{\Sigma}, \hat{S})$ is generated by separating twists. Then $\mathcal{K}(\Sigma, S)$ is generated by separating twists.*

*Remark.* In particular, this implies that if $\mathcal{K}(\Sigma_g)$ is generated by separating twists, then $\mathcal{K}(\Sigma_{g,1})$ is generated by separating twists.

Set $H = \mathrm{H}_1(\Sigma)$ and let $\rho : \pi_1(U\hat{\Sigma}) \to H$ be the projection. By Lemma 2.13, we have a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(U\hat{\Sigma}) \cap \mathcal{I}(\Sigma, S) & \longrightarrow & \mathcal{I}(\Sigma, S) & \longrightarrow & \mathcal{I}(\hat{\Sigma}, \hat{S}) & \longrightarrow & 1 \\ & & \downarrow \rho & & \downarrow \tau_{\Sigma,S} & & \downarrow \tau_{\hat{\Sigma},\hat{S}} & & \\ 0 & \longrightarrow & H & \longrightarrow & \wedge^3 H & \longrightarrow & (\wedge^3 H)/H & \longrightarrow & 0 \end{array}$$

Consider $f \in \mathcal{K}(\Sigma, S)$. By assumption, $\mathcal{K}(\hat{\Sigma}, \hat{S}) = \ker(\tau_{\hat{\Sigma},\hat{S}})$ is generated by separating twists, and each of these lifts to a separating twist in $\mathcal{K}(\Sigma, S)$. Using this, we can assume without loss of generality that $f \in \pi_1(U\hat{\Sigma}) \cap \mathcal{I}(\Sigma, S)$. There are 2 cases.

- If $\beta \subset S$, then $\ker(\rho|_{\pi_1(U\hat{\Sigma}) \cap \mathcal{I}(\Sigma,S)})$ is the image of $\mathcal{P}_{\mathcal{K}}(\Sigma, S; \beta) < \tilde{\mathcal{K}}(\Sigma, S)$ in $\mathcal{K}(\Sigma, S)$, and we can apply Proposition 4.2 to conclude that $f$ can be written as a product of separating twists.

- If $\beta \cap S = \emptyset$, then using Lemma 2.3 either $\pi_1(U\hat{\Sigma}) \cap \mathrm{Mod}(\Sigma, S) = 1$, in which case the desired result is trivial, or the component $C$ of $\Sigma \setminus S$ containing $\beta$ is homeomorphic to $\Sigma_{0,3}$ and

$$\pi_1(U\hat{\Sigma}) \cap \mathscr{I}(\Sigma, S) \cong \pi_1(U\hat{\Sigma}) \cap \mathrm{Mod}(\Sigma, S) = \langle T_x T_y^{-1} \rangle,$$

where $\partial C = \beta \cup x \cup y$. In this latter case, observe that $\tau_{\Sigma,S}(T_x T_y^{-1}) \neq 0$. Since $\tau_{\Sigma,S}(f) = 0$, we conclude that $f = 1$, and we are done.

**Step 3.** *Assume that $\Sigma$ is closed. If $\mathscr{K}(\Sigma_{g,1})$ is generated by separating twists and if $S \cong \Sigma_{g,1}$, then $\mathscr{K}(\Sigma, S)$ is generated by separating twists.*

Observe that the genus of $\Sigma$ is strictly larger than the genus of $S$. Set $H_\Sigma = \mathrm{H}_1(\Sigma)$ and $H_S = \mathrm{H}_1(S)$. Since the genus of $\Sigma$ is strictly larger than the genus of $S$, Lemma 3.2 implies that the induced map $i : \wedge^3 H_S \to (\wedge^3 H_\Sigma)/H_\Sigma$ is an injection. Moreover, by Theorem 2.1, we have that $\mathscr{I}(\Sigma, S) \cong \tilde{\mathscr{I}}(\Sigma, S)$, and since $H_S$ injects into $H_\Sigma$, we have $\tilde{\mathscr{I}}(\Sigma, S) \cong \mathscr{I}(S)$. We thus have a commutative diagram

$$\begin{array}{ccc} \mathscr{I}(S) & \xrightarrow{\tau_S} & \wedge^3 H_S \\ \downarrow \cong & & \downarrow i \\ \mathscr{I}(\Sigma, S) & \xrightarrow{\tau_{\Sigma,S}} & (\wedge^3 H_\Sigma)/H_\Sigma \end{array}$$

Since $\ker(\tau_S) \cong \mathscr{K}(S)$ is generated by separating twists and $i$ is an injection, we conclude that $\mathscr{I}(\Sigma, S)$ is generated by separating twists, as desired.

**Step 4.** *Assume that $S \cong \Sigma_{g,n}$. Also, assume that $\mathscr{K}(\Sigma, T)$ is generated by separating twists for all cleanly embedded subsurfaces $T \hookrightarrow \Sigma$ with $T \cong \Sigma_{g,1}$. Then $\mathscr{K}(\Sigma, S)$ is generated by separating twists.*

By Step 2, we can assume that $\Sigma$ is closed. The proof will be by induction on $n$. The base case $n = 1$ holds by assumption. Assume now that $n > 1$. Let $\beta$ be a boundary component of $S$ and let $S' \subset S$ be a splitting surface for $\beta$. Proposition 4.1 says that there is a split short exact sequence

$$1 \longrightarrow \mathscr{P}_{\mathscr{K}}(\Sigma, S; \beta) \longrightarrow \tilde{\mathscr{K}}(\Sigma, S) \longrightarrow \tilde{\mathscr{K}}(\Sigma, S') \longrightarrow 1.$$

Also, Propositions 4.2 and 4.3 say that the image of $\mathscr{P}_{\mathscr{K}}(\Sigma, S; \beta)$ in $\mathscr{I}(\Sigma, S)$ is contained in the subgroup generated by separating twists. Finally, the image of $\tilde{\mathscr{K}}(\Sigma, S')$ in $\mathscr{I}(\Sigma, S')$ is $\mathscr{K}(\Sigma, S')$, and since $S'$ has fewer boundary components than $S$, the inductive hypothesis says that this is generated by separating twists. The desired result follows.

**Step 5.** *If $\mathscr{K}(\Sigma_{g+1}, S)$ is generated by separating twists whenever $S \cong \Sigma_{g,2}$, then $\mathscr{K}(\Sigma_{g+1})$ is generated by separating twists.*

By Theorem 5.3, the group $\mathscr{K}(\Sigma_g)$ acts on $\mathscr{C}_g^{\mathrm{ns},2}$ without rotations. Using Theorem 5.1 and Proposition 5.2, we conclude that $\mathscr{K}(\Sigma_{g+1})$ is generated by the stabilizers of vertices of $\mathscr{C}_{g+1}^{\mathrm{ns},2}$. Letting $\gamma$ be the simple closed nonseparating curve corresponding to such a vertex, we have that $(\mathscr{K}(\Sigma_{g+1}))_\gamma = \mathscr{K}(\Sigma_{g+1}, S)$, where $S$ is the complement of a regular neighborhood of $\gamma$. Since such an $S$ is homeomorphic to $\Sigma_{g,2}$, these stabilizers are by assumption generated by separating twists, and we are done. □

## 6 The complex of curves

The goal of this section is to prove Proposition 5.2, which asserts that for $g \geq 3$, the complexes $\mathscr{C}_g^{\text{ns},2}$ and $\mathscr{C}_g^{\text{ns},2}/\mathscr{K}(\Sigma_g)$ are simply connected. These 2 facts are proven as Lemmas 6.1 and 6.5 in §6.1 and §6.3 below. The fact that $\mathscr{C}_g^{\text{ns},2}/\mathscr{K}(\Sigma_g)$ is simply connected is proven using the trick introduced in [26] to prove that the "complex of separating curves" is simply connected. This requires a presentation for $\text{Mod}(\Sigma_g)/\mathscr{K}(\Sigma_g)$, which we construct in §6.2.

A *simplicial path* in a simplicial complex $X$ is a sequence of vertices connected by edges. We will denote the path that starts at $v_1 \in X$, then goes to $v_2 \in X$, etc. and ends at $v_n \in X$ by $v_1 - v_2 - \cdots - v_n$. If $f : X \to Y$ is a simplicial map and $\ell$ is a simplicial path in $X$, then we will denote by $f_*(\ell)$ the induced path in $Y$.

### 6.1 The simple connectivity of $\mathscr{C}_g^{\textbf{ns},2}$

In this section, we prove the following lemma.

**Lemma 6.1.** *For $g \geq 3$, the complex $\mathscr{C}_g^{ns,2}$ is simply connected.*

For the proof, we will need the following theorem of Harer.

**Theorem 6.2** ([11, Theorem 1.1])**.** *For $g \geq 1$, the complex $\mathscr{C}_g^{ns}$ is $(g-2)$-connected.*

*Proof of Lemma 6.1.* For $g \geq 2$, the fact that $\mathscr{C}_g^{\text{ns}} \subset \mathscr{C}_g^{\text{ns},2}$ is connected implies that $\mathscr{C}_g^{\text{ns},2}$ is connected. Now assume that $g \geq 3$ and that $\ell$ is a loop in $\mathscr{C}_g^{\text{ns},2}$. We wish to prove that $\ell$ may be contracted to a point. By Theorem 6.2, it is enough to homotope $\ell$ into $\mathscr{C}_g^{\text{ns}} \subset \mathscr{C}_g^{\text{ns},2}$. Using the simplicial approximation theorem, we can assume that $\ell$ is a simplicial path, i.e. of the form $v_1 - v_2 - \cdots - v_n$ with $v_i$ an isotopy class of nonseparating simple closed curve on $\Sigma_g$ for $1 \leq i \leq n$ and $v_n = v_1$. Assume first that $i(v_i, v_{i+1}) \neq 0$ for some $1 \leq i < n$. An easy Euler characteristic argument (see, e.g., [8, Lemma 2.1]) shows that there is a simple closed curve $w$ such that $i(v_i, w) = i(v_{i+1}, w) = 0$. The loop $\ell$ can then be homotoped to $v_1 - \cdots - v_i - w - v_{i+1} - \cdots - v_n$. Repeating this, we can homotope $\ell$ such that $i(v_i, v_{i+1}) = 0$ for all $1 \leq i < n$. By a similar argument, we can homotope $\ell$ such that in addition $v_i \cup v_{i+1}$ does not separate $\Sigma_g$ for all $1 \leq i < n$; i.e. such that $\ell$ lies in $\mathscr{C}_g^{\text{ns}} \subset \mathscr{C}_g^{\text{ns},2}$. □

### 6.2 A presentation for $\mathscr{I}(\Sigma_g)/\mathscr{K}(\Sigma_g)$

#### 6.2.1 Generators

Begin by fixing a *geometric symplectic basis $B$* on $\Sigma_g$, i.e. a set $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ of nonseparating simple closed curves on $\Sigma_g$ such that

$$i(\alpha_i, \beta_i) = 1 \quad \text{and} \quad i(\alpha_i, \beta_j) = i(\alpha_i, \alpha_j) = i(\beta_i, \beta_j) = 0$$

for distinct $1 \leq i, j \leq g$ (see Figure 11.a). For $x \in B$, we will say that the *companion* of $x$ is $\beta_i$ if $x = \alpha_i$ and $\alpha_i$ if $x = \beta_i$. Letting $B' = \{\delta_1, \ldots, \delta_{g-1}\}$ be the curves in Figure 11.b, a theorem of Lickorish [21] says that $S_{\text{Mod}} = \{T_x \mid x \in B \cup B'\}$ generates $\text{Mod}(\Sigma_g)$.

To simplify our relations, we will add to $S_{\text{Mod}}$ a certain set $S_\wedge \subset \mathscr{I}(\Sigma_g)$. For distinct $x, y, z \in B$, the set $S_\wedge$ will contain an element $J_{\{x,y,z\}}$ with $\tau_{\Sigma_g}(J_{\{x,y,z\}}) = \pm[x] \wedge [y] \wedge [z]$. We emphasize that $J_{\{x,y,z\}}$ depends only on the (unordered) set $\{x, y, z\}$. Fix distinct $x, y, z \in B$ and define $J_{\{x,y,z\}}$ as

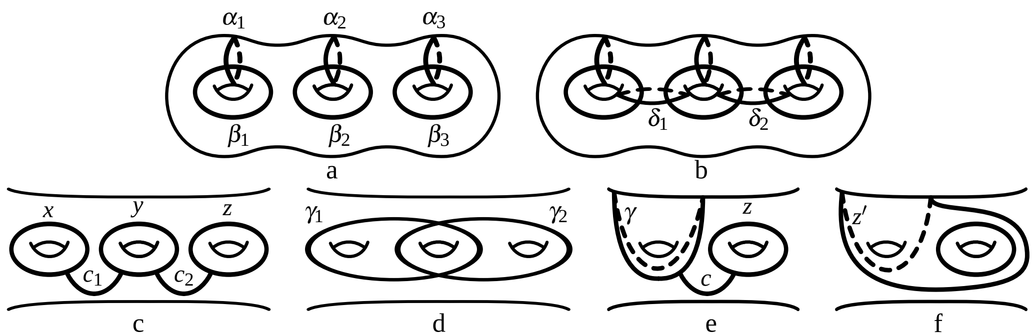



**Figure 11:** *a. A geometric symplectic basis b. Lickorish's generating set for* $\text{Mod}(\Sigma_g)$ *c–d. Constructing* $J_{\{x,y,z\}} = [T_{\gamma_1}, T_{\gamma_2}]$ *when* $x,y,z \in B$ *are disjoint. The interiors of the arcs* $c_i$ *do not intersect any elements of* $B$*. e–f. Constructing* $J_{\{x,y,z\}} = T_z T_{z'}^{-1}$ *when* $i(x,y) = 1$*. The curve* $\gamma$ *is a the boundary of a regular neighborhood of* $x \cup y$ *and the interior of the arc* $c$ *does not intersect any element of* $B$

follows. These definitions involve a number of arbitrary choices, but none of our results will depend on the choices we make.

- If $i(x,y) = i(x,z) = i(y,z) = 0$, then $J_{\{x,y,z\}}$ will be a simply intersecting pair map $[T_{\gamma_1}, T_{\gamma_2}]$ constructed as follows. Choose embedded disjoint arcs $c_1$ and $c_2$ such that $c_1$ starts at a point in $x$ and ends at a point in $y$, such that $c_2$ starts at a point of $y$ and ends at a point in $z$, and such that the interiors of the $c_i$ are disjoint from all elements of $B$ (see Figure 11.c). Let $N_1$ be a regular neighborhood of $x \cup c_1 \cup y$ and $N_2$ be a regular neighborhood of $y \cup c_2 \cup z$. Then $\gamma_1$ is the component of $\partial N_1$ that is not homotopic to $x$ or $y$ and $\gamma_2$ is the component of $\partial N_2$ that is not homotopic to $y$ or $z$ (see Figure 11.d).

- Otherwise, 2 elements of $\{x,y,z\}$ (say $x$ and $y$) are companions. In this case, $J_{\{x,y,z\}}$ will be a bounding pair map $T_z T_{z'}^{-1}$, where the curve $z'$ is constructed as follows. A regular neighborhood of $x \cup y$ is homeomorphic to $\Sigma_{1,1}$; let $\gamma$ be its boundary. We can assume that $\gamma$ is disjoint from all elements of $B$. Let $c$ be an embedded arc which goes from a point of $\gamma$ to a point of $z$ and whose interior is disjoint from $\gamma$ and all elements of $B$ (see Figure 11.e). Let $N$ be a regular neighborhood of $\gamma \cup c \cup z$. Then $z'$ is the component of $\partial N$ that is not homotopic to $\gamma$ or $z$ (see Figure 11.f).

It is clear that $\tau_{\Sigma_g}(J_{\{x,y,z\}}) = \pm[x] \wedge [y] \wedge [z]$. Let $S_\wedge = \{J_{\{x,y,z\}} \mid x,y,z \in B \text{ are distinct}\}$ and $S = S_{\text{Mod}} \cup S_\wedge$. The following lemma is immediate from our construction.

**Lemma 6.3.** *Consider* $s \in S$ *and* $b \in B$*.*

1. *If* $s \in S_{\text{Mod}}$*, then* $i(b, s(b)) \leq 1$*. Moreover, there are at most two* $b \in B$ *such that* $s(b) \neq b$*.*
2. *If* $s = J_{\{x,y,z\}} \in S_\wedge$*, then* $i(b, s(b)) \leq 2$*. Moreover,* $s(b) = b$ *if*

$$b \in \{x,y,z\} \cup \{x' \in B \mid \text{neither } x' \text{ nor the companion of } x' \text{ lies in } \{x,y,z\}\}.$$

### 6.2.2 Relations

We now construct the relations in our presentation. Let $B$ and $S = S_{\text{Mod}} \cup S_\wedge$ be as in §6.2.1 and let $H = \text{H}_1(\Sigma_g)$. Our strategy will be to add relations needed to yield the structure contained in the exact sequence

$$1 \longrightarrow (\wedge^3 H)/H \longrightarrow \mathrm{Mod}(\Sigma_g)/\mathscr{K}(\Sigma_g) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1$$

obtained by quotienting the exact sequence

$$1 \longrightarrow \mathscr{I}(\Sigma_g) \longrightarrow \mathrm{Mod}(\Sigma_g) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1$$

by $\mathscr{K}(\Sigma_g) < \mathscr{I}(\Sigma_g)$. Let $\phi : \langle S_\wedge \rangle \to \wedge^3 H$ be the natural map, where $\langle S_\wedge \rangle$ denotes the free group on $S_\wedge$. We then define the following sets of relations on the generators $S$.

- $R_{\mathrm{Mod}}$ is a set of relations such that $\langle S \mid R_{\mathrm{Mod}} \rangle \cong \mathrm{Mod}(\Sigma_g)$ (the exact form of the relations in $R_{\mathrm{Mod}}$ will not matter).
- $R_{\mathrm{comm}} = \{[J_{\{x_1,x_2,x_3\}}, J_{\{y_1,y_2,y_3\}}] \mid x_1,x_2,x_3 \in B$ are distinct and $y_1,y_2,y_3 \in B$ are distinct$\}$
- Recall that $H$ is embedded in $\wedge^3 H$ via the map $h \mapsto h \wedge \omega$, where $\omega = \sum_i [\alpha_i] \wedge [\beta_i]$. If $h \in \{\alpha_i, \beta_i\}$, then the corresponding element of $\wedge^3 H$ is $h \wedge \sum_{j \neq i} [\alpha_j] \wedge [\beta_j]$. We define

  $$R_H = \{\prod_{j \neq i} J^{\pm 1}_{\{x,\alpha_j,\beta_j\}} \mid x \in \{\alpha_i, \beta_i\} \text{ for some } 1 \leq i \leq g\},$$

  where the signs are chosen such that $\phi(\prod_{j \neq i} J^{\pm 1}_{\{x,\alpha_j,\beta_j\}}) = [x] \wedge \sum_{j \neq i} [\alpha_j] \wedge [\beta_j]$.
- For each $s \in S_{\mathrm{Mod}}$, the element $s$ acts on $\wedge^3 H$ via the symplectic representation. For distinct $x,y,z \in B$, let $W_{\{x,y,z\},s}$ be a minimal length word in $S_\wedge^{\pm 1}$ such that $\phi(W_{\{x,y,z\},s}) = s(\phi(J_{\{x,y,z\}}))$. We define

  $$R_{\mathrm{conj}} = \{s J_{\{x,y,z\}} s^{-1} W^{-1}_{\{x,y,z\},s} \mid x,y,z \in B \text{ are distinct and } s \in S_{\mathrm{Mod}}\}.$$

Set $R = R_{\mathrm{Mod}} \cup R_{\mathrm{comm}} \cup R_H \cup R_{\mathrm{conj}}$. We have the following.

**Lemma 6.4.** *We have* $\langle S \mid R \rangle \cong \mathrm{Mod}(\Sigma_g)/\mathscr{K}(\Sigma_g)$.

*Proof.* We clearly have a homomorphism $\psi : \langle S \mid R \rangle \to \mathrm{Mod}(\Sigma_g)/\mathscr{K}(\Sigma_g)$. Let $A < \langle S \mid R \rangle$ be the subgroup spanned by $S_\wedge$. Using the relations $R_{\mathrm{comm}}$ and $R_{\mathrm{conj}}$, the subgroup $A$ is abelian and normal. Also, we have

$$\langle S \mid R \rangle / A \cong \langle S \mid R \cup S_\wedge \rangle \cong \langle S \mid R_{\mathrm{Mod}} \cup S_\wedge \rangle.$$

By Theorem 2.5, the set $S_\wedge$ normally generates $\mathscr{I}(\Sigma_g) \lhd \mathrm{Mod}(\Sigma_g)$. We conclude that

$$\langle S \mid R \rangle / A \cong \mathrm{Mod}(\Sigma_g)/\mathscr{I}(\Sigma_g) \cong \mathrm{Sp}_{2g}(\mathbb{Z}).$$

The homomorphism $\psi$ restricts to a surjection $\psi' : A \to (\wedge^3 H)/H$. Using the relations $R_H$, we have that the rank of $A$ is at most the rank of $(\wedge^3 H)/H$, so $\psi'$ is an isomorphism. We have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & \langle S \mid R \rangle & \longrightarrow & \mathrm{Sp}_{2g}(\mathbb{Z}) & \longrightarrow & 1 \\ & & \downarrow \cong & & \downarrow \psi & & \downarrow \cong & & \\ 1 & \longrightarrow & (\wedge^3 H)/H & \longrightarrow & \mathrm{Mod}(\Sigma_g)/\mathscr{K}(\Sigma_g) & \longrightarrow & \mathrm{Sp}_{2g}(\mathbb{Z}) & \longrightarrow & 1 \end{array}$$

By the 5 lemma, $\psi$ is an isomorphism, and we are done. ☐

## 6.3 The simple connectivity of $\mathscr{C}_g^{\mathbf{ns},2}/\mathscr{K}(\Sigma_g)$

In this section, we will prove the following lemma.

**Lemma 6.5.** *For $g \geq 3$, the complex $\mathscr{C}_g^{\mathrm{ns},2}/\mathscr{K}(\Sigma_g)$ is simply connected.*

During the proof, we will make use of the following concept (see [12] for a survey).

**Definition.** Let $G$ be a group with a generating set $S$. The *Cayley graph* of $G$, denotes $\mathrm{Cay}(G,S)$, is the oriented graph whose vertices are elements of $G$ and where $g_1, g_2 \in G$ are connected by an edge if $g_2 = g_1 s$ for some $s \in S$.

*Remark.* The group $G$ acts on $\mathrm{Cay}(G,S)$ on the left.

*Proof of Lemma 6.5.* Set $\Gamma = \mathrm{Mod}(\Sigma_g)/\mathscr{K}(\Sigma_g)$. Let $S = S_{\mathrm{Mod}} \cup S_{\wedge}$ be the generating set for $\Gamma$ constructed in §6.2.1 and let $B = \{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ be the geometric symplectic basis used to construct $S$. The set $S$ is also a generating set for $\mathrm{Mod}(\Sigma_g)$. Let $\rho_1 : \mathrm{Cay}(\mathrm{Mod}(\Sigma_g),S) \to \mathrm{Cay}(\Gamma,S)$ and $\rho_2 : \mathscr{C}_g^{\mathrm{ns},2} \to \mathscr{C}_g^{\mathrm{ns},2}/\mathscr{K}(\Sigma_g)$ be the natural projections.

By Lemma 6.3, we have $i(\alpha_1, s(\alpha_1)) \leq 2$ for all $s \in S$. This implies that $i(g(\alpha_1), gs(\alpha_1)) \leq 2$ for all $s \in S$ and $g \in \mathrm{Mod}(\Sigma_g)$, so there is a simplicial map $\psi : \mathrm{Cay}(\mathrm{Mod}(\Sigma_g),S) \to \mathscr{C}_g^{\mathrm{ns},2}$ such that $\psi(g) = g(\alpha_1)$. This induces a map $\overline{\psi} : \mathrm{Cay}(\Gamma,S) \to \mathscr{C}_g^{\mathrm{ns},2}/\mathscr{K}(\Sigma_g)$ such that the diagram

$$\begin{array}{ccc} \mathrm{Cay}(\mathrm{Mod}(\Sigma_g),S) & \xrightarrow{\psi} & \mathscr{C}_g^{\mathrm{ns},2} \\ \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_2} \\ \mathrm{Cay}(\Gamma,S) & \xrightarrow{\overline{\psi}} & \mathscr{C}_g^{\mathrm{ns},2}/\mathscr{K}(\Sigma_g) \end{array}$$

commutes. Since $\overline{\psi}(1) = \rho_2(a_1)$, we get an induced map

$$\overline{\psi}_* : \pi_1(\mathrm{Cay}(\Gamma,S),1) \longrightarrow \pi_1(\mathscr{C}_g^{\mathrm{ns},2}/\mathscr{K}(\Sigma_g), \rho_2(a_1)).$$

To prove the lemma, it is enough to show that $\overline{\psi}_*$ is both surjective and the zero map.

**Claim 1.** *$\overline{\psi}_*$ is surjective.*

Consider $\ell \in \pi_1(\mathscr{C}_g^{\mathrm{ns},2}/\mathscr{K}(\Sigma_g), \rho_2(a_1))$, which we can assume is a simplicial loop. Lifting $\ell$ one edge at a time to $\mathscr{C}_g^{\mathrm{ns},2}$, we obtain a path $\tilde{\ell}$ in $\mathscr{C}_g^{\mathrm{ns},2}$ that begins at $a_1$ and ends at $f(a_1)$ for some $f \in \mathscr{K}(\Sigma_g)$ such that $(\rho_2)_*(\tilde{\ell}) = \ell$. There is a path $\tilde{\delta}$ in $\mathrm{Cay}(\mathrm{Mod}(\Sigma_g),S)$ that starts at 1 and ends at $f$. Observe that $(\rho_1)_*(\tilde{\delta}) \in \pi_1(\mathrm{Cay}(\Gamma,S),1)$. Since $\tilde{\ell}$ and $\psi_*(\tilde{\delta})$ have the same starting and ending points, Lemma 6.1 implies that they are homotopic while fixing their endpoints. Projecting this homotopy to $\mathscr{C}_g^{\mathrm{ns},2}/\mathscr{K}(\Sigma_g)$, we can use the commutativity of the above diagram to conclude that $\overline{\psi}_*((\rho_1)_*(\tilde{\delta}))$ is homotopic to $\ell$ while fixing the basepoint, and we are done.

**Claim 2.** *$\overline{\psi}_*$ is the zero map.*

Let $R = R_{\mathrm{Mod}} \cup R_{\mathrm{conj}} \cup R_{\mathrm{comm}} \cup R_H$ be the relations for $\Gamma$ given by Lemma 6.4. For each relation $s_1 \cdots s_k \in R$, we get an element of $\pi_1(\mathrm{Cay}(\Gamma,S),1)$, namely $1 - s_1 - s_1 s_2 - \cdots - s_1 \cdots s_k = 1$. The $\Gamma$-orbits of these loops "essentially" generate $\pi_1(\mathrm{Cay}(\Gamma,S),1)$ (there is a small issue with basepoints). One precise version of this statement is that we can construct a simply connected complex $X$ from $\mathrm{Cay}(\Gamma,S)$ by attaching discs to the $\Gamma$-orbits of the loops associated to the elements of $R$ (see [12,

§V.13]). We will show that the images in $\mathscr{C}_g^{\text{ns},2}/\mathscr{K}(\Sigma_g)$ of the loops in $\text{Cay}(\Gamma,S)$ associated to the elements of $R$ are contractible. This will imply that we can extend $\overline{\psi}$ to $X$. Since $X$ is simply connected, we will be able to conclude that $\overline{\psi}_*$ is the zero map, as desired.

Consider $r \in R$. If $r = s_1 \cdots s_k$ with $s_i \in S^{\pm 1}$ for $1 \leq i \leq k$, then the loop $\ell$ in $\text{Cay}(\Gamma,S)$ associated to $r$ is $1 - s_1 - s_1 s_2 - \cdots - s_1 s_2 \cdots s_k = 1$. We will verify that $\overline{\psi}_*(\ell)$ is contractible. We begin with some general remarks. Using Lemma 6.3, the only $s \in S^{\pm 1}$ such that $s(\alpha_1) \neq \alpha_1$ are $T_{\beta_1}^{\pm 1}$ and $J_{\beta_1,x,y}^{\pm 1}$ with $x, y \neq \alpha_1$. We will call these the *interesting generators*. If none of the $s_i$ are interesting, then $\overline{\psi}_*(\ell)$ is the constant loop, so there is nothing to prove. If exactly 1 of the $s_i$ is interesting, the $\overline{\psi}_*(\ell)$ is a loop $\rho_2(\alpha_1) - v$ with $v \neq \rho_2(\alpha_1)$, which is impossible. If exactly two $s_i$ are interesting, then $\overline{\psi}_*(\ell)$ is of the form $\rho_2(\alpha_1) - v - \rho_2(\alpha_1)$, so it is trivially contractible. We thus only need to worry about the $r \in R$ that use at least 3 interesting generators. We will call these the *interesting relations*. To deal with the interesting $r \in R$, we must consider several cases.

- $r \in R_{\text{Mod}}$. In this case, we can lift $\overline{\psi}_*(\ell)$ to a loop in $\mathscr{C}_g^{\text{ns},2}$. The resulting loop is contractible by Lemma 6.1, and by projecting this contraction down to $\mathscr{C}_g^{\text{ns},2}/\mathscr{K}(\Sigma_g)$, we obtain the desired null-homotopy.

- $r \in R_{\text{comm}}$. The interesting relations here are of the form $[J_{\beta_1,v_1,v_2}, J_{\beta_1,v_3,v_4}]$ with $v_i \notin \{\alpha_1, \beta_1\}$ for $1 \leq i \leq 4$. By Lemma 6.3, we have $J_{\beta_1,v_1,v_2}(\beta_1) = \beta_1$ and $J_{\beta_1,v_3,v_4}(\beta_1) = \beta_1$. Since $i(\alpha_1,\beta_1) = 1$, we deduce that for every subword $w$ of $r$, we have $i(w(\alpha_1),\beta_1) = 1$. This implies that $\rho_2(\beta_1)$ is adjacent to every vertex of $\overline{\psi}_*(\ell)$, so we can contract $\overline{\psi}_*(\ell)$ to $\rho_2(\beta_1)$.

- $r \in R_H$. In this case, there is only 1 interesting relation, namely $\prod_{i=2}^{g} J_{\beta_1,\alpha_i,\beta_i}$. Lemma 6.3 says that $J_{\beta_1,\alpha_i,\beta_i}(\beta_1) = \beta_1$ for $2 \leq i \leq g$, so an argument like in the $R_{\text{comm}}$ case shows that $\overline{\psi}_*(\ell)$ can be contracted to $\rho_2(\beta_1)$.

- $r \in R_{\text{conj}}$. Write $r = T_\gamma J_{\{y_1,y_2,y_3\}} T_\gamma^{-1} W_{\{y_1,y_2,y_3\},\gamma}^{-1}$. By Lemma 6.3, the mapping class $T_\gamma$ fixes all but at most 2 elements of $B$. Reordering the $y_i$ if necessary, we can assume that $T_\gamma(y_3) = y_3$. In particular, when we expand out $T_\gamma([y_1] \wedge [y_2] \wedge [y_3])$ in terms of the basis $\{[b] \mid b \in B\}$, every term is of the form $c[b_1] \wedge [b_2] \wedge [y_3]$ with $c \in \mathbb{Z}$ and $b_1, b_2 \in B$ distinct and not equal to $y_3$. The upshot is that every generator used in $W_{\{y_1,y_2,y_3\},\gamma}$ is of the form $J_{\{b_1,b_2,y_3\}}^{\pm 1}$ with $b_1, b_2 \in B$ distinct and not equal to $y_3$. By Lemma 6.3, we conclude that every generator used in $r$ fixes $y_3$, so by the argument used in the $R_{\text{comm}}$ case, we can contract $\overline{\psi}_*(\ell)$ to $\rho_2(y_3)$. $\square$

# 7 Finite index subgroups of the Torelli group

In this section, we prove Theorem 1.2, whose statement we now recall. Let $\Sigma$ be a surface with at most 1 boundary component and let $S \hookrightarrow \Sigma$ be a cleanly embedded subsurface whose genus is at least 3. Let $\Gamma$ be a finite-index subgroup of $\mathscr{I}(\Sigma,S)$ with $\mathscr{K}(\Sigma,S) < \Gamma$. Then Theorem 1.2 asserts that $\text{H}_1(\Gamma;\mathbb{Q}) \cong \tau_\Sigma(\mathscr{I}(\Sigma,S)) \otimes \mathbb{Q}$.

The target of $\tau_\Sigma$ is a free abelian group and $[\mathscr{I}(\Sigma,S) : \Gamma] < \infty$, so

$$\tau_\Sigma(\mathscr{I}(\Sigma,S)) \otimes \mathbb{Q} \cong \tau_\Sigma(\Gamma) \otimes \mathbb{Q}.$$

Since $\mathscr{K}(\Sigma,S) < \Gamma$, we have $\ker(\tau_\Sigma|_\Gamma) = \mathscr{K}(\Sigma,S)$, so Theorem 1.1 says that $\ker(\tau_\Sigma|_\Gamma)$ is generated by separating twists. We conclude that Theorem 1.2 is equivalent to the following lemma. If $G$ is a group and $g \in G$, then denote by $[g]_G$ the associated element of $\text{H}_1(G;\mathbb{Q})$.

**Lemma 7.1** (Separating twists vanish)**.** *Let $\Sigma$ be a surface with at most* 1 *boundary component and let $S \hookrightarrow \Sigma$ be a cleanly embedded subsurface whose genus is at least* 3*. Let $\Gamma$ be a finite-index subgroup of $\mathscr{I}(\Sigma,S)$ with $\mathscr{K}(\Sigma,S) < \Gamma$ and let $\beta$ be a simple closed separating curve in $S$. Then $[T_\beta]_\Gamma = 0$.*

*Remark.* In the special case that $S = \Sigma$ and $\Gamma = \mathscr{I}(\Sigma,S)$, this was proven by Johnson in [20]. Our proof builds on Johnson's proof.

The proof of Lemma 7.1 is contained in §7.2. This is proceeded by §7.1, which contains a group-theoretic lemma.

## 7.1 A bit of group cohomology

We will need the following lemma, which is an abstraction of the main idea of the proof of [28, Theorem 1.1].

**Lemma 7.2.** *Let $G$ be a group, let $G' < G$ be a finite index subgroup, and let $g \in G$ be a central element. Assume that $[g]_G = 0$ and that $g \in G'$. Then $[g]_{G'} = 0$.*

*Proof.* We can assume that $g$ has infinite order, so $\langle g \rangle \cong \mathbb{Z}$. Since $g$ is central, we have $\langle g \rangle \lhd G$. Define $\overline{G} = G/\langle g \rangle$ and $\overline{G}' = G'/\langle g \rangle$. We have a commutative diagram of central extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & G' & \longrightarrow & \overline{G}' & \longrightarrow & 1 \\ & & \downarrow\cong & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & G & \longrightarrow & \overline{G} & \longrightarrow & 1 \end{array}$$

There is an associated map of 5-term exact sequences in rational group homology (see [5, Corollary VII.6.4]), which takes the form

$$\begin{array}{ccccccccccc} \mathrm{H}_2(G';\mathbb{Q}) & \longrightarrow & \mathrm{H}_2(\overline{G}';\mathbb{Q}) & \xrightarrow{j'} & \mathbb{Q} & \xrightarrow{i'} & \mathrm{H}_1(G';\mathbb{Q}) & \longrightarrow & \mathrm{H}_1(\overline{G}';\mathbb{Q}) & \longrightarrow & 0 \\ \downarrow & & \downarrow\phi & & \downarrow\cong & & \downarrow & & \downarrow & & \\ \mathrm{H}_2(G;\mathbb{Q}) & \longrightarrow & \mathrm{H}_2(\overline{G};\mathbb{Q}) & \xrightarrow{j} & \mathbb{Q} & \xrightarrow{i} & \mathrm{H}_1(G;\mathbb{Q}) & \longrightarrow & \mathrm{H}_1(\overline{G};\mathbb{Q}) & \longrightarrow & 0 \end{array}$$

By assumption, the map $i$ in this diagram is the 0 map, so $j$ is surjective. Since $\overline{G}'$ is a finite index subgroup of $\overline{G}$, we can make use of the transfer map in group homology (see [5, Chapter III.9]) to deduce that $\phi$ is surjective. We conclude that $j'$ is surjective, so $i' = 0$, and the lemma follows. □

## 7.2 A vanishing result

We now turn to Lemma 7.1. For the proof, we will need the following lemma. If $S$ is a surface, then define $\hat{\mathrm{H}}_1(S) = \mathrm{H}_1(S)/X$, where $X = \langle [c] \mid c \text{ a component of } \partial S \rangle$. Also, if $c$ is a simple closed curve in $S$, then denote by $\overline{[c]}$ the associated element of $\hat{\mathrm{H}}_1(S)$.

**Lemma 7.3** (Conjugacy classes of BP maps)**.** *Let $\Sigma$ be a surface with at most* 1 *boundary component and let $S \hookrightarrow \Sigma$ be a cleanly embedded subsurface. Assume that each component of $\partial S$ is a separating curve in $\Sigma$, and let $\beta$ be a component of $\partial S$. Let $T_{x_1}T_{y_1}^{-1}$ and $T_{x_2}T_{y_2}^{-1}$ be* 2 *bounding pair maps in $\tilde{\mathscr{I}}(\Sigma,S)$ that satisfy the following* 2 *conditions.*

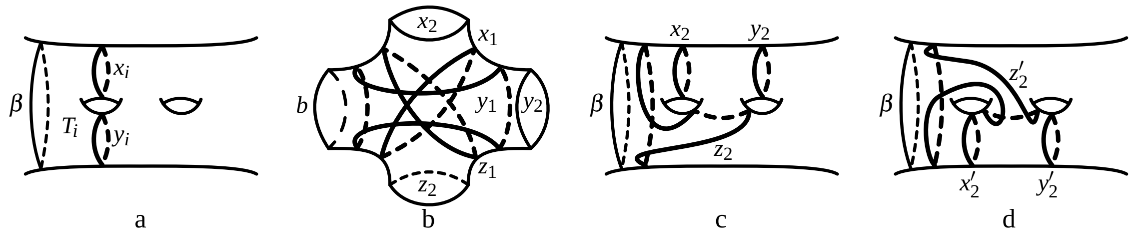



**Figure 12:** *a. Curves like in Lemma 7.3 b. A lantern relation* $T_b = (T_{x_1}T_{x_2}^{-1})(T_{y_1}T_{y_2}^{-1})(T_{z_1}T_{z_2}^{-1})$ *c,d. Two lantern relations as indicated in the proof of Lemma 7.1. We only draw the* 4 *"exterior" curves, namely* $\beta \cup x_2 \cup y_2 \cup z_2$ *and* $\beta \cup x_2' \cup y_2' \cup z_2'$.

1. *For* $1 \le i \le 2$, *the* 1*-submanifold* $\beta \cup x_i \cup y_i$ *of* $S$ *bounds a subsurface* $T_i$ *homeomorphic to* $\Sigma_{0,3}$ *(see Figure 12.a).*
2. $\overline{[x_1]} = \overline{[x_2]}$, *where the curves* $x_i$ *are oriented such that* $T_i$ *lies to the left of* $x_i$.

*Then* $T_{x_1}T_{y_1}^{-1}$ *and* $T_{x_2}T_{y_2}^{-1}$ *are conjugate elements of* $\tilde{\mathscr{I}}(\Sigma, S)$.

*Remark.* This was proven by Johnson [16, Theorem 1B] in the special case that $S = \Sigma$. Our proof is a slight elaboration of his. We remark that in the statement of [16, Theorem 1B], there is an extra condition about a certain splitting of homology – in our situation, that condition is trivial.

*Proof of Lemma 7.3.* By condition 1 and an Euler characteristic calculation, we obtain homeomorphic disconnected surfaces when we cut $S$ along $x_i \cup y_i$ for $1 \le i \le 2$. Using such a homeomorphism, we obtain an $f \in \mathrm{Mod}(S)$ such that $f(x_1) = x_2$ and $f(y_1) = y_2$. We have

$$fT_{x_1}T_{y_1}^{-1}f^{-1} = T_{f(x_1)}T_{f(y_1)}^{-1} = T_{x_2}T_{y_2}^{-1},$$

so if $f \in \tilde{\mathscr{I}}(\Sigma, S)$, then we are done. Our goal is to modify $f$ so that it lies in $\tilde{\mathscr{I}}(\Sigma, S)$.

The group $\mathrm{Mod}(S)$ acts on $\hat{\mathrm{H}}_1(S)$, and by [25, Theorem 1.1], the group $\tilde{\mathscr{I}}(\Sigma, S)$ is the kernel of this action (we remark that this is also easy to prove directly). Define $\hat{S}$ to be $S$ with discs glued to all boundary components except for $\beta$. Observe that $\hat{\mathrm{H}}_1(S) \cong \mathrm{H}_1(\hat{S})$. Let $\pi : \mathrm{Mod}(S) \to \mathrm{Mod}(\hat{S})$ be the induced map and let $\phi : \mathrm{Mod}(\hat{S}) \to \mathrm{Aut}(\mathrm{H}_1(\hat{S}))$ be the symplectic representation. Also, for $1 \le i \le 2$ let $\hat{x}_i$ and $\hat{y}_i$ be the curves in $\hat{S}$ which are the images of $x_i$ and $y_i$, respectively. Observe that by condition 2, the symplectic group element $\phi(\pi(f))$ acts as the identity on

$$[\hat{x}_1] = [\hat{x}_2] = -[\hat{y}_1] = -[\hat{y}_2].$$

Denote by $\mathrm{Mod}_{x_2,y_2}(S)$ (resp. $\mathrm{Mod}_{\hat{x}_2,\hat{y}_2}(\hat{S})$) the pointwise stabilizer in $\mathrm{Mod}(S)$ (resp. $\mathrm{Mod}(\hat{S})$) of $x_2 \cup y_2$ (resp. $\hat{x}_2 \cup \hat{y}_2$). The map $\pi$ restricts to a surjection $\mathrm{Mod}_{x_2,y_2}(S) \to \mathrm{Mod}_{\hat{x}_2,\hat{y}_2}(\hat{S})$. It is easy to see that $\phi(\mathrm{Mod}_{\hat{x}_2,\hat{y}_2}(\hat{S})) < \mathrm{Aut}(\mathrm{H}_1(\hat{S}))$ is exactly the stabilizer of $[\hat{x}_2] = -[\hat{y}_2]$. Thus there exists some $\hat{g} \in \mathrm{Mod}_{\hat{x}_2,\hat{y}_2}(\hat{S})$ such that $\phi(\hat{g}) = \phi(\pi(f))$. Letting $g \in \mathrm{Mod}_{x_2,y_2}(S)$ be any lift of $\hat{g}$, we deduce that $f' = fg^{-1}$ satisfies $f' \in \tilde{\mathscr{I}}(\Sigma, S)$. Moreover, $f'(x_1) = x_2$ and $f'(y_1) = y_2$, and the lemma follows. □

*Proof of Lemma 7.1.* Since $S$ has genus at least 3, we can find a cleanly embedded genus 2 subsurface $S'$ of $S$ all of whose boundary components separate $\Sigma$ such that $\beta \subset \partial S'$. Setting $\Gamma' =$

$\Gamma \cap \mathscr{I}(\Sigma, S')$, it is enough to show that $[T_\beta]_{\Gamma'} = 0$. Since $T_\beta$ is central in $\mathscr{I}(\Sigma, S')$ and $\Gamma'$ is a finite-index subgroup of $\mathscr{I}(\Sigma, S')$, Lemma 7.2 says that it is enough to prove that $[T_\beta]_{\mathscr{I}(\Sigma,S')} = 0$. To simplify our notation, set $G = \mathscr{I}(\Sigma, S')$.

We will use the *lantern relation*, which is the relation

$$T_b = (T_{x_1} T_{x_2}^{-1})(T_{y_1} T_{y_2}^{-1})(T_{z_1} T_{z_2}^{-1}),$$

where the curves $x_i$ and $y_i$ are as in Figure 12.b (see [7, §7g]). The key observation (see Figures 12.c–d) is that in $S'$ there are 2 lantern relations

$$T_\beta = (T_{x_1} T_{x_2}^{-1})(T_{y_1} T_{y_2}^{-1})(T_{z_1} T_{z_2}^{-1}) \quad \text{and} \quad T_\beta = (T_{x'_1} T_{x'_2}^{-1})(T_{y'_1} T_{y'_2}^{-1})(T_{z'_1} T_{z'_2}^{-1})$$

such that

$$\overline{[x_1]} = -\overline{[x'_1]} \quad \text{and} \quad \overline{[y_1]} = -\overline{[y'_1]} \quad \text{and} \quad \overline{[z_1]} = -\overline{[z'_1]};$$

here the curves $x_1$, $y_1$, etc. are oriented so that $\beta$ lies in the component of $S'$ to the left of $x_1 \cup x_2$, $y_1 \cup y_2$, etc. By Lemma 7.3 we have

$$[T_{x_1} T_{x_2}^{-1}]_G = -[T_{x'_1} T_{x'_2}^{-1}]_G \quad \text{and} \quad [T_{y_1} T_{y_2}^{-1}]_G = -[T_{y'_1} T_{y'_2}^{-1}]_G \quad \text{and} \quad [T_{z_1} T_{z_2}^{-1}]_G = -[T_{z'_1} T_{z'_2}^{-1}]_G,$$

so

$$2[T_\beta] = ([T_{x_1} T_{x_2}^{-1}]_G + [T_{y_1} T_{y_2}^{-1}]_G + [T_{z_1} T_{z_2}^{-1}]_G) + ([T_{x'_1} T_{x'_2}^{-1}]_G + [T_{y'_1} T_{y'_2}^{-1}]_G + [T_{z'_1} T_{z'_2}^{-1}]_G) = 0,$$

as desired. □

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