Abelian covers of surfaces and the homology of the level L mapping class group

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Abstract

We calculate the first homology group of the mapping class group with coefficients in the first rational homology group of the universal abelian $\mathbb{Z}/L\mathbb{Z}$-cover of the surface. If the surface has one marked point, then the answer is $\mathbb{Q}^{\tau(L)}$, where $\tau(L)$ is the number of positive divisors of L. If the surface instead has one boundary component, then the answer is $\mathbb{Q}$. We also perform the same calculation for the level L subgroup of the mapping class group. Set $H_L = \mathrm{H}_1(\Sigma_g;\mathbb{Z}/L\mathbb{Z})$. If the surface has one marked point, then the answer is $\mathbb{Q}[H_L]$, the rational group ring of H_L. If the surface instead has one boundary component, then the answer is $\mathbb{Q}$.

1 Introduction

Let $\Sigma_{g,b}^n$ be an oriented genus g surface with b boundary components and n marked points and let $\mathrm{Mod}_{g,b}^n$ be its *mapping class group*. This is the group of homotopy classes of orientation-preserving diffeomorphisms of $\Sigma_{g,b}^n$ that act as the identity on the boundary components and marked points. We will usually omit the b and the n if they vanish. The homology groups of $\mathrm{Mod}_{g,b}^n$, which play important roles in both algebraic geometry and low-dimensional topology, have been studied intensely for the past 40 years. The culmination of much recent work is the resolution of the Mumford conjecture by Madsen and Weiss [16], which identifies $\mathrm{H}^*(\mathrm{Mod}_{g,b}^n;\mathbb{Q})$ in a stable range.

Twisted coefficient systems. For many applications, it is important to know the homology groups of $\mathrm{Mod}_{g,b}^n$ with respect to various twisted coefficient systems. For simplicity, assume that $(b,n) \in \{(0,0),(1,0),(0,1)\}$. A lot is known about coefficient systems that factor through the standard *symplectic representation* of $\mathrm{Mod}_{g,b}^n$. This is the natural representation $\mathrm{Mod}_{g,b}^n \to \mathrm{Sp}_{2g}(\mathbb{Z})$ that arises from the action of $\mathrm{Mod}_{g,b}^n$ on $\mathrm{H}_1(\Sigma_{g,b}^n;\mathbb{Z})$. Its target is the symplectic group because the action preserves the algebraic intersection form. For any rational representation V of the algebraic group Sp_{2g}, Looijenga [14] has completely determined $\mathrm{H}^*(\mathrm{Mod}_g;V)$ as a module over $\mathrm{H}^*(\mathrm{Mod}_g;\mathbb{Q})$ in a stable range. Over $\mathbb{Z}$, a bit less is known. Morita [17] has calculated $\mathrm{H}_1(\mathrm{Mod}_{g,b}^n;\mathrm{H}_1(\Sigma_{g,b}^n;\mathbb{Z}))$ for $g \geq 3$. For $b \geq 1$, this was later generalized by Kawazumi [13], who calculated $\mathrm{H}^*(\mathrm{Mod}_{g,b}^n;(\mathrm{H}^1(\Sigma_{g,b}^n;\mathbb{Z}))^{\otimes k})$ as a module over $\mathrm{H}^*(\mathrm{Mod}_{g,b}^n;\mathbb{Z})$ in a stable range.

Fix some $L \geq 2$. To simplify our notation, we will denote $\mathbb{Z}/L\mathbb{Z}$ by $\mathbb{Z}_L$. In this paper, we calculate the first homology group of the mapping class group with coefficients in the first rational homology group of the universal abelian $\mathbb{Z}_L$-cover of the surface (see below for the definition). We remark that this representation does *not* factor through $\mathrm{Sp}_{2g}(\mathbb{Z})$. Our techniques also give results for certain finite-index subgroups of the mapping class group. These results play an important technical role in a recent pair of papers by the author [19, 20] that study the second cohomology group and Picard group of the moduli space of curves with level L structures.

Universal abelian $\mathbb{Z}_L$-cover. Let K_g^L be the kernel of the natural map $\pi_1(\Sigma_g) \to \mathrm{H}_1(\Sigma_g; \mathbb{Z}_L)$. The group K_g^L is the fundamental group of the universal abelian $\mathbb{Z}_L$-cover of Σ_g. Since Mod_g^1 fixes a basepoint on Σ_g, it acts on $\pi_1(\Sigma_g)$. This action preserves K_g^L. We thus obtain an action of Mod_g^1 on $\mathrm{H}_1(K_g^L; \mathbb{Q})$, the first homology group of the universal abelian $\mathbb{Z}_L$-cover of Σ_g. This representation has been previously studied by Looijenga [15], who essentially determined its image.

Remark. In [15], Looijenga more generally studied the actions of appropriate finite-index subgroups of Mod_g^1 on the first rational homology groups V of arbitrary finite abelian covers of Σ_g. Letting $\mathrm{Mod}_g^1(L)$ denote the level L subgroup of Mod_g^1 (see below), we can choose L so that $\mathrm{Mod}_g^1(L)$ acts on V. It then follows from Lemma 2.2 below that V appears a direct summand in the $\mathrm{Mod}_g^1(L)$-module $\mathrm{H}_1(K_g^L; \mathbb{Q})$, so one can use our results to study V as well.

Statements of theorems. Let $\tau(L)$ be the number of positive divisors of L (including 1 and L). Our first theorem is as follows.

Theorem A. *For $g \geq 4$ and $L \geq 2$, we have* $\mathrm{H}_1(\mathrm{Mod}_g^1; \mathrm{H}_1(K_g^L; \mathbb{Q})) \cong \mathbb{Q}^{\tau(L)}$.

In fact, our proof of Theorem A also gives a result for the *level L subgroup* of Mod_g^1, denoted $\mathrm{Mod}_g^1(L)$. This is the kernel of the action of Mod_g^1 on $\mathrm{H}_1(\Sigma_g^1; \mathbb{Z}_L)$. Far less is known about its homology. The only previous result of which the author is aware is a paper of Hain [8] that calculates $\mathrm{H}_1(\mathrm{Mod}_{g,b}^n(L); V)$ for rational representations V of the algebraic group Sp_{2g}. Our theorem is as follows.

Theorem B. *For $g \geq 4$ and $L \geq 2$, we have $\mathrm{H}_1(\mathrm{Mod}_g^1(L); \mathrm{H}_1(K_g^L; \mathbb{Q})) \cong \mathbb{Q}[H_L]$, where H_L equals $\mathrm{H}_1(\Sigma_g; \mathbb{Z}_L)$ and $\mathbb{Q}[H_L]$ is the rational group ring of the abelian group H_L.*

Remark. Both $\mathrm{H}_1(\mathrm{Mod}_g^1(L); \mathrm{H}_1(K_g^L; \mathbb{Q}))$ and $\mathbb{Q}[H_L]$ possess natural Mod_g^1-actions. The action on $\mathrm{H}_1(\mathrm{Mod}_g^1(L); \mathrm{H}_1(K_g^L; \mathbb{Q}))$ comes from conjugation, and the action on $\mathbb{Q}[H_L]$ factors through the symplectic group. The isomorphism in Theorem B is equivariant with respect to these actions.

Somewhat surprisingly, things are quite different for surfaces with boundary. Define $\mathrm{Mod}_{g,1}(L)$ to be the kernel of the action of $\mathrm{Mod}_{g,1}$ on $\mathrm{H}_1(\Sigma_{g,1}; \mathbb{Z}_L)$. Fixing a basepoint for $\pi_1(\Sigma_{g,1})$ on $\partial \Sigma_{g,1}$, the groups $\mathrm{Mod}_{g,1}$ and $\mathrm{Mod}_{g,1}(L)$ act on $\pi_1(\Sigma_{g,1})$. Define $K_{g,1}^L$ to be the kernel of the map $\pi_1(\Sigma_{g,1}) \to \mathrm{H}_1(\Sigma_{g,1}; \mathbb{Z}_L)$. The group $K_{g,1}^L$ is the fundamental group of the universal abelian $\mathbb{Z}_L$-cover of $\Sigma_{g,1}$ and is preserved by the actions of $\mathrm{Mod}_{g,1}$ and $\mathrm{Mod}_{g,1}(L)$. We then have the following theorem.

Theorem C. *For $g \geq 4$ and $L \geq 2$, we have*

$$\mathrm{H}_1(\mathrm{Mod}_{g,1}; \mathrm{H}_1(K_{g,1}^L; \mathbb{Q})) \cong \mathrm{H}_1(\mathrm{Mod}_{g,1}(L); \mathrm{H}_1(K_{g,1}^L; \mathbb{Q})) \cong \mathbb{Q}.$$

Remark. The group Mod_g does not act on the universal abelian $\mathbb{Z}_L$-cover of Σ_g. Each individual mapping class can be lifted to a diffeomorphism of the cover, but a fixed basepoint is necessary to make this lift canonical and thereby provide a representation of the entire group.

Comments on the proofs. The key observation underlying the proofs of our theorems is as follows. The group Mod_g^1 contains a natural copy of $\pi_1(\Sigma_g)$, known as the "point-pushing subgroup" (see §2.4 below). This fits into the Birman exact sequence

$$1 \longrightarrow \pi_1(\Sigma_g) \longrightarrow \mathrm{Mod}_g^1 \longrightarrow \mathrm{Mod}_g \longrightarrow 1.$$

While the action of Mod_g^1 on K_g^L is very complicated, the action of $\pi_1(\Sigma_g)$ on $K_g^L \lhd \pi_1(\Sigma_g)$ is simply conjugation. Moreover, it turns out that in some vague sense the action of Mod_g^1 on $\mathrm{H}_1(K_g^L;\mathbb{Q})$ is "concentrated" in the action of $\pi_1(\Sigma_g)$ on $\mathrm{H}_1(K_g^L;\mathbb{Q})$. Intuitively, this happens because (as noted in the remark above) the quotient of Mod_g^1 by $\pi_1(\Sigma_g)$, namely Mod_g, does *not* act in any natural way on K_g^L. We prove our theorems by carefully examining all of these groups and actions.

Some related results. Some additional related results should be mentioned. First, Ivanov [10] has proven a homological stability result for the homology of $\mathrm{Mod}_{g,1}$ with respect to very general systems of coefficients (those of "bounded degree"; the system $\mathrm{H}_1(K_{g,1}^L;\mathbb{Z})$ satisfies this condition). This generalizes Harer's [9] well-known untwisted homological stability theorem for the mapping class group. Ivanov's theorem has been extended to $\Sigma_{g,b}$ for $b > 1$ by Boldsen [5]. We remark that such a result is false for closed surfaces. Indeed, in [17, Corollary 5.4], Morita showed that

$$\mathrm{H}_1(\mathrm{Mod}_g;\mathrm{H}_1(\Sigma_g;\mathbb{Z})) \cong \mathbb{Z}/(2g-2)\mathbb{Z}$$

for $g \geq 2$. In a somewhat different direction, a recent series of papers by Anderson and Villemoes [1, 2, 3] calculate the first homology groups of $\mathrm{Mod}_{g,b}^n$ with coefficients in certain spaces of functions on representations varieties of $\mathrm{Mod}_{g,b}^n$.

Outline of paper. In §2, we discuss some background results about group cohomology and the mapping class group. Next, in §3 we introduce a number of groups and group actions that will play important roles in our paper. In §4 we prove a number of preliminary lemmas needed for our proofs. Two key results (Lemmas 4.5 and 4.6) have lengthy proofs, so we postpone proving them until the end of the paper. In §5, we prove our main theorems. Finally, in §6 – §8 we give the proofs of Lemmas 4.5 and 4.6.

Notation and conventions. To simplify our notation, we will occasionally write $\mathrm{Mod}_{g,1}(1)$ and $\mathrm{Mod}_g^1(1)$ for $\mathrm{Mod}_{g,1}$ and Mod_g^1. We will denote by $i(x,y) \in \mathbb{Z}_L$ the algebraic intersection number of $x, y \in \mathrm{H}_1(\Sigma_g;\mathbb{Z}_L)$. Finally, if G is a group, then we define $[g_1,g_2] = g_1^{-1}g_2^{-1}g_1g_2$ and $g_1^{g_2} = g_2^{-1}g_1g_2$ for $g_1, g_2 \in G$.

2 Preliminaries

2.1 Group homology

We begin by reviewing some facts about group homology and establishing some notation (see [6] for more details).

Degree zero. Let G be a group and M be a G-module. The *coinvariants* of M, denoted M_G, is the quotient M/K, where K is the submodule spanned by the set $\{g \cdot x - x \mid x \in M,\, g \in G\}$. We have $\mathrm{H}_0(G;M) = M_G$.

The five-term exact sequence. Let

$$1 \longrightarrow K \longrightarrow G \longrightarrow Q \longrightarrow 1$$

be a short exact sequence of groups and let M be a G-module. We then have a 5-term exact sequence

$$\mathrm{H}_2(G;M) \longrightarrow \mathrm{H}_2(Q;M_K) \longrightarrow (\mathrm{H}_1(K;M))_Q \longrightarrow \mathrm{H}_1(G;M) \longrightarrow \mathrm{H}_1(Q;M_K) \longrightarrow 0.$$

The long exact sequence. Let G be a group and let

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

be a short exact sequence of G-modules. Then there is a long exact sequence of the form

$$\cdots \longrightarrow \mathrm{H}_k(G;M_1) \longrightarrow \mathrm{H}_k(G;M_2) \longrightarrow \mathrm{H}_k(G;M_3) \longrightarrow \mathrm{H}_{k-1}(G;M_1) \longrightarrow \cdots.$$

The transfer map. If $G_2 < G_1$ are groups satisfying $[G_1 : G_2] < \infty$ and M is a G_1-module, then for all k there exists a *transfer map* of the form $t : \mathrm{H}_k(G_1;M) \to \mathrm{H}_k(G_2;M)$ (see, e.g., [6, Chapter III.9]). The key property of t (see [6, Proposition III.9.5]) is that if $i : \mathrm{H}_k(G_2;M) \to \mathrm{H}_k(G_1;M)$ is the map induced by the inclusion, then $i \circ t : \mathrm{H}_k(G_1;M) \to \mathrm{H}_k(G_1;M)$ is multiplication by $[G_1 : G_2]$. In particular, if M is a G_1-vector space over $\mathbb{Q}$, then we obtain a right inverse $\frac{1}{[G_1:G_2]}t$ to i. This yields the following standard lemma.

Lemma 2.1. *Let $G_2 < G_1$ be groups satisfying $[G_1 : G_2] < \infty$ and let M be a G_1-vector space over $\mathbb{Q}$. Then the map $\mathrm{H}_k(G_2;M) \to \mathrm{H}_k(G_1;M)$ is surjective for all $k \geq 0$.*

Assume now that $M = \mathbb{Q}$ and that Γ is a group acting on G_2 and G_1 such that the inclusion is Γ-equivariant. The induced map $\mathrm{H}_k(G_2;M) \to \mathrm{H}_k(G_1;M)$ is therefore Γ-equivariant. Moreover, the map $t : \mathrm{H}_k(G_1;M) \to \mathrm{H}_k(G_2;M)$ is also Γ-equivariant, so the surjection $\mathrm{H}_k(G_2;M) \to \mathrm{H}_k(G_1;M)$ splits in a Γ-equivariant manner. We obtain the following lemma.

Lemma 2.2. *Fix $k \geq 0$, and let $G_2 < G_1$ be groups satisfying $[G_1 : G_2] < \infty$. Let Γ be a group acting on G_1 and G_2 such that the inclusion map $G_2 \to G_1$ is Γ-equivariant. Define C to be the kernel of the surjection $\mathrm{H}_k(G_2;\mathbb{Q}) \to \mathrm{H}_k(G_1;\mathbb{Q})$. We then have a Γ-invariant splitting $\mathrm{H}_k(G_2;\mathbb{Q}) \cong \mathrm{H}_k(G_1;\mathbb{Q}) \oplus C$.*

Finally, for finite-index normal subgroups, the Hochschild-Serre spectral sequences implies the following strengthening of Lemma 2.1.

Lemma 2.3. *Let $G_2 \lhd G_1$ be groups satisfying $[G_1 : G_2] < \infty$ and let M be a G_1-vector space over $\mathbb{Q}$. Then $\mathrm{H}_k(G_1;M) \cong (\mathrm{H}_k(G_2;M))_{G_1}$ for all $k \geq 0$.*

2.2 Rational group rings

Let G be a finite group and let $\mathbb{Q}[G]$ be the rational group ring of G. We will consider $\mathbb{Q}[G]$ to be a left G-module. Let $\varepsilon : \mathbb{Q}[G] \to \mathbb{Q}$ be the *augmentation map*, i.e. the unique linear map such that $\varepsilon(g) = 1$ for all $g \in G$. The map ε is a map of G-modules, where $\mathbb{Q}$ has the trivial G-action. Its kernel is the *augmentation ideal* $I(G)$. We thus have a short exact sequence of G-modules

$$0 \longrightarrow I(G) \longrightarrow \mathbb{Q}[G] \xrightarrow{\varepsilon} \mathbb{Q} \longrightarrow 0. \tag{1}$$

Set $\theta = \sum_{g \in G} g \in \mathbb{Q}[G]$. The element θ is invariant under G, and the exact sequence (1) splits via the G-equivariant map $\psi : \mathbb{Q} \to \mathbb{Q}[G]$ defined by $\psi(1) = \frac{1}{|G|}\theta$. The associated projection $\phi : \mathbb{Q}[G] \to I(G)$ satisfies $\ker(\phi) = \langle \theta \rangle$. From these considerations, we obtain the following lemma.

Lemma 2.4. *Let G be a finite group. Then $\mathbb{Q}[G] \cong \mathbb{Q} \oplus I(G)$, where $I(G)$ is isomorphic to the quotient of $\mathbb{Q}[G]$ by $\langle \theta \rangle$.*

2.3 The first homology groups of subgroups of groups

The group K_g^L is a subgroup of $\pi_1(\Sigma_g)$ that contains $[\pi_1(\Sigma_g), \pi_1(\Sigma_g)]$. A similar comment applies to $K_{g,1}^L$. We now discuss some generalities about the first homology groups of subgroups of a group G that contain $[G,G]$. Throughout this section, G is a group with a fixed generating set S and $R \in \{\mathbb{Z}, \mathbb{Q}\}$.

We begin with some notation. Consider $x,y,z \in G$ (resp. $a \in [G,G]$ and $b \in G$). Recalling that g^f means $f^{-1}gf$, we will denote by $[x,y]^z_{\text{ab}}$ (resp. $[a]^b_{\text{ab}}$) the image of $[x,y]^z$ (resp. a^b) in $\mathrm{H}_1([G,G];R)$.

Lemma 2.5.

1. *$\mathrm{H}_1([G,G];R)$ is generated as an R-module by the set $T = \{[x,y]^f_{ab} \mid x,y \in S,\ f \in G\}$.*
2. *If $f_1, f_2 \in G$ have the same image in $\mathrm{H}_1(G;\mathbb{Z})$, then $[a]^{f_1}_{ab} = [a]^{f_2}_{ab}$ for all $a \in [G,G]$.*

Proof. By definition, $\mathrm{H}_1([G,G];R)$ is generated by the set $T' = \{[x,y]_{\text{ab}} \mid x,y \in G\}$. Using the commutator identities $[xy,z] = [x,z]^y[y,z]$ and $[x^{-1},y] = [y,x]^{x^{-1}}$ and $[x,y]^{-1} = [y,x]$, we can expand out any element of T' as a sum of elements of T, proving the first claim. As for the second claim, by assumption there is some $c \in [G,G]$ such that $f_1 = cf_2$. We thus have

$$[a]^{f_1}_{\text{ab}} = [a]^{cf_2}_{\text{ab}} = [c^{-1}ac]^{f_2}_{\text{ab}} = [c^{-1}]^{f_2}_{\text{ab}} + [a]^{f_2}_{\text{ab}} + [c]^{f_2}_{\text{ab}} = [a]^{f_2}_{\text{ab}}. \qquad \square$$

Denote by $\widehat{x}$ the image of $x \in G$ in $\mathrm{H}_1(G;\mathbb{Z})$. Lemma 2.5 implies that if $x,y,z \in G$ (resp. $a \in [G,G]$ and $b \in G$), then we may unambiguously define $[x,y]^{\widehat{z}}_{\text{ab}}$ (resp. $[a]^{\widehat{b}}_{\text{ab}}$) to equal $[x,y]^z_{\text{ab}}$ (resp. $[a]^b_{\text{ab}}$).

Lemma 2.6.

1. *For $a \in [G,G]$ and $x \in G$, we have $[a,x]_{ab} = [a]^{\widehat{x}}_{ab} - [a]_{ab}$.*
2. *For $x,y \in G$, we have $[x,y]_{ab} = -[y,x]_{ab}$.*
3. *For $x,y,z \in G$, we have $[xy,z]_{ab} = [x,z]^{\widehat{y}}_{ab} + [y,z]_{ab}$. As an immediate consequence, we have $[y^{-1},z]_{ab} = -[y,z]^{-\widehat{y}}_{ab}$.*

Proof. Items 1 and 2 are obvious, while item 3 follows from the identity $[xy,z] = [x,z]^y[y,z]$. $\square$

We now introduce some more notation. Let H be a subgroup of $\mathrm{H}_1(G;\mathbb{Z})$. We will denote by G_H the preimage of H under the natural map $G \to \mathrm{H}_1(G;\mathbb{Z})$ and by C_H the kernel of the map $\mathrm{H}_1(G_H;R) \to \mathrm{H}_1(H;R)$. We thus have exact sequences

$$1 \longrightarrow [G,G] \longrightarrow G_H \longrightarrow H \longrightarrow 1 \tag{2}$$

and

$$0 \longrightarrow C_H \longrightarrow \mathrm{H}_1(G_H;R) \longrightarrow \mathrm{H}_1(H;R) \longrightarrow 0.$$

Finally, if $x,y,z \in G$ (resp. $a \in G_H$ and $b \in G$), then we will denote by $\langle\langle x,y\rangle\rangle_H^z$ (resp. $\langle\langle a\rangle\rangle_H^b$) the image of $[x,y]^z$ (resp. a^b) in $\mathrm{H}_1(G_H;R)$.

Lemma 2.7. *Let H be a subgroup of $\mathrm{H}_1(G;\mathbb{Z})$. Then the following hold.*

1. *C_H is generated as an R-module by the set $\{\langle\langle x,y\rangle\rangle_H^f \mid x,y \in S,\ f \in G\}$.*
2. *If $f_1, f_2 \in G$ have the same image in $\mathrm{H}_1(G;\mathbb{Z})/H$, then $\langle\langle a\rangle\rangle_H^{f_1} = \langle\langle a\rangle\rangle_H^{f_2}$ for all $a \in G_H$.*
3. *For all $x,y \in G_H$, we have $\langle\langle x,y\rangle\rangle_H = 0$.*

Proof. Observe first that it is enough to prove the lemma for $R = \mathbb{Z}$, as the case $R = \mathbb{Q}$ easily follows.

Associated to (2) is a 5-term exact sequence in homology, the last 3 terms of which are

$$(\mathrm{H}_1([G,G];\mathbb{Z}))_H \longrightarrow \mathrm{H}_1(G_H;\mathbb{Z}) \longrightarrow \mathrm{H}_1(H;\mathbb{Z}) \longrightarrow 0.$$

We conclude that C_H is a quotient of $(\mathrm{H}_1([G,G];\mathbb{Z}))_H$, so the first conclusion follows from Lemma 2.5. For the second conclusion, since G_H is the pullback to G of H, there exists some $c \in G_H$ such that $f_1 = cf_2$. We then have

$$\langle\langle a\rangle\rangle_H^{f_1} = \langle\langle a\rangle\rangle_H^{cf_2} = \langle\langle c^{-1}ac\rangle\rangle_H^{f_2} = \langle\langle c^{-1}\rangle\rangle_H^{f_2} + \langle\langle a\rangle\rangle_H^{f_2} + \langle\langle c\rangle\rangle_H^{f_2} = \langle\langle a\rangle\rangle_H^{f_2}.$$

For the third conclusion, for $x,y \in G_H$ we have

$$\langle\langle x,y\rangle\rangle_H = \langle\langle x^{-1}y^{-1}xy\rangle\rangle_H = -\langle\langle x\rangle\rangle_H - \langle\langle y\rangle\rangle_H + \langle\langle x\rangle\rangle_H + \langle\langle y\rangle\rangle_H = 0. \qquad \square$$

Denote by $\bar{x}$ the image of $x \in G$ in $\mathrm{H}_1(G;\mathbb{Z})/H$. Lemma 2.5 implies that if $x,y,z \in G$ (resp. $a \in G_H$ and $b \in G$), then we may unambiguously define $\langle\langle x,y\rangle\rangle_H^{\bar{z}}$ (resp. $\langle\langle a\rangle\rangle_H^{\bar{b}}$) to equal $\langle\langle x,y\rangle\rangle_H^z$ (resp. $\langle\langle a\rangle\rangle_H^b$). The following set of identities then follow from Lemma 2.6.

Lemma 2.8.

1. *For $a \in G_H$ and $x \in G$, we have $\langle\langle a,x\rangle\rangle_H = \langle\langle a\rangle\rangle_H^{\bar{x}} - \langle\langle a\rangle\rangle_H$.*
2. *For $x,y \in G$, we have $\langle\langle x,y\rangle\rangle_H = -\langle\langle y,x\rangle\rangle_H$.*
3. *For $x,y,z \in G$, we have $\langle\langle xy,z\rangle\rangle_H = \langle\langle x,z\rangle\rangle_H^{\bar{y}} + \langle\langle y,z\rangle\rangle_H$. As an immediate consequence, we have $\langle\langle y^{-1},z\rangle\rangle_H = -\langle\langle y,z\rangle\rangle_H^{-\bar{y}}$.*

2.4 The mapping class group

The Birman exact sequence. For simplicity, assume that $g \geq 2$. The Birman exact sequence describes the effect on the mapping class group of deleting a marked point or gluing a disc to a boundary component. The first version, due to Birman (see [4]), is of the form

$$1 \longrightarrow \pi_1(\Sigma_{g,b}^p, *) \longrightarrow \mathrm{Mod}_{g,b}^{p+1} \longrightarrow \mathrm{Mod}_{g,b}^p \longrightarrow 1. \tag{3}$$

Here $*$ is a marked point and the map $\mathrm{Mod}_{g,b}^{p+1} \to \mathrm{Mod}_{g,b}^p$ comes from deleting $*$. For $\gamma \in \pi_1(\Sigma_{g,b}^p, *)$, the associated mapping class in the kernel of (3) "pushes" the deleted marked point around the path γ. For this reason, the kernel $\pi_1(\Sigma_{g,b}^p, *)$ is known as the "point-pushing subgroup". If $b \geq 1$, then (3) splits in the following way. Let $\Sigma_{g,b}^p \hookrightarrow \Sigma_{g,b}^{p+1}$ be an embedding such that $\overline{\Sigma_{g,b}^{p+1} \setminus \Sigma_{g,b}^p}$ is homeomorphic to $\Sigma_{0,2}^1$ and the deleted marked point lies in $\Sigma_{g,b}^{p+1} \setminus \Sigma_{g,b}^p$ (see Figure 2.a for an example). Then the splitting map $\mathrm{Mod}_{g,b}^p \hookrightarrow \mathrm{Mod}_{g,b}^{p+1}$ is induced by the map that extends a mapping class on $\Sigma_{g,b}^p$ by the identity.

The second form of the Birman exact sequence, due to Johnson [12], is of the form

$$1 \longrightarrow \pi_1(U\Sigma_{g,b}^p) \longrightarrow \mathrm{Mod}_{g,b+1}^p \longrightarrow \mathrm{Mod}_{g,b}^p \longrightarrow 1. \tag{4}$$

Here $U\Sigma_{g,b}^p$ is the unit tangent bundle of $\Sigma_{g,b}^p$ and the map $\mathrm{Mod}_{g,b+1}^p \to \mathrm{Mod}_{g,b}^p$ comes from gluing a disc to a boundary component β of $\Sigma_{g,b+1}^p$ and extending mapping classes by the identity. The fiber of the kernel $\pi_1(U\Sigma_{g,b}^p)$ corresponds to the mapping class T_β. Again, if $b \geq 1$, then (4) splits.

The level L subgroup. Now assume that $b = p = 0$ and fix some $L \geq 2$. The kernels of (3) and (4) both lie in the level L subgroup of the mapping class group. We thus have short exact sequences

$$1 \longrightarrow \pi_1(\Sigma_g) \longrightarrow \mathrm{Mod}_g^1(L) \longrightarrow \mathrm{Mod}_g(L) \longrightarrow 1$$

and

$$1 \longrightarrow \pi_1(U\Sigma_g) \longrightarrow \mathrm{Mod}_{g,1}(L) \longrightarrow \mathrm{Mod}_g(L) \longrightarrow 1.$$

We will also refer to these as Birman exact sequences.

We will also need two cohomological results about the level L subgroups of the mapping class group, both of which are due to Hain.

Theorem 2.9 (Hain, [8])**.** *For $g \geq 3$ and $L \geq 1$, we have* $\mathrm{H}_1(\mathrm{Mod}_{g,1}(L);\mathbb{Q}) = 0$.

Theorem 2.10 (Hain, [8])**.** *For $g \geq 3$ and $L \geq 1$, we have*

$$\mathrm{H}_1(\mathrm{Mod}_{g,1}(L);\mathrm{H}_1(\Sigma_g;\mathbb{Q})) \cong \mathrm{H}_1(\mathrm{Mod}_g^1(L);\mathrm{H}_1(\Sigma_g;\mathbb{Q})) \cong \mathbb{Q}.$$

Remark. In fact, Hain calculated $\mathrm{H}_1(\mathrm{Mod}_{g,b}^p(L);M)$ for all rational representations M of the algebraic group Sp_{2g}.

3 The cast of characters

We now discuss several objects that will be used throughout this paper. Fix $g \geq 1$ and $L \geq 1$.

Fundamental groups of abelian covers. As in the introduction, define

$$K_g^L = \ker(\pi_1(\Sigma_g) \to \mathrm{H}_1(\Sigma_g;\mathbb{Z}_L)) \quad \text{and} \quad K_{g,1}^L = \ker(\pi_1(\Sigma_{g,1}) \to \mathrm{H}_1(\Sigma_{g,1};\mathbb{Z}_L)).$$

Here the basepoint for $\pi_1(\Sigma_{g,1})$ lies on $\partial\Sigma_{g,1}$. We have actions of Mod_g^1 and $\mathrm{Mod}_{g,1}$ on K_g^L and $K_{g,1}^L$, respectively. Next, define

$$C_g^L = \ker(\mathrm{H}_1(K_g^L;\mathbb{Q}) \to \mathrm{H}_1(\Sigma_g;\mathbb{Q})) \quad \text{and} \quad C_{g,1}^L = \ker(\mathrm{H}_1(K_{g,1}^L;\mathbb{Q}) \to \mathrm{H}_1(\Sigma_{g,1};\mathbb{Q})).$$

By Lemma 2.2, we have mapping class group invariant decompositions

$$\mathrm{H}_1(K_g^L;\mathbb{Q}) \cong \mathrm{H}_1(\Sigma_g;\mathbb{Q}) \oplus C_{g,1}^L \quad \text{and} \quad \mathrm{H}_1(K_{g,1}^L;\mathbb{Q}) \cong \mathrm{H}_1(\Sigma_{g,1};\mathbb{Q}) \oplus C_{g,1}^L.$$

We now observe that we are in the situation discussed in §2.3. Indeed, in the notation of that section, if $G = \pi_1(\Sigma_g)$ and $H = L \cdot \mathrm{H}_1(\Sigma_g;\mathbb{Z})$, then $G_H = K_g^L$ and $C_H = C_g^L$. Similarly, if $G = \pi_1(\Sigma_{g,1})$ and $H = L \cdot \mathrm{H}_1(\Sigma_{g,1};\mathbb{Z})$, then $G_H = K_{g,1}^L$ and $C_H = C_{g,1}^L$. In both of these cases, we will throughout this paper denote $\langle\!\langle \cdot,\cdot \rangle\!\rangle_H$ and $\langle\!\langle \cdot \rangle\!\rangle_H$ by $\langle\!\langle \cdot,\cdot \rangle\!\rangle$ and $\langle\!\langle \cdot \rangle\!\rangle$, respectively. For instance, if $x, y \in \pi_1(\Sigma_g)$ and $z \in \mathrm{H}_1(\Sigma_g;\mathbb{Z}_L)$, then we will denote by $\langle\!\langle x,y \rangle\!\rangle^z$ the element of C_g^L associated to $[x,y]^{\tilde{z}} \in K_g^L$, where $\tilde{z} \in \pi_1(\Sigma_g)$ is any lift of $z \in \mathrm{H}_1(\Sigma_g;\mathbb{Z}_L)$. The domains of the arguments of our expressions will determine whether we are discussing K_g^L or $K_{g,1}^L$. Similarly, if $z \in \pi_1(\Sigma_g)$, then $\widehat{z}$ and $\bar{z}$ will denote the associated elements of $\mathrm{H}_1(\Sigma_g;\mathbb{Z})$ and $\mathrm{H}_1(\Sigma_g;\mathbb{Z}_L)$, respectively, and similarly for $z \in \pi_1(\Sigma_{g,1})$. For $R \in \{\mathbb{Q},\mathbb{Z},\mathbb{Z}_L\}$, we will often identify the naturally isomorphic groups $\mathrm{H}_1(\Sigma_g;R)$ and $\mathrm{H}_1(\Sigma_{g,1};R)$.

Filling in the boundary. The surjective map $\pi_1(\Sigma_{g,1}) \to \pi_1(\Sigma_g)$ induced by gluing a disc to the boundary component of $\Sigma_{g,1}$ restricts to a surjection $K_{g,1}^L \to K_g^L$. This induces a surjective map $C_{g,1}^L \to C_g^L$. Let I_g^L be its kernel, so we have a short exact sequence

$$0 \longrightarrow I_g^L \longrightarrow C_{g,1}^L \longrightarrow C_g^L \longrightarrow 0.$$

Let $\gamma \in \pi_1(\Sigma_{g,1})$ be the curve that goes once around $\partial\Sigma_{g,1}$ with the interior of the surface to its left. The kernel of the map $\pi_1(\Sigma_{g,1}) \to \pi_1(\Sigma_g)$ is normally generated by $\gamma \in [\pi_1(\Sigma_{g,1}),\pi_1(\Sigma_{g,1})]$, so I_g^L is generated by the set

$$\{\langle\!\langle \gamma \rangle\!\rangle^z \mid z \in \mathrm{H}_1(\Sigma_{g,1};\mathbb{Z}_L)\}.$$

For $z \in \mathrm{H}_1(\Sigma_{g,1};\mathbb{Z}_L)$, we will throughout this paper denote $\langle\!\langle \gamma \rangle\!\rangle^z$ by $[\![z]\!]$. We now prove the following.

Lemma 3.1. *For $g \geq 1$ and $L \geq 2$, the vector space I_g^L is isomorphic as a $\mathrm{Mod}_{g,1}$-module to the augmentation ideal of $\mathbb{Q}[H_L]$, where $H_L = \mathrm{H}_1(\Sigma_{g,1};\mathbb{Z}_L)$.*

Proof. Let $* \in \partial\Sigma_{g,1}$ be the basepoint and let $\pi : (\tilde{\Sigma}, \tilde{*}) \to (\Sigma_{g,1}, *)$ be the based cover corresponding to $K_{g,1}^L$. Thus $K_{g,1} = \pi_1(\tilde{\Sigma}, \tilde{*})$ and $\mathrm{H}_1(K_{g,1};\mathbb{Q}) = \mathrm{H}_1(\tilde{\Sigma};\mathbb{Q})$. The subgroup $I_g^L < \mathrm{H}_1(\tilde{\Sigma};\mathbb{Q})$ is exactly the subgroup generated by the homology classes of the boundary components of $\tilde{\Sigma}$. The group of deck transformations H_L acts on these boundary components. Since γ lifts to a simple closed curve in $\tilde{\Sigma}$, this action is free. Let $\tilde{\gamma}$ be the lift of γ starting at $\tilde{*}$. For $v \in H_L$, the boundary component corresponding to $[\![v]\!] \in I_g^L$ is $v(\tilde{\gamma})$. The only relation between the homology classes of the boundary components of a surface with boundary is that their sum is 0. We conclude that I_g^L is isomorphic to the quotient of the $\mathbb{Q}$-vector space with basis the formal symbols $\{[\![v]\!] \mid v \in H_L\}$ by the 1-dimensional subspace generated by $\sum_{v \in H_L} [\![v]\!]$. This is exactly the augmentation ideal of $\mathbb{Q}[H_L]$, and we are done. □

Remark. An immediate corollary of Lemma 3.1 is that $\mathrm{Mod}_{g,1}(L)$ acts trivially on I_g^L.

Filling in the boundary II. Let β be the boundary component of $\Sigma_{g,1}$. We then have a short exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Mod}_{g,1} \longrightarrow \mathrm{Mod}_g^1 \longrightarrow 1,$$

where $\mathbb{Z} = \langle T_\beta \rangle$. Recalling that $\gamma \in \pi_1(\Sigma_{g,1})$ is the loop around the boundary component, the mapping class T_β acts on $\pi_1(\Sigma_{g,1})$ as conjugation by γ. Since $\gamma \in K_{g,1}^L$, the mapping class T_β acts trivially on $\mathrm{H}_1(K_{g,1}^L;\mathbb{Q})$ and $C_{g,1}^L$. We thus get an induced action of Mod_g^1 on $\mathrm{H}_1(K_{g,1}^L;\mathbb{Q})$ and $C_{g,1}^L$.

Embedding into the mapping class group. The kernel of the Birman exact sequence for Mod_g^1 (the "point-pushing subgroup") is isomorphic to $\pi_1(\Sigma_g)$. We will frequently identify $K_g^L < \pi_1(\Sigma_g)$ with the corresponding subgroup of Mod_g^1. The action of K_g^L (as a subgroup of Mod_g^1) on K_g^L is simply the conjugation action. By the previous paragraph, we also have a natural action of K_g^L on $\mathrm{H}_1(K_{g,1}^L;\mathbb{Q})$ and $C_{g,1}^L$. While the action of K_g^L on $\mathrm{H}_1(K_g^L;\mathbb{Q})$ and C_g^L is trivial, our calculations below will show that the action of K_g^L on $\mathrm{H}_1(K_{g,1}^L;\mathbb{Q})$ and $C_{g,1}^L$ is non-trivial.

The kernel of the Birman exact sequence for $\mathrm{Mod}_{g,1}$ is isomorphic to $\pi_1(U\Sigma_g)$. The fiber of $\pi_1(U\Sigma_g)$ corresponds to the mapping class T_β, where β is the boundary component of $\Sigma_{g,1}$. Define $\overline{K}_g^L < \pi_1(U\Sigma_g)$ to be the pullback of $K_g^L < \pi_1(\Sigma_g)$ under the projection $\pi_1(U\Sigma_g) \to \pi_1(\Sigma_g)$. We will identify $\overline{K}_g^L$ with the corresponding subgroup of $\mathrm{Mod}_{g,1}$. The group $\overline{K}_g^L$ thus acts on $K_{g,1}^L$ and $C_{g,1}^L$. This action factors through the projection $\overline{K}_g^L \to K_g^L$, whose kernel is generated by T_β.

4 Some preliminary lemmas

In this section, we prove a number of preliminary lemmas needed in the proofs of our main theorems.

4.1 Two key technical lemmas to be proven later

We start by stating two key technical lemmas whose lengthy proofs are postponed until later. We preface these lemmas with three sets of definitions.

Definition 4.1. Let $(\Sigma, *)$ be a based compact surface (possibly with boundary) and let $x \in \pi_1(\Sigma, *)$. To simplify many of our statements, we will say that x is a *simple closed curve* if x can be realized by a simple closed curve. We will say that x is a *nonseparating simple closed curve* or a *separating simple closed curve* if x can be realized by a nonseparating or separating simple closed curve, respectively. We will say that x is a *simple closed genus h separating curve* if x can be realized by a simple closed curve that cuts Σ into two components X_1 and X_2 such that $\partial \Sigma \subset X_1$ and $X_2 \cong \Sigma_{h,1}$.

Remark. The condition $\partial \Sigma \subset X_1$ is redundant except when $* \in \partial \Sigma$.

Definition 4.2. Let $* \in \Sigma_g$ be a marked point. We will say that a set $\{x_1, \ldots, x_k\} \subset \pi_1(\Sigma_g, *)$ of curves *has the same unoriented intersection pattern* as a set $\{x_1', \ldots, x_k'\} \subset \pi_1(\Sigma_g, *)$ of curves if there is some $f \in \mathrm{Mod}_g^1$ such that $f(x_i) = x_i'^{\pm 1}$ for $1 \leq i \leq k$. If f can be chosen such that $f(x_i) = x_i'$ for all $1 \leq i \leq k$, then we will simply say that the curves *have the same intersection pattern*.

Remark. We will frequently assert without proof that two sets of curves have the same (possibly oriented) intersection pattern. In all these cases, the assertion can be proved via the "change of coordinates" principle from [7, §1.3].

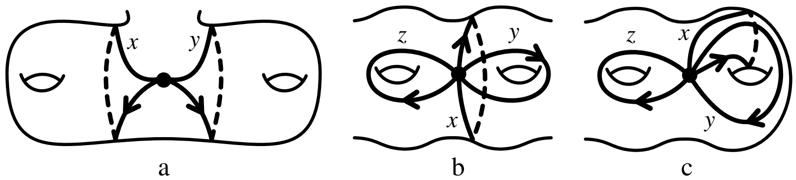

Figure 1: *a-c. Examples of curves used in $\mathscr{S}_g^L(1)$-$\mathscr{S}_g^L(3)$*

Definition 4.3. Let $* \in \Sigma_g$ be a marked point. We will say that two curves $x, y \in \pi_1(\Sigma_g, *)$ are *essentially separate* if there are subsurfaces X and X' of Σ_g satisfying the following two properties.

- $X \cup X' = \Sigma_g$ and $X \cap X'$ is a simple closed curve containing the basepoint.
- $x \in \text{Image}(\pi_1(X) \to \pi_1(\Sigma_g))$ and $y \in \text{Image}(\pi_1(X') \to \pi_1(\Sigma_g))$.

Remark. Observe that a simple closed separating curve is essentially separate from itself.

Our lemmas concern the short exact sequence

$$0 \longrightarrow I_g^L \longrightarrow C_{g,1}^L \longrightarrow C_g^L \longrightarrow 0 \tag{5}$$

of $\text{Mod}_g^1(L)$ modules. Restrict the action to $K_g^L < \text{Mod}_g^1(L)$. Since K_g^L acts trivially on I_g^L, we have $\text{H}_0(K_g^L; I_g^L) \cong I_g^L$. The long exact sequence in K_g^L-homology associated to (5) thus contains the segment

$$\cdots \longrightarrow \text{H}_1(K_g^L; C_{g,1}^L) \xrightarrow{i} \text{H}_1(K_g^L; C_g^L) \xrightarrow{\partial} I_g^L \longrightarrow \cdots.$$

Since $C_g^L < \text{H}_1(K_g^L; \mathbb{Q})$, the action of K_g^L on C_g^L is trivial, so $\text{H}_1(K_g^L; C_g^L) \cong \text{H}_1(K_g^L; \mathbb{Z}) \otimes C_g^L$. We make the following definition.

Definition 4.4. Define $\mathscr{S}_g^L \subset \text{H}_1(K_g^L; C_g^L)$ to equal $\mathscr{S}_g^L(1) \cup \cdots \cup \mathscr{S}_g^L(4)$, where the $\mathscr{S}_g^L(i)$ are as follows. To simplify our notation, we will denote $\pi_1(\Sigma_g)$ by π.

$$\begin{aligned}
\mathscr{S}_g^L(1) = \{ \langle\!\langle x \rangle\!\rangle^{\overline{f}_1} \otimes \langle\!\langle y \rangle\!\rangle^{\overline{f}_2} \mid\ & f_1, f_2 \in \pi,\ x \in K_g^L,\ y \in [\pi, \pi], \text{ and } x \text{ and } y \text{ are} \\
& \text{essentially separate (see Figure 1.a)} \}, \\
\mathscr{S}_g^L(2) = \{ \langle\!\langle x \rangle\!\rangle^{\overline{f}_1} \otimes \langle\!\langle y^L, z \rangle\!\rangle^{\overline{f}_2} \mid\ & x, y, z, f_1, f_2 \in \pi \text{ and } \{x, y, z\} \text{ has the same unoriented} \\
& \text{intersection pattern as the curves in Figure 1.b.} \}, \\
\mathscr{S}_g^L(3) = \{ \langle\!\langle x^L \rangle\!\rangle^{\overline{f}_1} \otimes \langle\!\langle y, z^L \rangle\!\rangle^{\overline{f}_2} \mid\ & x, y, z, f_1, f_2 \in \pi \text{ and } \{x, y, z\} \text{ has the same unoriented} \\
& \text{intersection pattern as the curves in Figure 1.c.} \}, \\
\mathscr{S}_g^L(4) = \{ \langle\!\langle x \rangle\!\rangle \otimes \langle\!\langle y \rangle\!\rangle - \langle\!\langle \phi(x) \rangle\!\rangle & \otimes \langle\!\langle \phi(y) \rangle\!\rangle \mid x \in K_g^L,\ y \in [\pi, \pi],\ \phi \in \text{Mod}_g^1(L) \}.
\end{aligned}$$

Remark. Since $\text{Mod}_g^1(L)$ contains all inner automorphisms of $\pi_1(\Sigma_g)$, the set $\mathscr{S}_g^L(4)$ contains the subset

$$\{ \langle\!\langle x \rangle\!\rangle \otimes \langle\!\langle y \rangle\!\rangle - \langle\!\langle x \rangle\!\rangle^{\overline{f}} \otimes \langle\!\langle y \rangle\!\rangle^{\overline{f}} \mid x \in K_g^L,\ y \in [\pi_1(\Sigma_g), \pi_1(\Sigma_g)],\ f \in \pi_1(\Sigma_g) \}.$$

We will frequently make use of this subset in the following way. For $x, y, f \in \pi_1(\Sigma_g)$, the elements $\langle\!\langle x \rangle\!\rangle^{\overline{f}} \otimes \langle\!\langle y \rangle\!\rangle$ and $\langle\!\langle x \rangle\!\rangle \otimes \langle\!\langle y \rangle\!\rangle^{-\overline{f}}$ of $\text{H}_1(K_g^L; C_g^L)$ are equal modulo $\mathscr{S}_g^L(4)$. We will say that they differ by an *exponent trade*.

Our first lemma is as follows. It implies that the image of i contains $\mathscr{S}_g^L$. Its proof is in §6.

Lemma 4.5. *For $g \geq 4$, we have $\partial(\mathscr{S}_g^L) = 0$.*

Our second lemma is as follows. It shows that the image of i is spanned by $\mathscr{S}_g^L$. Its proof is in §7.

Lemma 4.6. *For $g \geq 4$, the map $\mathrm{H}_1(K_g^L; C_g^L)/\langle \mathscr{S}_g^L \rangle \to I_g^L$ induced by ∂ is an isomorphism.*

4.2 Two calculations

We now perform two necessary calculations.

Lemma 4.7. *For $g \geq 4$ and $L \geq 2$, we have $(C_{g,1}^L)_{\pi_1(\Sigma_g)} = (C_g^L)_{\pi_1(\Sigma_g)} = 0$.*

Proof. Recall that if M is a G-module, then $\mathrm{H}_0(G;M) = G_M$. The group K_g^L acts trivially on I_g^L and C_g^L, so $(I_g^L)_{K_g^L} = I_g^L$ and $(C_g^L)_{K_g^L} = C_g^L$. The long exact sequence in K_g^L-homology associated to (5) thus contains the segment

$$\mathrm{H}_1(K_g^L; C_g^L) \xrightarrow{\partial} I_g^L \longrightarrow (C_{g,1}^L)_{K_g^L} \longrightarrow C_g^L \longrightarrow 0.$$

By Lemma 4.6, the map ∂ is a surjection. We conclude that $(C_{g,1}^L)_{K_g^L} \cong C_g^L$. This implies that $(C_{g,1}^L)_{\pi_1(\Sigma_g)} = (C_g^L)_{\pi_1(\Sigma_g)}$, so it is enough to prove that $(C_g^L)_{\pi_1(\Sigma_g)} = 0$.

Consider $x, y \in \pi_1(\Sigma_g)$. By the first conclusion of Lemma 2.7, it is enough to show that the image of $\langle\!\langle x, y \rangle\!\rangle$ in $(C_g^L)_{\pi_1(\Sigma_g)}$ is 0. Clearly $x^L, y^L \in K_g^L$, so by the third conclusion of Lemma 2.7 we have $\langle\!\langle x^L, y^L \rangle\!\rangle = 0$. Using Lemma 2.8, we have

$$0 = \frac{1}{L^2} \langle\!\langle x^L, y^L \rangle\!\rangle = \frac{1}{L^2} \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} \langle\!\langle x, y \rangle\!\rangle^{i \cdot \bar{x} + j \cdot \bar{y}}.$$

Modulo the action of $\pi_1(\Sigma_g)$ on C_g^L, this equals

$$\frac{1}{L^2} \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} \langle\!\langle x, y \rangle\!\rangle = \langle\!\langle x, y \rangle\!\rangle,$$

as desired. □

Lemma 4.8. *For $g \geq 4$ and $L \geq 2$, the natural map $\mathrm{H}_1(\mathrm{Mod}_{g,1}(L); C_g^L) \to \mathrm{H}_1(\mathrm{Mod}_g^1(L); C_g^L)$ is an isomorphism.*

Proof. Let β be the boundary component of $\Sigma_{g,1}$. We have a short exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Mod}_{g,1}(L) \longrightarrow \mathrm{Mod}_g^1(L) \longrightarrow 1,$$

where $\mathbb{Z} = \langle T_\beta \rangle$. The last 3 terms of the associated 5-term exact sequence with coefficients C_g^L are

$$(C_g^L)_{\mathrm{Mod}_g^1(L)} \longrightarrow \mathrm{H}_1(\mathrm{Mod}_{g,1}(L); C_g^L) \longrightarrow \mathrm{H}_1(\mathrm{Mod}_g^1(L); C_g^L) \longrightarrow 0.$$

By Lemma 4.7, we have $(C_g^L)_{\mathrm{Mod}_g^1(L)} = 0$, and the lemma follows. □

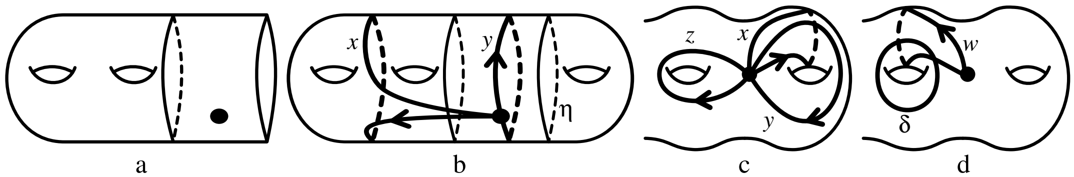

Figure 2: *a. A surface* $\Sigma^1_{g-1,1}$*. The Birman exact sequence for* $\mathrm{Mod}^1_{g-1,1}$ *splits via the indicated inclusion* $\mathrm{Mod}_{g,1,1} \hookrightarrow \mathrm{Mod}^1_{g-1,1}$*. b. Mapping* $\mathrm{Mod}^1_{g-1,1}$ *into* Mod_g*. c.* $\langle\!\langle x^L \rangle\!\rangle^{\overline{f}_1} \otimes \langle\!\langle y, z^L \rangle\!\rangle^{\overline{f}_2} \in \mathscr{S}^L_g(3)$ *d.* $T^L_\delta(w) = wz^L$

4.3 Eliminating $\mathscr{S}^L_g$

In this section, we show that the image of $\mathscr{S}^L_g \subset \mathrm{H}_1(K^L_g; C^L_g)$ in $\mathrm{H}_1(\mathrm{Mod}^1_g(L); C^L_g)$ is 0. We will need the following lemma.

Lemma 4.9 ([18, Lemma A.1])**.** *Let* $(\Sigma, *)$ *be a based compact surface of genus at least* 1 *(possibly with boundary). Then* $[\pi_1(\Sigma, *), \pi_1(\Sigma, *)]$ *is generated by the set*

$$\{x^y \mid x, y \in \pi_1(\Sigma, *) \text{ and } x \text{ is a simple closed genus } 1 \text{ separating curve}\}$$

Remark. The original statement of [18, Lemma A.1] requires that $* \in \mathrm{Int}(\Sigma)$ and the original conclusion does not require the conjugating elements y. The only use in the proof of the fact that $* \in \mathrm{Int}(\Sigma)$ is the fact that in this case any conjugate of a simple closed curve is simple – one can easily check that the proof gives the indicated conclusion in the general case.

Lemma 4.10. *The image of* $\mathscr{S}^L_g \subset \mathrm{H}_1(K^L_g; C^L_g)$ *in* $\mathrm{H}_1(\mathrm{Mod}^1_g(L); C^L_g)$ *is* 0*.*

Proof. Let $j : \mathrm{H}_1(K^L_g; C^L_g) \to \mathrm{H}_1(\mathrm{Mod}^1_g(L); C^L_g)$ be the inclusion map. Our goal is to show that $j(\mathscr{S}^L_g(i)) = 0$ for $1 \le i \le 4$. There are three steps.

Step 1. $j(\mathscr{S}^L_g(4)) = 0$

This follows from the fact that inner automorphisms act trivially on homology.

Step 2. $j(\mathscr{S}^L_g(1)) = 0$

Consider $\langle\!\langle x \rangle\!\rangle^{\overline{f}_1} \otimes \langle\!\langle y \rangle\!\rangle^{\overline{f}_2} \in \mathscr{S}^L_g(1)$. Since x and y are essentially separate, there exist subsurfaces X and X' of Σ_g with the following two properties.

- $X \cup X' = \Sigma_g$ and $X \cap X'$ is a simple closed curve containing the basepoint.
- $x \in \mathrm{Image}(\pi_1(X) \to \pi_1(\Sigma_g))$ and $y \in \mathrm{Image}(\pi_1(X') \to \pi_1(\Sigma_g))$.

Lemma 4.9 says that $[\pi_1(X'), \pi_1(X')]$ is generated by the conjugates of simple closed genus 1 separating curves. Using this fact, we can assume that y is the image of a genus 1 separating curve in $\pi_1(X')$. Also, using the fact that $j(\mathscr{S}^L_g(4)) = 0$, we can perform an exponent trade and assume that $f_1 = 1$.

Let η be a simple closed curve on Σ_g that is freely homotopic to y and disjoint from realizations of x and y (see Figure 2.b; the curve η will lie on $X' \subset \Sigma_g$). Let X'' be the component of Σ_g cut along η that contains the basepoint. From the conditions on η, we have $x, y \in \mathrm{Image}(\pi_1(X'') \to \pi_1(\Sigma_g))$. We can assume that $X'' \cong \Sigma_{g-1,1}$; the only problematic case is when $x = y$, and in this case we merely have to choose η so that $x \cup y$ is *not* contained in the component of Σ_g cut along η that is homeomorphic to $\Sigma_{1,1}$.

The Birman exact sequence says that there is a split exact sequence

$$1 \longrightarrow \pi_1(\Sigma_{g-1,1}, *) \longrightarrow \text{Mod}_{g-1,1}^1 \longrightarrow \text{Mod}_{g-1,1} \longrightarrow 1,$$

where $*$ is the marked point. Choosing a splitting of this exact sequence as in Figure 2.a, we obtain a subgroup $\text{Mod}_{g-1,1} < \text{Mod}_{g-1,1}^1$ such that $\text{Mod}_{g-1,1}^1$ is the semidirect product of $\text{Mod}_{g-1,1}$ and the "point-pushing subgroup" $\pi_1(\Sigma_{g-1,1}, *)$. The surface $\Sigma_{g-1,1}^1$ can be deformation retracted to the splitting surface, and after this deformation retraction the marked point ends up on $\partial\Sigma_{g-1,1}$. We deduce that the action of $\text{Mod}_{g-1,1}$ on $\pi_1(\Sigma_{g-1,1}, *)$ arising from our semidirect product decomposition of $\text{Mod}_{g-1,1}^1$ can be identified with the action of $\text{Mod}_{g-1,1}$ on $\pi_1(\Sigma_{g-1,1})$, where the basepoint for $\pi_1(\Sigma_{g-1,1})$ lies on $\partial\Sigma_{g-1,1}$. The subgroup $K_{g-1,1}^L < \pi_1(\Sigma_{g-1,1})$ is preserved by the action of $\text{Mod}_{g-1,1}$. Define $\Gamma < \text{Mod}_{g-1,1}^1$ to be the semidirect product of $\text{Mod}_{g-1,1}(L) < \text{Mod}_{g-1,1}$ and $K_{g-1,1}^L$.

As shown in Figures 2.a–b, there is an injection $\phi : \text{Mod}_{g-1,1}^1 \hookrightarrow \text{Mod}_g^1$ with the following properties.

- $\phi(\Gamma) \subset \text{Mod}_g^1(L)$.
- The image under ϕ of the subgroup $\text{Mod}_{g-1,1}(L) < \Gamma$ fixes y.

By the definition of $K_{g-1,1}^L$, the image under ϕ of the subgroup $K_{g-1,1}^L < \Gamma$ fixes $\langle\langle y \rangle\rangle^{\overline{f}_2}$. We deduce that $\phi(\Gamma)$ fixes $\langle\langle y \rangle\rangle^{\overline{f}_2}$. Letting $\mathbb{Q} \cdot \langle\langle y \rangle\rangle^{\overline{f}_2}$ be the 1-dimensional trivial Γ-module over $\mathbb{Q}$, we obtain a natural map $\psi : \text{H}_1(\Gamma; \mathbb{Q} \cdot \langle\langle y \rangle\rangle^{\overline{f}_2}) \to \text{H}_1(\text{Mod}_g^1(L); C_g^L)$. By the definition of $K_{g-1,1}^L$, the subgroup $\phi(\Gamma) < \text{Mod}_g^1(L)$ contains x (considered as an element of the "point-pushing" group of Mod_g^1). The upshot of this is that $j(\langle\langle x \rangle\rangle^{\overline{f}_1} \otimes \langle\langle y \rangle\rangle^{\overline{f}_2})$ is contained in the image of ψ. It is therefore enough to show that $\text{H}_1(\Gamma; \mathbb{Q}) = 0$.

Since Γ is the semidirect product of $\text{Mod}_{g-1,1}(L)$ and $K_{g-1,1}^L$, we have

$$\text{H}_1(\Gamma; \mathbb{Q}) \cong \text{H}_1(\text{Mod}_{g-1,1}(L); \mathbb{Q}) \oplus (\text{H}_1(K_{g-1,1}^L; \mathbb{Q}))_{\text{Mod}_{g-1,1}(L)}.$$

Theorem 2.9 says that $\text{H}_1(\text{Mod}_{g-1,1}(L); \mathbb{Q}) = 0$. Also, using Lemmas 2.2 and 4.7, we have

$$(\text{H}_1(K_{g-1,1}^L; \mathbb{Q}))_{\text{Mod}_{g-1,1}(L)} \cong (C_{g-1,1}^L)_{\text{Mod}_{g-1,1}(L)} \oplus (\text{H}_1(\Sigma_g; \mathbb{Q}))_{\text{Mod}_{g-1,1}(L)} = 0.$$

We conclude that $\text{H}_1(\Gamma; \mathbb{Q}) = 0$, as desired.

Step 3. $j(\mathscr{S}_g^L(2)) = 0$ *and* $j(\mathscr{S}_g^L(3)) = 0$

The proofs in these two cases are similar. We will do the case of $\mathscr{S}_g^L(3)$ and leave the case of $\mathscr{S}_g^L(2)$ to the reader. Consider $\langle\langle x^L \rangle\rangle^{\overline{f}_1} \otimes \langle\langle y, z^L \rangle\rangle^{\overline{f}_2} \in \mathscr{S}_g^L(3)$. We will assume that the intersection pattern of $\{x, y, z\}$ is the same as the intersection pattern of the curves in Figure 2.c (including orientations); the other cases are handled similarly. There is then some $w \in \pi_1(\Sigma_g)$ together with a simple closed curve δ such that δ is disjoint from simple closed curve realizations of x and y and such that $T_\delta^L(w) = wz^L$ (see Figure 2.d). Since $T_\delta^L \in \text{Mod}_g^1(L)$, it follows that T_δ^L acts trivially on $\overline{f}_1$ and $\overline{f}_2$. Thus T_δ^L takes $\langle\!\langle x^L \rangle\!\rangle^{\overline{f}_1} \otimes \langle\!\langle y, w \rangle\!\rangle^{\overline{f}_2}$ to

$$\begin{aligned}\langle\!\langle x^L \rangle\!\rangle^{\overline{f}_1} \otimes \langle\!\langle y, wz^L \rangle\!\rangle^{\overline{f}_2} &= \langle\!\langle x^L \rangle\!\rangle^{\overline{f}_1} \otimes (\langle\!\langle y, w \rangle\!\rangle^{L\cdot\overline{z}+\overline{f}_2} + \langle\!\langle y, z^L \rangle\!\rangle^{\overline{f}_2}) \\ &= \langle\!\langle x^L \rangle\!\rangle^{\overline{f}_1} \otimes \langle\!\langle y, w \rangle\!\rangle^{\overline{f}_2} + \langle\!\langle x^L \rangle\!\rangle^{\overline{f}_1} \otimes \langle\!\langle y, z^L \rangle\!\rangle^{\overline{f}_2}.\end{aligned}$$

The first calculation here uses Lemma 2.8 and the second the fact that $L \cdot \overline{z} = 0$. Since $\text{Mod}_g^1(L)$ acts trivially on the image of j, we conclude that

$$j(\langle\!\langle x^L \rangle\!\rangle^{\overline{f}_1} \otimes \langle\!\langle y, w \rangle\!\rangle^{\overline{f}_2}) = j(\langle\!\langle x^L \rangle\!\rangle^{\overline{f}_1} \otimes \langle\!\langle y, w \rangle\!\rangle^{\overline{f}_2} + \langle\!\langle x^L \rangle\!\rangle^{\overline{f}_1} \otimes \langle\!\langle y, z^L \rangle\!\rangle^{\overline{f}_2});$$

i.e. that $j(\langle\!\langle x^L \rangle\!\rangle^{\overline{f}_1} \otimes \langle\!\langle y, z^L \rangle\!\rangle^{\overline{f}_2}) = 0$, as desired. □

5 Proofs of the main theorems

We now turn to the proofs of our main theorems.

5.1 Surfaces with boundary

We begin with Theorem C, which asserts that if $g \geq 4$ and $L \geq 2$, then

$$\text{H}_1(\text{Mod}_{g,1}; \text{H}_1(K_{g,1}^L; \mathbb{Q})) \cong \text{H}_1(\text{Mod}_{g,1}(L); \text{H}_1(K_{g,1}^L; \mathbb{Q})) \cong \mathbb{Q}.$$

As was noted in §3, we can use Lemma 2.2 to obtain a $\text{Mod}_{g,1}$-invariant decomposition

$$\text{H}_1(K_{g,1}^L; \mathbb{Q}) \cong \text{H}_1(\Sigma_{g,1}; \mathbb{Q}) \oplus C_{g,1}^L.$$

This implies that

$$\text{H}_1(\text{Mod}_{g,1}; \text{H}_1(K_{g,1}^L; \mathbb{Q})) \cong \text{H}_1(\text{Mod}_{g,1}; \text{H}_1(\Sigma_{g,1}; \mathbb{Q})) \oplus \text{H}_1(\text{Mod}_{g,1}; C_{g,1}^L),$$

and similarly for $\text{Mod}_{g,1}(L)$. Theorem 2.10 says that

$$\text{H}_1(\text{Mod}_{g,1}; \text{H}_1(\Sigma_{g,1}; \mathbb{Q})) \cong \text{H}_1(\text{Mod}_{g,1}(L); \text{H}_1(\Sigma_{g,1}; \mathbb{Q})) \cong \mathbb{Q}.$$

To prove Theorem C, therefore, it is enough to prove the following theorem.

Theorem 5.1. *For $g \geq 4$ and $L \geq 2$, we have*

$$\text{H}_1(\text{Mod}_{g,1}; C_{g,1}^L) \cong \text{H}_1(\text{Mod}_{g,1}(L); C_{g,1}^L) = 0.$$

Proof. Since $\text{Mod}_{g,1}(L)$ is a finite-index subgroup of $\text{Mod}_{g,1}$, Lemma 2.1 implies that it is enough to prove that $\text{H}_1(\text{Mod}_{g,1}(L); C_{g,1}^L) = 0$. Associated to the Birman exact sequence

$$1 \longrightarrow \pi_1(U\Sigma_g) \longrightarrow \text{Mod}_{g,1}(L) \longrightarrow \text{Mod}_g(L) \longrightarrow 1$$

is a 5-term exact sequence in homology with coefficients in $C_{g,1}^L$. The last 3 terms of this are

$$(\text{H}_1(\pi_1(U\Sigma_g); C_{g,1}^L))_{\text{Mod}_{g,1}(L)} \xrightarrow{f} \text{H}_1(\text{Mod}_{g,1}(L); C_{g,1}^L) \longrightarrow \text{H}_1(\text{Mod}_g(L); (C_{g,1}^L)_{\pi_1(U\Sigma_g)}) \longrightarrow 0.$$

Lemma 4.7 says that

$$(C^L_{g,1})_{\pi_1(U\Sigma_g)} = (C^L_{g,1})_{\pi_1(\Sigma_g)} = 0.$$

To prove the theorem, therefore, it is enough to show that $f = 0$.

Consider the sequence of maps

$$\mathrm{H}_1(\overline{K}^L_g; C^L_{g,1}) \xrightarrow{f'} \mathrm{H}_1(\pi_1(U\Sigma_g); C^L_{g,1}) \xrightarrow{f''} (\mathrm{H}_1(U\Sigma_g; C^L_{g,1}))_{\mathrm{Mod}_{g,1}(L)} \xrightarrow{f} \mathrm{H}_1(\mathrm{Mod}_{g,1}(L); C^L_{g,1}).$$

Since $\overline{K}^L_g$ is a finite-index subgroup of $\pi_1(U\Sigma_g)$, Lemma 2.1 implies that f' is surjective. Since f'' is also surjective, to prove that $f = 0$ it is enough to prove that the map

$$\mathrm{H}_1(\overline{K}^L_g; C^L_{g,1}) \longrightarrow \mathrm{H}_1(\mathrm{Mod}_{g,1}(L); C^L_{g,1}) \tag{6}$$

is the zero map.

We now simplify the target of this map. From the short exact sequence

$$0 \longrightarrow I^L_g \longrightarrow C^L_{g,1} \longrightarrow C^L_g \longrightarrow 0$$

of $\mathrm{Mod}_{g,1}(L)$-modules we obtain a long exact sequence in homology. This long exact sequence contains the segment

$$\mathrm{H}_1(\mathrm{Mod}_{g,1}(L); I^L_g) \longrightarrow \mathrm{H}_1(\mathrm{Mod}_{g,1}(L); C^L_{g,1}) \longrightarrow \mathrm{H}_1(\mathrm{Mod}_{g,1}(L); C^L_g).$$

Since $\mathrm{Mod}_{g,1}(L)$ acts trivially on I^L_g, Theorem 2.9 implies that $\mathrm{H}_1(\mathrm{Mod}_{g,1}(L); I^L_g) = 0$. Thus the map $\mathrm{H}_1(\mathrm{Mod}_{g,1}(L); C^L_{g,1}) \to \mathrm{H}_1(\mathrm{Mod}_{g,1}(L); C^L_g)$ is injective. By Lemma 4.8, the map

$$\mathrm{H}_1(\mathrm{Mod}_{g,1}(L); C^L_g) \to \mathrm{H}_1(\mathrm{Mod}^1_g(L); C^L_g)$$

is an isomorphism, so we deduce that to prove that the map in (6) is the zero map, it is enough to prove that the map

$$\mathrm{H}_1(\overline{K}^L_g; C^L_{g,1}) \longrightarrow \mathrm{H}_1(\mathrm{Mod}^1_g(L); C^L_g) \tag{7}$$

is the zero map.

Observe now the factorization

$$\mathrm{H}_1(\overline{K}^L_g; C^L_{g,1}) \longrightarrow \mathrm{H}_1(K^L_g; C^L_{g,1}) \xrightarrow{i} \mathrm{H}_1(K^L_g; C^L_g) \xrightarrow{j} \mathrm{H}_1(\mathrm{Mod}^1_g(L); C^L_g)$$

of the map in (7). Lemmas 4.5 and 4.6 imply that the image of i is spanned by the set $\mathscr{S}^L_g$ from §4.1. Lemma 4.10 says that $j(\mathscr{S}^L_g) = 0$, so we conclude that the map in (7) is the zero map, as desired. □

5.2 Closed surfaces

We now turn to Theorems A and B, which assert that if $g \geq 4$ and $L \geq 2$, then

$$\mathrm{H}_1(\mathrm{Mod}^1_g; \mathrm{H}_1(K^L_g; \mathbb{Q})) \cong \mathbb{Q}^{\tau(L)} \quad \text{and} \quad \mathrm{H}_1(\mathrm{Mod}^1_g(L); \mathrm{H}_1(K^L_g; \mathbb{Q})) \cong \mathbb{Q}[H_L].$$

Here $\tau(L)$ is the number of positive divisors of L and $H_L \cong \mathrm{H}_1(\Sigma_g; \mathbb{Z}_L)$. Also, the second isomorphism should be equivariant with respect to Mod^1_g actions on $\mathrm{H}_1(\mathrm{Mod}^1_g(L); \mathrm{H}_1(K^L_g; \mathbb{Q}))$ and $\mathbb{Q}[H_L]$. We begin by deriving Theorem A from Theorem B.

Proof of Theorem A, assuming Theorem B. Lemma 2.3 implies that

$$\mathrm{H}_1(\mathrm{Mod}_g^1; \mathrm{H}_1(K_g^L; \mathbb{Q})) \cong (\mathrm{H}_1(\mathrm{Mod}_g^1(L); \mathrm{H}_1(K_g^L; \mathbb{Q})))_{\mathrm{Mod}_g^1}.$$

Applying Theorem B, we must show that $(\mathbb{Q}[H_L])_{\mathrm{Mod}_g^1} \cong \mathbb{Q}^{\tau(L)}$. The vector space $\mathbb{Q}[H_L]$ has a basis that is permuted by the action of Mod_g^1, namely the elements of H_L. It is enough, therefore, to show that there are $\tau(L)$ orbits of the action of Mod_g^1 on H_L. This action factors through the surjection $\mathrm{Mod}_g^1 \to \mathrm{Sp}_{2g}(\mathbb{Z}_L)$. Let $v \in H_L$ be a fixed primitive vector, and set

$$X = \{cv \mid c \text{ is a positive divisor of } L\} \subset H_L.$$

The set X has cardinality $\tau(L)$, and clearly no two elements of X are in the same $\mathrm{Sp}_{2g}(\mathbb{Z}_L)$-orbit. Also, if $w \in H_L$, then there is a primitive vector w' and a positive divisor c of L such that $w = cw'$. Since $\mathrm{Sp}_{2g}(\mathbb{Z}_L)$ acts transitively on the set of primitive vectors, there is some $\phi \in \mathrm{Sp}_{2g}(\mathbb{Z}_L)$ such that $v = \phi(w')$. Thus w is in the same $\mathrm{Sp}_{2g}(\mathbb{Z}_L)$-orbit as $cv \in X$. We conclude that X contains a unique representative from every $\mathrm{Sp}_{2g}(\mathbb{Z}_L)$-orbit, and we are done. □

Finally, we discuss Theorem B. Lemma 2.2 implies that there is a Mod_g^1-invariant decomposition

$$\mathrm{H}_1(K_g^L; \mathbb{Q}) \cong \mathrm{H}_1(\Sigma_g; \mathbb{Q}) \oplus C_g^L,$$

so we have a Mod_g^1-invariant decomposition

$$\mathrm{H}_1(\mathrm{Mod}_g^1(L); \mathrm{H}_1(K_g^L; \mathbb{Q})) \cong \mathrm{H}_1(\mathrm{Mod}_g^1(L); \mathrm{H}_1(\Sigma_g; \mathbb{Q})) \oplus \mathrm{H}_1(\mathrm{Mod}_g^1(L); C_g^L).$$

Also, Theorem 2.10 says that

$$\mathrm{H}_1(\mathrm{Mod}_g^1(L); \mathrm{H}_1(\Sigma_g; \mathbb{Q})) \cong \mathbb{Q},$$

so we obtain a Mod_g^1-invariant decomposition

$$\mathrm{H}_1(\mathrm{Mod}_g^1(L); \mathrm{H}_1(K_g^L; \mathbb{Q})) \cong \mathbb{Q} \oplus \mathrm{H}_1(\mathrm{Mod}_g^1(L); C_g^L).$$

By Lemma 2.4, we have $\mathbb{Q}[H_L] \cong \mathbb{Q} \oplus I$, where I is the augmentation ideal of $\mathbb{Q}[H_L]$. Lemma 3.1 says that I_g^L is isomorphic to the augmentation ideal of $\mathbb{Q}[H_L]$, so we conclude that it is enough to prove the following theorem.

Theorem 5.2. *For $g \geq 4$ and $L \geq 2$, we have a Mod_g^1-equivariant isomorphism $\mathrm{H}_1(\mathrm{Mod}_g^1(L); C_g^L) \cong I_g^L$.*

Proof. Lemma 4.8 says that there is an isomorphism

$$\mathrm{H}_1(\mathrm{Mod}_{g,1}(L); C_g^L) \cong \mathrm{H}_1(\mathrm{Mod}_g^1(L); C_g^L). \tag{8}$$

The action of $\mathrm{Mod}_{g,1}$ on $\mathrm{H}_1(\mathrm{Mod}_{g,1}(L); C_g^L)$ factors through Mod_g^1, and it is easy to see that the isomorphism in (8) is Mod_g^1-equivariant. We deduce that it is enough to construct a $\mathrm{Mod}_{g,1}$-equivariant isomorphism $\mathrm{H}_1(\mathrm{Mod}_{g,1}(L); C_g^L) \cong I_g^L$. The long exact sequence in $\mathrm{Mod}_{g,1}(L)$-homology associated to the short exact sequence

$$0 \longrightarrow I_g^L \longrightarrow C_{g,1}^L \longrightarrow C_g^L \longrightarrow 0$$

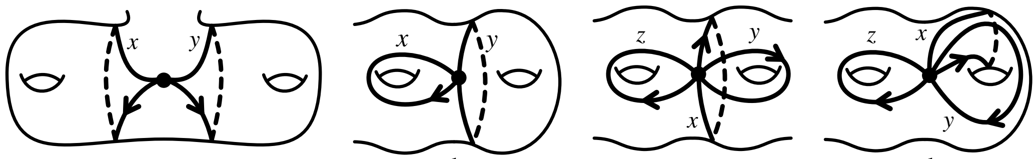

Figure 3: *Example of curves used in* $\mathscr{T}_g^L$

of $\mathrm{Mod}_{g,1}$-modules contains the segment

$$\mathrm{H}_1(\mathrm{Mod}_{g,1}(L);C_{g,1}^L) \longrightarrow \mathrm{H}_1(\mathrm{Mod}_{g,1}(L);C_g^L) \longrightarrow I_g^L \longrightarrow (C_{g,1}^L)_{\mathrm{Mod}_{g,1}(L)}.$$

Here we are using the fact that $\mathrm{Mod}_{g,1}(L)$ acts trivially on I_g^L, so

$$\mathrm{H}_0(\mathrm{Mod}_{g,1}(L);I_g^L) = (I_g^L)_{\mathrm{Mod}_{g,1}(L)} = I_g^L.$$

Theorem 5.1 says that $\mathrm{H}_1(\mathrm{Mod}_{g,1}(L);C_{g,1}^L) = 0$, and Lemma 4.7 implies that $(C_{g,1}^L)_{\mathrm{Mod}_{g,1}(L)} = 0$. We obtain an isomorphism $\mathrm{H}_1(\mathrm{Mod}_{g,1}(L);C_g^L) \cong I_g^L$, which is easily verified to be $\mathrm{Mod}_{g,1}$-equivariant. The theorem follows. □

6 Some boundary map calculations

Throughout this section, we will fix some $g \geq 4$ and $L \geq 2$.

Let $\partial : \mathrm{H}_1(K_g^L;C_g^L) \to I_g^L$ be as in §4.1. In this section, we do two things. First, we we calculate the image of elements of a certain set $\mathscr{T}_g^L$ under ∂. Second, we use this calculation to prove Lemma 4.5, which we recall says that $\partial(\mathscr{S}_g^L) = 0$.

We start by defining $\mathscr{T}_g^L \subset \mathrm{H}_1(K_g^L;C_g^L)$. Recall that $\mathrm{H}_1(K_g^L;C_g^L) \cong \mathrm{H}_1(K_g^L;\mathbb{Z}) \otimes C_g^L$.

Definition 6.1. Set $\mathscr{T}_g^L = \mathscr{T}_g^L(1) \cup \mathscr{T}_g^L(1') \cup \mathscr{T}_g^L(2) \cup \mathscr{T}_g^L(3)$, where the $\mathscr{T}_g^L(i)$ are defined as follows. To simplify our notation, we will denote $\pi_1(\Sigma_g)$ by π.

$$\mathscr{T}_g^L(1) = \{\langle\!\langle x \rangle\!\rangle \otimes \langle\!\langle y \rangle\!\rangle^{\overline{f}} \mid x,y,f \in \pi \text{ and } x \text{ and } y \text{ are essentially separate genus 1 separating curves (see Figure 3.a)}\},$$

$$\mathscr{T}_g^L(1') = \{\langle\!\langle x^L \rangle\!\rangle \otimes \langle\!\langle y \rangle\!\rangle^{\overline{f}} \mid x,y,f \in \pi, x \text{ is a nonseparating simple closed curve, } y \text{ is a genus 1 separating curve, and } x \text{ and } y \text{ are essentially separate (see Figure 3.b)}\},$$

$$\mathscr{T}_g^L(2) = \{\langle\!\langle x \rangle\!\rangle \otimes \langle\!\langle y,z \rangle\!\rangle^{\overline{f}} \mid x,y,z,f \in \pi \text{ and } \{x,y,z\} \text{ has the same unoriented intersection pattern as the curves in Figure 3.c}\},$$

$$\mathscr{T}_g^L(3) = \{\langle\!\langle x^L \rangle\!\rangle \otimes \langle\!\langle y,z \rangle\!\rangle^{\overline{f}} \mid x,y,z,f \in \pi \text{ and } \{x,y,z\} \text{ has the same unoriented intersection pattern as the curves in Figure 3.d}\}.$$

Our first result gives the images of elements of $\mathscr{T}_g^L$ under ∂.

Lemma 6.2.

1. *If* $\delta \in \mathscr{T}_g^L(1) \cup \mathscr{T}_g^L(1')$*, then* $\partial(\delta) = 0$.

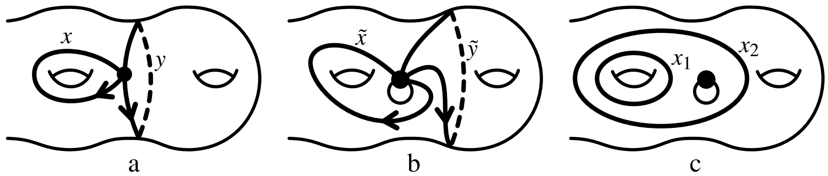

Figure 4: *Curves needed to deal with* $\mathscr{T}_g^L(1')$

2. *If* $\langle\langle x\rangle\rangle \otimes \langle\langle y,z\rangle\rangle^{\overline{f}} \in \mathscr{T}_g^L(2)$ *and* $\{x,y,z\}$ *has the same intersection pattern as the curves in Figure 3.c, then* $\partial(\langle\langle x\rangle\rangle \otimes \langle\langle y,z\rangle\rangle^{\overline{f}}) = [\![\overline{f}]\!] - [\![\overline{f}+\overline{y}]\!] - [\![\overline{f}+\overline{z}]\!] + [\![\overline{f}+\overline{y}+\overline{z}]\!]$.

3. *If* $\langle\langle x^L\rangle\rangle \otimes \langle\langle y,z\rangle\rangle^{\overline{f}} \in \mathscr{T}_g^L(3)$ *and* $\{x,y,z\}$ *has the same intersection pattern as the curves in Figure 3.d, then* $\partial(\langle\langle x^L\rangle\rangle \otimes \langle\langle y,z\rangle\rangle^{\overline{f}}) = \sum_{k=0}^{L-1}([\![\overline{f}+k\cdot\overline{x}+\overline{y}]\!] - [\![\overline{f}+k\cdot\overline{x}+\overline{y}+\overline{z}]\!])$.

Proof. Consider $\delta \in \mathscr{T}_g^L$. Write $\delta = \langle\langle a\rangle\rangle \otimes b$ with $a \in K_g^L$ and $b \in C_g^L$. Unwrapping the definitions, to calculate $\partial(\langle\langle a\rangle\rangle \otimes b)$, we must do the following (see [6, §III.7] for a discussion of ∂).

1. Choose a lift $\phi \in \mathrm{Mod}_{g,1}(L)$ of $a \in K_g^L \subset \mathrm{Mod}_g^1(L)$ together with a lift $\tilde{b} \in C_{g,1}^L$ of $b \in C_g^L$.

2. Calculate $\phi(\tilde{b}) - \tilde{b}$, which lies in $I_g^L = \ker(C_{g,1}^L \to C_g^L)$ since K_g^L acts trivially on C_g^L. This is $\partial(\langle\langle a\rangle\rangle \otimes b)$.

Let $\gamma \in \pi_1(\Sigma_{g,1})$ be the simple closed curve that goes once around the boundary component with the interior of the surface to its left, so $\langle\langle\gamma\rangle\rangle^v = [\![v]\!]$ for all $v \in \mathrm{H}_1(\Sigma_g;\mathbb{Z}_L)$. There are 3 cases.

Case 1. $\delta \in \mathscr{T}_g^L(1)$ or $\delta \in \mathscr{T}_g^L(1')$.

We will assume that $\delta \in \mathscr{T}_g^L(1')$; the other case is similar. We thus have $\delta = \langle\langle x^L\rangle\rangle \otimes \langle\langle y\rangle\rangle^{\overline{f}}$ for some $x,y,f \in \pi_1(\Sigma_g)$ such that $\{x,y\}$ has the same unoriented intersection pattern as the curves in Figure 4.a. Possibly changing the orientations of x and y (and thus inverting δ), we can assume that $\{x,y\}$ has the same intersection pattern as the curves in Figure 4.a. Our lift ϕ of x will be $(T_{x_1}T_{x_2}^{-1})^L$, where x_1 and x_2 are the curves in Figure 4.c. Our lift of $\langle\langle y\rangle\rangle^{\overline{f}}$ will be $\langle\langle\tilde{y}\rangle\rangle^{\overline{f}}$, where $\tilde{y}$ is as in Figure 4.b.

Let $\tilde{x}$ be the curve in Figure 4.b. The mapping class $T_{x_1}T_{x_2}^{-1}$ fixes $\tilde{x}$ and acts on $\tilde{y}$ as conjugation by $\tilde{x}$. Also, x_1 and x_2 are homologous, so T_{x_1} and T_{x_2} act identically on $\mathrm{H}_1(\Sigma_{g,1};\mathbb{Z})$. Thus $T_{x_1}T_{x_2}^{-1}$ acts trivially on $\mathrm{H}_1(\Sigma_{g,1};\mathbb{Z})$, so $T_{x_1}T_{x_2}^{-1}(\overline{f}) = \overline{f}$. We deduce that

$$(T_{x_1}T_{x_1}^{-1})^L(\langle\langle\tilde{y}\rangle\rangle^{\overline{f}}) = \langle\langle\tilde{y}\rangle\rangle^{\overline{f}+L\cdot\overline{x}} = \langle\langle\tilde{y}\rangle\rangle^{\overline{f}},$$

so $\partial(\langle\langle x^L\rangle\rangle \otimes \langle\langle y\rangle\rangle^{\overline{f}}) = 0$, as desired.

Case 2. $\delta \in \mathscr{T}_g^L(2)$ and $\delta = \langle\langle x\rangle\rangle \otimes \langle\langle y,z\rangle\rangle^{\overline{f}}$ for some $x,y,z,f \in \pi_1(\Sigma_g)$ such that $\{x,y,z\}$ has the same intersection pattern as the curves in Figure 5.a.

Our lift ϕ of x will be $T_{x_1}T_{x_2}^{-1}$, where x_1 and x_2 are the curves in Figure 5.c. Our lift of $\langle\langle y,z\rangle\rangle^{\overline{f}}$ will be $\langle\langle\tilde{y},\tilde{z}\rangle\rangle^{\overline{f}}$, where $\tilde{y}$ and $\tilde{z}$ are as in Figure 5.b. Let $\tilde{x}_1$ and $\tilde{x}_2$ be the curves in Figure 5.b. Observe that $\tilde{x}_2 = \tilde{x}_1\cdot\gamma$. Also, the mapping class $T_{x_1}T_{x_2}^{-1}$ acts on $\tilde{y}$ as conjugation by $\tilde{x}_1$ and on $\tilde{z}$ as conjugation by $\tilde{x}_2$. Finally, $T_{x_1}T_{x_2}^{-1}$ acts trivially on $\mathrm{H}_1(\Sigma_{g,1};\mathbb{Z})$, so $T_{x_1}T_{x_2}^{-1}(\overline{f}) = \overline{f}$. Making use of the fact that γ and

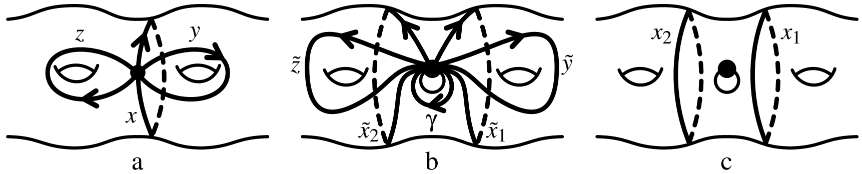

Figure 5: *Curves needed to deal with* $\mathcal{T}_g^L(2)$

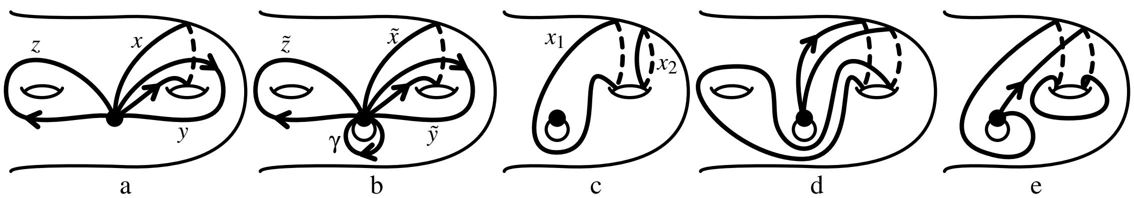

Figure 6: *a-c. Curves needed to deal with* $\mathcal{T}_g^L(3)$ *d.* $T_{x_1}T_{x_2}^{-1}(\tilde{z}) = \tilde{x}^{-1}\gamma\tilde{z}\gamma^{-1}\tilde{x}$ *e.* $T_{x_1}T_{x_2}^{-1}(\tilde{y}) = \tilde{x}^{-1}\tilde{y}\tilde{x}\gamma^{-1}$

the $\tilde{x}_i$ are null-homologous, we can apply Lemma 2.8 several times to deduce that $T_{x_1}T_{x_2}^{-1}(\langle\!\langle \tilde{y},\tilde{z}\rangle\!\rangle^{\overline{f}})$ equals

$$\begin{aligned}\langle\!\langle \tilde{x}_1^{-1}\tilde{y}\tilde{x}_1, \gamma^{-1}\tilde{x}_1^{-1}\tilde{z}\tilde{x}_1\gamma\rangle\!\rangle^{\overline{f}} &= \langle\!\langle \tilde{x}_1^{-1}\tilde{y}\tilde{x}_1, \gamma^{-1}\rangle\!\rangle^{\overline{f}+\overline{z}} + \langle\!\langle \tilde{x}_1^{-1}\tilde{y}\tilde{x}_1, \tilde{x}_1^{-1}\tilde{z}\tilde{x}_1\rangle\!\rangle^{\overline{f}} + \langle\!\langle \tilde{x}_1^{-1}\tilde{y}\tilde{x}_1, \gamma\rangle\!\rangle^{\overline{f}}\\ &= \langle\!\langle \tilde{x}_1^{-1}\tilde{y}\tilde{x}_1, \tilde{x}_1^{-1}\tilde{z}\tilde{x}_1\rangle\!\rangle^{\overline{f}} + \langle\!\langle \gamma, \tilde{x}_1^{-1}\tilde{y}\tilde{x}_1\rangle\!\rangle^{\overline{f}+\overline{z}} - \langle\!\langle \gamma, \tilde{x}_1^{-1}\tilde{y}\tilde{x}_1\rangle\!\rangle^{\overline{f}}\\ &= \langle\!\langle \tilde{y},\tilde{z}\rangle\!\rangle^{\overline{f}} + ([\![\overline{f}+\overline{z}+\overline{y}]\!] - [\![\overline{f}+\overline{z}]\!]) - ([\![\overline{f}+\overline{y}]\!] - [\![\overline{f}]\!]).\end{aligned}$$

We conclude that $\partial\,(\langle\!\langle x\rangle\!\rangle \otimes \langle\!\langle y,z\rangle\!\rangle^{\overline{f}}) = [\![\overline{f}]\!] - [\![\overline{f}+\overline{y}]\!] - [\![\overline{f}+\overline{z}]\!] + [\![\overline{f}+\overline{z}+\overline{y}]\!]$, as desired.

Case 3. $\delta \in \mathcal{T}_g^L(3)$ and $\delta = \langle\!\langle x^L\rangle\!\rangle \otimes \langle\!\langle y,z\rangle\!\rangle^{\overline{f}}$ for some $x,y,z,f \in \pi_1(\Sigma_g)$ such that $\{x,y,z\}$ has the same intersection pattern as the curves in Figure 6.a.

Our lift ϕ of x^L will be $(T_{x_1}T_{x_2}^{-1})^L$, where x_1 and x_2 are the curves in Figure 6.c. Our lift of $\langle\!\langle y,z\rangle\!\rangle^{\overline{f}}$ will be $\langle\!\langle \tilde{y},\tilde{z}\rangle\!\rangle^{\overline{f}}$, where $\tilde{y}$ and $\tilde{z}$ are as in Figure 6.b. Let $\tilde{x}$ be the other curve in Figure 6.b. As shown in Figures 6.d–e, we have $T_{x_1}T_{x_2}^{-1}(\tilde{z}) = \tilde{x}^{-1}\gamma\tilde{z}\gamma^{-1}\tilde{x}$ and $T_{x_1}T_{x_2}^{-1}(\tilde{y}) = \tilde{x}^{-1}\tilde{y}\tilde{x}\gamma^{-1}$. Also, $T_{x_1}T_{x_2}^{-1}$ acts trivially on $\mathrm{H}_1(\Sigma_{g,1};\mathbb{Z})$, so $T_{x_1}T_{x_2}^{-1}(\overline{f}) = \overline{f}$. We can thus apply Lemma 2.8 several times to deduce that the image of $\langle\!\langle \tilde{y},\tilde{z}\rangle\!\rangle^{\overline{f}}$ under $T_{x_1}T_{x_2}^{-1}$ equals

$$\begin{aligned}\langle\!\langle \tilde{x}^{-1}\tilde{y}\tilde{x}\gamma^{-1}, \tilde{x}^{-1}\gamma\tilde{z}\gamma^{-1}\tilde{x}\rangle\!\rangle^{\overline{f}} =&\langle\!\langle \tilde{y}, \gamma\tilde{z}\gamma^{-1}\rangle\!\rangle^{\overline{f}+\overline{x}} + \langle\!\langle \gamma^{-1}, \tilde{x}^{-1}\gamma\tilde{z}\gamma^{-1}\tilde{x}\rangle\!\rangle^{\overline{f}}\\ =&-\langle\!\langle \gamma,\tilde{y}\rangle\!\rangle^{\overline{f}+\overline{x}+\overline{z}} + \langle\!\langle \tilde{y},\tilde{z}\rangle\!\rangle^{\overline{f}+\overline{x}} + \langle\!\langle \gamma,\tilde{y}\rangle\!\rangle^{\overline{f}+\overline{x}} - \langle\!\langle \gamma, \tilde{x}^{-1}\gamma\tilde{z}\gamma^{-1}\tilde{x}\rangle\!\rangle^{\overline{f}}\\ =&\langle\!\langle \tilde{y},\tilde{z}\rangle\!\rangle^{\overline{f}+\overline{x}} - ([\![\overline{f}+\overline{x}+\overline{z}+\overline{y}]\!] - [\![\overline{f}+\overline{x}+\overline{z}]\!])\\ &+([\![\overline{f}+\overline{x}+\overline{y}]\!] - [\![\overline{f}+\overline{x}]\!]) - ([\![\overline{f}+\overline{z}]\!] - [\![\overline{f}]\!])\end{aligned}$$

Iterating this and using the fact that $T_{x_1}T_{x_2}^{-1}$ acts trivially on all the $[\![\cdot]\!]$ terms, we see that

$$\begin{aligned}(T_{x_1}T_{x_2}^{-1})^L(\langle\!\langle \tilde{y},\tilde{z}\rangle\!\rangle^{\overline{f}}) = \langle\!\langle \tilde{y},\tilde{z}\rangle\!\rangle^{\overline{f}+L\cdot\overline{x}} + \sum_{k=0}^{L-1}\big(&-[\![\overline{f}+(k+1)\cdot\overline{x}+\overline{z}+\overline{y}]\!] + [\![\overline{f}+(k+1)\cdot\overline{x}+\overline{z}]\!]\\ &+[\![\overline{f}+(k+1)\cdot\overline{x}+\overline{y}]\!] - [\![\overline{f}+(k+1)\cdot\overline{x}]\!]\\ &-[\![\overline{f}+\overline{z}+k\cdot\overline{x}]\!] + [\![\overline{f}+k\cdot\overline{x}]\!]\big)\end{aligned}$$

Since $L \cdot \overline{x} = 0$, the first term equals $\langle\langle \tilde{y}, \tilde{z} \rangle\rangle^{\overline{f}}$. In the summation, the $[\![\overline{f} + k \cdot \overline{x}]\!]$ and the $[\![\overline{f} + (k+1) \cdot \overline{x}]\!]$ terms cancel. Similarly, the $[\![\overline{f} + \overline{z} + k \cdot \overline{x}]\!]$ and the $[\![\overline{f} + (k+1) \cdot \overline{x} + \overline{z}]\!]$ terms cancel. The desired result now follows from the reparametrization $k \mapsto k-1$. □

Next, we prove Lemma 4.5, which says that $\partial(\mathscr{S}_g^L) = 0$.

Proof of Lemma 4.5. There are four cases.

Case 1. $\partial(\mathscr{S}_g^L(4)) = 0$.

Consider $\langle\langle x \rangle\rangle \otimes \langle\langle y \rangle\rangle - \langle\langle x \rangle\rangle^{\overline{f}} \otimes \langle\langle y \rangle\rangle^{\overline{f}} \in \mathscr{S}_g^L(4)$. Since $\text{Mod}_g^1(L)$ acts trivially on I_g^L, the long exact sequence in $\text{Mod}_g^1(L)$-homology associated to the short exact sequence

$$0 \longrightarrow I_g^L \longrightarrow C_{g,1}^L \longrightarrow C_g^L \longrightarrow 0$$

contains a map $\partial : \text{H}_1(\text{Mod}_g^1(L); C_g^L) \to I_g^L$. This fits into a commutative diagram

$$\begin{array}{ccc} \text{H}_1(K_g^L; C_g^L) & \xrightarrow{\partial} & I_g^L \\ \downarrow & & \| \\ \text{H}_1(\text{Mod}_g^1(L); C_g^L) & \xrightarrow{\partial} & I_g^L \end{array}$$

Since inner automorphisms act trivially on homology, the image of $\mathscr{S}_g^L(4)$ in $\text{H}_1(\text{Mod}_g^1(L); C_g^L)$ is zero, and the result follows.

Case 2. $\partial(\mathscr{S}_g^L(1)) = 0$.

Consider $\langle\langle x \rangle\rangle^{\overline{f}_1} \otimes \langle\langle y \rangle\rangle^{\overline{f}_2} \in \mathscr{S}_g^L(1)$. Using the essential separateness of x and y together with Lemma 4.9, we can write $\langle\langle x \rangle\rangle^{\overline{f}_1} = \langle\langle x_1 \rangle\rangle^{\overline{g}_1} + \cdots + \langle\langle x_k \rangle\rangle^{\overline{g}_k}$, where for $1 \leq i \leq k$ we have the following.

- $g_i \in \pi_1(\Sigma_g)$.
- The curve x_i is essentially separate from y.
- The curve x_i is either a genus 1 separating curve or the L^{th} power of a simple closed nonseparating curve.

For $1 \leq i \leq k$, we can then use the essential separateness of x_i and y together with Lemma 4.9 to write $\langle\langle y \rangle\rangle^{\overline{f}_2} = \langle\langle y_{i,1} \rangle\rangle^{\overline{h}_{i,1}} + \cdots + \langle\langle y_{i,l_i} \rangle\rangle^{\overline{h}_{i,l_i}}$, where for $1 \leq j \leq l_i$ we have the following.

- $h_{i,j} \in \pi_1(\Sigma_g)$.
- $y_{i,j}$ is essentially separate from x_i.
- $y_{i,j}$ is a genus 1 separating curve.

We then have

$$\langle\langle x \rangle\rangle^{\overline{f}_1} \otimes \langle\langle y \rangle\rangle^{\overline{f}_2} = \sum_{i=1}^{k} \left(\sum_{j=1}^{l_i} \langle\langle x_i \rangle\rangle^{\overline{g}_i} \otimes \langle\langle y_{i,j} \rangle\rangle^{\overline{h}_{i,j}} \right).$$

For $1 \leq i \leq k$ and $1 \leq j \leq l_i$, the element $\langle\langle x_i \rangle\rangle^{\overline{g}_i} \otimes \langle\langle y_{i,j} \rangle\rangle^{\overline{h}_{i,j}}$ differs from an element of $\mathscr{T}_g^L(1) \cup \mathscr{T}_g^L(1')$ by an element of $\mathscr{S}_g^L(4)$ (via an exponent trade), so using Lemma 6.2 we conclude that $\partial(\langle\langle x \rangle\rangle^{\overline{f}_1} \otimes \langle\langle y \rangle\rangle^{\overline{f}_2}) = 0$, as desired.

Case 3. $\partial(\mathscr{S}_g^L(2)) = 0$.

Consider $\langle\langle x\rangle\rangle^{\overline{f}_1} \otimes \langle\langle y^L, z\rangle\rangle^{\overline{f}_2} \in \mathscr{S}_g^L(2)$. Since $\partial(\mathscr{S}_g^L(4)) = 0$, we can perform an exponent trade and assume that $f_1 = 1$. Also, by inverting x, y, or z if necessary, we can assume that $\{x,y,z\}$ has the same intersection pattern as the curves in Figure 1.b. Indeed, inverting x merely inverts $\langle\langle x\rangle\rangle^{\overline{f}_1} \otimes \langle\langle y^L, z\rangle\rangle^{\overline{f}_2}$, and by Lemma 2.8 inverting y or z merely changes $\overline{f}_2$ and inverts $\langle\langle x\rangle\rangle^{\overline{f}_1} \otimes \langle\langle y^L, z\rangle\rangle^{\overline{f}_2}$. Using Lemma 2.8, we see that

$$\langle\langle x\rangle\rangle \otimes \langle\langle y^L, z\rangle\rangle^{\overline{f}_2} = \sum_{k=0}^{L-1} \langle\langle x\rangle\rangle \otimes \langle\langle y, z\rangle\rangle^{k\cdot\overline{y}+\overline{f}_2}.$$

Observe that $\langle\langle x\rangle\rangle \otimes \langle\langle y, z\rangle\rangle^{k\cdot\overline{y}+\overline{f}_2} \in \mathscr{T}_g^L(2)$ for all $k \in \mathbb{Z}$. Using Lemma 6.2, we see that

$$\partial(\langle\langle x\rangle\rangle \otimes \langle\langle y^L, z\rangle\rangle^{\overline{f}_2}) = \sum_{k=0}^{L-1} ([\![k\cdot\overline{y}+\overline{f}_2]\!] - [\![k\cdot\overline{y}+\overline{f}_2+\overline{y}]\!] - [\![k\cdot\overline{y}+\overline{f}_2+\overline{z}]\!] + [\![k\cdot\overline{y}+\overline{f}_2+\overline{y}+\overline{z}]\!]).$$

The $[\![k\cdot\overline{y}+\overline{f}_2]\!]$ terms and the $[\![k\cdot\overline{y}+\overline{f}_2+\overline{y}]\!]$ terms cancel each other. Similarly, the $[\![k\cdot\overline{y}+\overline{f}_2+\overline{z}]\!]$ terms and the $[\![k\cdot\overline{y}+\overline{f}_2+\overline{y}+\overline{z}]\!]$ terms cancel each other. We are left with 0, as desired.

Case 4. $\partial(\mathscr{S}_g^L(3)) = 0$.

Consider $\langle\langle x^L\rangle\rangle^{\overline{f}_1} \otimes \langle\langle y, z^L\rangle\rangle^{\overline{f}_2} \in \mathscr{S}_g^L(3)$. Like in Case 3, we can assume that $f_1 = 1$ and that $\{x,y,z\}$ has the same intersection pattern as the curves in Figure 1.c. Using Lemma 2.8, we see that

$$\langle\langle x^L\rangle\rangle \otimes \langle\langle y, z^L\rangle\rangle^{\overline{f}_2} = \sum_{k=0}^{L-1} \langle\langle x^L\rangle\rangle \otimes \langle\langle y, z\rangle\rangle^{k\cdot\overline{z}+\overline{f}_2}$$

Observe that $\langle\langle x^L\rangle\rangle \otimes \langle\langle y, z\rangle\rangle^{k\cdot\overline{z}+\overline{f}_2} \in \mathscr{T}_g^L(3)$ for all $k \in \mathbb{Z}$. Using Lemma 6.2, we see that

$$\partial(\langle\langle x^L\rangle\rangle \otimes \langle\langle y, z^L\rangle\rangle^{\overline{f}_2}) = \sum_{k'=0}^{L-1}\sum_{k=0}^{L-1} ([\![k\cdot\overline{z}+\overline{f}_2+k'\cdot\overline{x}+\overline{y}]\!] - [\![k\cdot\overline{z}+\overline{f}_2+k'\cdot\overline{x}+\overline{y}+\overline{z}]\!]).$$

The inner sum telescopes and thus vanishes, as desired. □

7 The isomorphism between $\mathrm{H}_1(K_g^L; C_g^L)/\langle\mathscr{S}_g^L\rangle$ and I_g^L

Throughout this section, we fix some $g \geq 4$ and $L \geq 2$.

Let $\partial : \mathrm{H}_1(K_g^L; C_g^L) \to I_g^L$ be as in §4.1. Define $Q_g^L = \mathrm{H}_1(K_g^L; C_g^L)/\langle\mathscr{S}_g^L\rangle$. By Lemma 4.5, the map ∂ induces a homomorphism $Q_g^L \to I_g^L$. In this section, we prove (modulo a certain linear-algebraic calculation) that this homomorphism is an isomorphism (Lemma 4.6). There are three subsections. In §7.1 we determine generators for Q_g^L, in §7.2 we find some relations between these generators, and in §7.3 we put these pieces together to prove Lemma 4.6. Let $\rho : \mathrm{H}_1(K_g^L; C_g^L) \to Q_g^L$ be the projection.

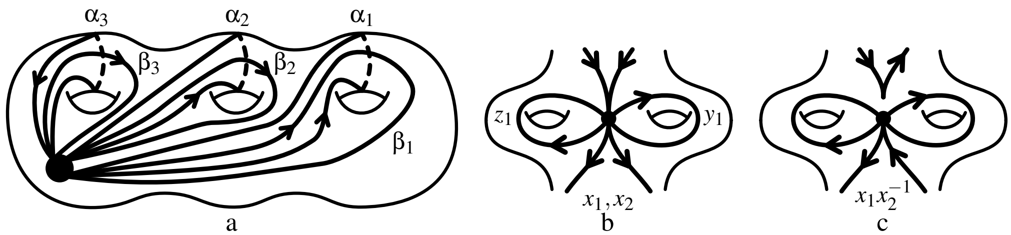

Figure 7: *a. A standard basis for $\pi_1(\Sigma_g)$ b. Two different x curves for an element of $\mathscr{T}_g^L(2)$. The order in which they exit and enter the basepoint is arbitrary. c. You can homotope $x_1x_2^{-1}$ to a configuration like this. Observe that the orientations for the "top" and "bottom" (or, better, "end" and "middle') portions can be arbitrary – we just made one possible choice here*

7.1 Generators

Recall that K_g^L acts trivially on C_g^L, so $\mathrm{H}_1(K_g^L;C_g^L) \cong K_g^L \otimes C_g^L$. Set $C_g^L(\mathbb{Z}) = \ker(\mathrm{H}_1(K_g^L;\mathbb{Z}) \to \mathrm{H}_1(\Sigma_g;\mathbb{Z}))$. We begin with the following.

Lemma 7.1. *We have $\rho\,(C_g^L(\mathbb{Z}) \otimes C_g^L) = Q_g^L$.*

Proof. Set $\Gamma = Q_g^L/\rho\,(C_g^L(\mathbb{Z}) \otimes C_g^L)$. Our goal is to show that $\Gamma = 0$. Since C_g^L is a vector space over $\mathbb{Q}$, we have a short exact sequence

$$0 \longrightarrow C_g^L(\mathbb{Z}) \otimes C_g^L \longrightarrow K_g^L \otimes C_g^L \longrightarrow H_L \otimes C_g^L \longrightarrow 0,$$

where $H_L = L \cdot \mathrm{H}_1(\Sigma_g;\mathbb{Z})$. Let $\rho' : H_L \otimes C_g^L \to \Gamma$ be the induced surjection. We will show that $\rho' = 0$.

For $x \in \pi_1(\Sigma_g)$, recall that $\widehat{x} \in \mathrm{H}_1(\Sigma_g;\mathbb{Z})$ is the associated homology class. We begin with the following observation. Consider $x,y,z,f \in \pi_1(\Sigma_g)$ such that $\langle\!\langle x^L \rangle\!\rangle \otimes \langle\!\langle y,z \rangle\!\rangle^{\overline{f}} \in \mathscr{T}_g^L(3)$. Since ρ' vanishes on the image in $H_L \otimes C_g^L$ of $\mathscr{S}_g^L(4)$, for all $j \in \mathbb{Z}$ we can perform an exponent trade to deduce that

$$\rho'(\widehat{x}^L \otimes \langle\!\langle y,z \rangle\!\rangle^{\overline{f}+j\cdot\overline{z}}) = \rho'((\widehat{z^jxz^{-j}})^L \otimes \langle\!\langle y,z \rangle\!\rangle^{\overline{f}}) = \rho'(\widehat{x}^L \otimes \langle\!\langle y,z \rangle\!\rangle^{\overline{f}}).$$

Consequently,

$$\rho'(\widehat{x}^L \otimes \langle\!\langle y,z \rangle\!\rangle^{\overline{f}}) = \frac{1}{L}\sum_{j=0}^{L-1} \rho'(\widehat{x}^L \otimes \langle\!\langle y,z \rangle\!\rangle^{\overline{f}+j\cdot\overline{z}}) = \frac{1}{L}\rho'(\widehat{x}^L \otimes \langle\!\langle y,z^L \rangle\!\rangle^{\overline{f}}) = 0.$$

This last equality follows from the fact that $\langle\!\langle x^L \rangle\!\rangle \otimes \langle\!\langle y,z^L \rangle\!\rangle^{\overline{f}} \in \mathscr{S}_g^L(3)$.

Now fix some $\alpha_1 \in \pi_1(\Sigma_g)$. Our goal is to show that $\rho'(\widehat{\alpha}_1^L \otimes c) = 0$ for all $c \in C_g^L$. We can assume without loss of generality that α_1 is a simple closed nonseparating curve. Define $U = U_1 \cup U_2$, where the $U_i \subset C_g^L$ are as follows.

$$U_1 = \{\langle\!\langle y,z \rangle\!\rangle^{\overline{f}} \mid y,z,f \in \pi_1(\Sigma_g) \text{ and the curves } \alpha_1 \text{ and } [y,z] \text{ are essentially separate}\},$$
$$U_2 = \{\langle\!\langle y,z \rangle\!\rangle^{\overline{f}} \mid y,z,f \in \pi_1(\Sigma_g) \text{ and } \langle\!\langle \alpha_1^L \rangle\!\rangle \otimes \langle\!\langle y,z \rangle\!\rangle^{\overline{f}} \in \mathscr{T}_g^L(3)\}.$$

Since $\langle\!\langle \alpha_1^L \rangle\!\rangle \otimes c \in \mathscr{S}_g^L(1)$ for all $c \in U_1$, we have $\rho'(\widehat{\alpha}_1^L \otimes c) = 0$ for all $c \in U_1$. Also, we showed above that $\rho'(\widehat{\alpha}_1^L \otimes c) = 0$ for all $c \in U_2$. We will show that $C_g^L = \langle U_1, U_2 \rangle$. It will then follow that $\rho'(\widehat{\alpha}_1^L \otimes c) = 0$ for all $c \in C_g^L$, as desired.

Extend $\{\alpha_1\}$ to a standard basis $B = \{\alpha_1, \beta_1, \dots, \alpha_g, \beta_g\}$ for $\pi_1(\Sigma_g)$ as shown in Figure 7.a. The set

$$X := \{\langle\!\langle y,z \rangle\!\rangle^{\overline{f}} \mid f \in \pi_1(\Sigma_g) \text{ and } y,z \in B \text{ and } y \neq z\}$$

spans C_g^L. Consider $\langle\!\langle y,z \rangle\!\rangle^{\overline{f}} \in X$. We will express $\langle\!\langle y,z \rangle\!\rangle^{\overline{f}}$ as a sum of elements of U. There are three cases. In the first, we either have $\{y,z\} = \{\alpha_1, \beta_1\}$ or $y,z \notin \{\alpha_1, \beta_1\}$. Hence $[y,z]$ is essentially separate from α_1, so $\langle\!\langle y,z \rangle\!\rangle^{\overline{f}} \in U_1$. In the second case, $\{y,z\} \cap \{\alpha_1, \beta_1\} = \{\beta_1\}$, so either $\langle\!\langle y,z \rangle\!\rangle^{\overline{f}} \in U_2$ or $\langle\!\langle z,y \rangle\!\rangle^{\overline{f}} \in U_2$. In the final case, $\{y,z\} \cap \{\alpha_1, \beta_1\} = \{\alpha_1\}$. Assume that $y = \alpha_1$; the other case is similar. We then have

$$\langle\!\langle \alpha_1\beta_1, z \rangle\!\rangle^{\overline{f}-\overline{\beta}_1} = \langle\!\langle \alpha_1, z \rangle\!\rangle^{\overline{f}} + \langle\!\langle \beta_1, z \rangle\!\rangle^{\overline{f}-\overline{\beta}_1}.$$

Since $\langle\!\langle \alpha_1\beta_1, z \rangle\!\rangle^{\overline{f}-\overline{\beta}_1}$ and $\langle\!\langle \beta_1, z \rangle\!\rangle^{\overline{f}-\overline{\beta}_1}$ are both elements of U_2, we are done. □

Next, we prove the following.

Lemma 7.2. *Fix $f \in \pi_1(\Sigma_g)$. For $1 \leq i \leq 2$, let $x_i, y_i, z_i \in \pi_1(\Sigma_g)$ be such that $\{x_i, y_i, z_i\}$ has the same intersection pattern as the curves in Figure 3.c. Assume that $\overline{y}_1 = \overline{y}_2$ and $\overline{z}_1 = \overline{z}_2$. Then $\rho(\langle\!\langle x_1 \rangle\!\rangle \otimes \langle\!\langle y_1, z_1 \rangle\!\rangle^{\overline{f}}) = \rho(\langle\!\langle x_2 \rangle\!\rangle \otimes \langle\!\langle y_2, z_2 \rangle\!\rangle^{\overline{f}}$.*

Proof. It is easy to see that there exists some $\phi \in \mathrm{Mod}_g^1(L)$ such that $\phi(y_1) = y_2$ and $\phi(z_1) = z_2$ (the proof of this is a slight variation on the proof of [19, Proposition 6.7], which proves the analogous result for unbased curves). Using the fact that $\rho(\mathscr{S}_g^L(4)) = 0$, we reduce to the special case that $y_1 = y_2$ and $z_1 = z_2$.

In this case, observe (see Figure 7.b,c) that we can decompose Σ_g^1 into two subsurfaces Σ and Σ' such that the following hold.

- $\mathrm{Int}(\Sigma) \cap \mathrm{Int}(\Sigma') = \emptyset$ while $* \in \Sigma \cap \Sigma'$.
- $\Sigma \cong \Sigma_{0,3}$ and $\Sigma' \cong \Sigma_{g-3,3}$.
- $[y_1, z_1] \in \mathrm{Image}(\pi_1(\Sigma) \to \pi_1(\Sigma_g))$ and $x_1 x_2^{-1} \in \mathrm{Image}(\pi_1(\Sigma') \to \pi_1(\Sigma_g))$.
- The map $\mathrm{H}_1(\Sigma'; \mathbb{Z}) \to \mathrm{H}_1(\Sigma_g; \mathbb{Z})$ is injective.

Since $x_1 x_2^{-1} \in [\pi_1(\Sigma_g), \pi_1(\Sigma_g)]$, Lemma 4.9 and the injectivity of $\mathrm{H}_1(\Sigma'; \mathbb{Z}) \to \mathrm{H}_1(\Sigma_g; \mathbb{Z})$ imply that there exist $w_1, g_1, \dots, w_n, g_n \in \pi_1(\Sigma_g)$ with the following properties.

- $x_1 x_2^{-1} = w_1^{g_1} \cdots w_n^{g_n}$
- For $1 \leq i \leq n$, the element w_i is the image of a simple closed genus 1 separating curve in $\pi_1(\Sigma')$.

For $1 \leq i \leq n$, the curve w_i is essentially separate from $[y_1, z_1]$, so $\rho(\langle\!\langle w_i \rangle\!\rangle^{\overline{g}_i} \otimes \langle\!\langle y_1, z_1 \rangle\!\rangle^{\overline{f}}) = 0$. We conclude that

$$\begin{aligned} \rho(\langle\!\langle x_1 \rangle\!\rangle \otimes \langle\!\langle y_1, z_1 \rangle\!\rangle^{\overline{f}}) - \rho(\langle\!\langle x_2 \rangle\!\rangle \otimes \langle\!\langle y_2, z_2 \rangle\!\rangle^{\overline{f}}) &= \rho(\langle\!\langle x_1 x_2^{-1} \rangle\!\rangle \otimes \langle\!\langle y_1, z_1 \rangle\!\rangle^{\overline{f}}) \\ &= \sum_{i=1}^{n} \rho(\langle\!\langle w_i \rangle\!\rangle^{\overline{g}_i} \otimes \langle\!\langle y_1, z_1 \rangle\!\rangle^{\overline{f}}) = 0, \end{aligned}$$

as desired. □

We now make the following definitions. Recall that $i(\cdot,\cdot)$ is the algebraic intersection pairing.

Definition 7.3. A k-element set $\{w_1,\dots,w_k\} \subset \mathrm{H}_1(\Sigma_g;\mathbb{Z}_L)$ will be said to be *isotropic* if $i(w_i,w_j) = 0$ for all $1 \le i,j \le k$ and *unimodular* if $\langle w_1,\dots,w_k\rangle$ is direct summand of $\mathrm{H}_1(\Sigma_g;\mathbb{Z}_L)$ that is isomorphic to a k-dimensional free $\mathbb{Z}_L$-submodule.

Remark. A set $\{w_1,\dots,w_k\} \subset \mathrm{H}_1(\Sigma_g;\mathbb{Z}_L)$ is isotropic and unimodular if and only if there exist disjoint simple closed oriented curves $\{x_1,\dots,x_k\}$ on Σ_g with the following properties. First, $x_1 \cup \cdots \cup x_k$ does not separate Σ_g. Second, $\bar{x}_i = w_i$ for $1 \le i \le k$.

Definition 7.4. Let $\{w_1,w_2\} \subset \mathrm{H}_1(\Sigma_g;\mathbb{Z}_L)$ be isotropic and unimodular and let $v \in \mathrm{H}_1(\Sigma_g;\mathbb{Z}_L)$ be arbitrary. We define an element $\mathbf{X}(v,w_1,w_2) \in Q_g^L$ as follows. Choose elements $x,y,z \in \pi_1(\Sigma_g)$ such that $\{x,y,z\}$ has the same intersection pattern as the curves in Figure 3.c and such that $\bar{y} = w_1$ and $\bar{z} = w_2$. Then $\mathbf{X}(v,w_1,w_2) = \rho(\langle\langle x\rangle\rangle \otimes \langle\langle y,z\rangle\rangle^v) \in Q_g^L$.

By Lemma 7.2, the element $\mathbf{X}(v,w_1,w_2)$ is well-defined. Using Lemma 7.1, we could deduce that the elements $\mathbf{X}(v,w_1,w_2)$ generate Q_g^L. Below in Lemma 7.7 we will prove a more precise version of this.

7.2 Relations

The following lemma lists some fundamental properties of $\mathbf{X}(\cdot,\cdot,\cdot)$.

Lemma 7.5. *Let $\{w_1,w_2\} \subset \mathrm{H}_1(\Sigma_g;\mathbb{Z}_L)$ be an isotropic and unimodular set. Then the following hold for all $v \in \mathrm{H}_1(\Sigma_g;\mathbb{Z}_L)$.*

1. $\mathbf{X}(v,w_1,w_2) = \mathbf{X}(v,w_2,w_1)$
2. $\mathbf{X}(v,-w_1,w_2) = -\mathbf{X}(v-w_1,w_1,w_2)$
3. $\sum_{i=0}^{L-1} \mathbf{X}(v+i\cdot w_1,w_1,w_2) = 0$
4. *Let $w_3 \in \mathrm{H}_1(\Sigma_g;\mathbb{Z}_L)$ be such that $\{w_1,w_2,w_3\}$ is unimodular, such that $i(w_2,w_3) = 0$, and such that $-1 \le i(w_1,w_3) \le 1$. Then $\mathbf{X}(v,w_1+w_3,w_2) = \mathbf{X}(v,w_1,w_2) + \mathbf{X}(v+w_1,w_3,w_2)$.*

Proof. Let $x,y,z \in \pi_1(\Sigma_g)$ be curves such that $\{x,y,z\}$ has the same intersection pattern as the curves in Figure 3.c and such that $\bar{y} = w_1$ and $\bar{z} = w_2$. Hence $\mathbf{X}(v,w_1,w_2) = \rho(\langle\langle x\rangle\rangle \otimes \langle\langle y,z\rangle\rangle^v)$. For item 1, observe that if we flip y and z, then our curves no longer have the same intersection pattern as the curves in Figure 3.c. To restore the correct intersection pattern, we must reverse x. In other words,

$$\mathbf{X}(v,w_2,w_1) = \rho(\langle\langle x^{-1}\rangle\rangle \otimes \langle\langle z,y\rangle\rangle^v) = \rho((-\langle\langle x\rangle\rangle) \otimes (-\langle\langle y,z\rangle\rangle^v)) = \rho(\langle\langle x\rangle\rangle \otimes \langle\langle y,z\rangle\rangle^v) = \mathbf{X}(v,w_1,w_2),$$

as desired. For item 2, observe that the set of curves $\{x,y^{-1}x,z\}$ has the same intersection pattern as $\{x,y,z\}$. Also, since $\bar{x} = 0$, we have $\overline{y^{-1}x} = -\bar{y} = -w_1$. Thus we can apply Lemma 2.8 to get that

$$\begin{aligned}\mathbf{X}(v,-w_1,w_2) &= \rho(\langle\langle x\rangle\rangle \otimes \langle\langle y^{-1}x,z\rangle\rangle^v) = \rho(\langle\langle x\rangle\rangle \otimes \langle\langle y^{-1},z\rangle\rangle^{v+\bar{x}}) + \rho(\langle\langle x\rangle\rangle \otimes \langle\langle x,z\rangle\rangle^v)\\ &= -\rho(\langle\langle x\rangle\rangle \otimes \langle\langle y,z\rangle\rangle^{v-w_1}) + \rho(\langle\langle x\rangle\rangle \otimes \langle\langle x\rangle\rangle^{v+\bar{z}}) - \rho(\langle\langle x\rangle\rangle \otimes \langle\langle x\rangle\rangle^v).\end{aligned}$$

Since both $\langle\langle x\rangle\rangle \otimes \langle\langle x\rangle\rangle^{v+\bar{z}}$ and $\langle\langle x\rangle\rangle \otimes \langle\langle x\rangle\rangle^v$ lie in $\mathscr{S}_g^L(1)$, this equals

$$\rho(\langle\langle x\rangle\rangle \otimes \langle\langle y,z\rangle\rangle^{v-w_1}) = \mathbf{X}(v-w_1,w_1,w_2),$$

as desired. For item 3, we have $\langle\langle x\rangle\rangle \otimes \langle\langle y^L, z\rangle\rangle^v \in \mathscr{S}_g^L(2)$, so by Lemma 2.8 we have

$$0 = \rho\left(\langle\langle x\rangle\rangle \otimes \langle\langle y^L, z\rangle\rangle^v\right) = \sum_{i=0}^{L-1} \rho\left(\langle\langle x\rangle\rangle \otimes \langle\langle y, z\rangle\rangle^{v+i\cdot\bar{y}}\right) = \sum_{i=0}^{L-1} \mathbf{X}(v + i \cdot w_1, w_1, w_2),$$

as desired.

We conclude with item 4. The curve x can be realized by a curve that splits Σ_g into two subsurfaces X_1 and X_2 with $y \in \text{Image}(\pi_1(X_1) \to \pi_1(\Sigma_g))$ and $z \in \text{Image}(\pi_1(X_2) \to \pi_1(\Sigma_g))$. Modifying x if necessary, we can assume that $w_3 \in \text{Image}(\text{H}_1(X_1; \mathbb{Z}_L) \to \text{H}_1(\Sigma_g; \mathbb{Z}_L))$. There is then a simple closed curve $w \in \text{Image}(\pi_1(X_1) \to \pi_1(\Sigma_g))$ with $\overline{w} = w_3$ such that both $\{x, w, z\}$ and $\{x, wy, z\}$ have the same intersection pattern as $\{x, y, z\}$. Using Lemma 2.8, we conclude that

$$\begin{aligned}\mathbf{X}(v, w_1 + w_3, w_2) &= \rho\left(\langle\langle x\rangle\rangle \otimes \langle\langle wy, z\rangle\rangle^v\right) = \rho\left(\langle\langle x\rangle\rangle \otimes \langle\langle w, z\rangle\rangle^{v+\bar{y}}\right) + \rho\left(\langle\langle x\rangle\rangle \otimes \langle\langle y, z\rangle\rangle^v\right) \\ &= \mathbf{X}(v + w_1, w_3, w_2) + \mathbf{X}(v, w_1, w_2),\end{aligned}$$

as desired. □

7.3 Summary

We now make the following definition. In it, the vector space $\mathscr{A}_g$ depends on L, but we will suppress this dependence to simplify our notation.

Definition 7.6. Let $\mathscr{A}_g$ be the $\mathbb{Q}$-vector space defined by generators and relations as follows. The generators are the abstract symbols $X(v, w_1, w_2)$ for all $v, w_1, w_2 \in \text{H}_1(\Sigma_g; \mathbb{Z}_L)$ such that $\{w_1, w_2\}$ is isotropic and unimodular. The relations are as follows for all $v, w_1, w_2 \in \text{H}_1(\Sigma_g; \mathbb{Z}_L)$ such that $\{w_1, w_2\}$ is isotropic and unimodular.

1. $X(v, w_1, w_2) = X(v, w_2, w_1)$
2. $X(v, -w_1, w_2) = -X(v - w_1, w_1, w_2)$
3. $\sum_{i=0}^{L-1} X(v + i \cdot w_1, w_1, w_2) = 0$
4. Let $w_3 \in \text{H}_1(\Sigma_g; \mathbb{Z}_L)$ be such that $\{w_1, w_2, w_3\}$ is unimodular, such that $i(w_2, w_3) = 0$, and such that $-1 \le i(w_1, w_3) \le 1$. Then $X(v, w_1 + w_3, w_2) = X(v, w_1, w_2) + X(v + w_1, w_3, w_2)$.

By Lemma 7.5 the map $X(v, w_1, w_2) \mapsto \mathbf{X}(v, w_1, w_2)$ induces a map $\phi : \mathscr{A}_g \to Q_g^L$.

Let $B = \{a_1, b_1, \ldots, a_g, b_g\}$ be a symplectic basis for $\text{H}_1(\Sigma_g; \mathbb{Z}_L)$. Define $V = V_1 \cup V_2 \cup V_3 \subset \mathscr{A}_g$, where the V_i are as follows.

$$\begin{aligned}V_1 &= \{X(v, s_1, s_2) \mid v \in \text{H}_1(\Sigma_g; \mathbb{Z}_L),\ s_1, s_2 \in B \text{ distinct},\ i(s_1, s_2) = 0\}, \\ V_2 &= \{X(v, s_1, s_1 + es_2) \mid v \in \text{H}_1(\Sigma_g; \mathbb{Z}_L),\ s_1, s_2 \in B \text{ distinct},\ e \in \{-1, 1\},\ i(s_1, s_2) = 0\}, \\ V_3 &= \{X(v, s_1 + es_2, s_3 + e's_4) \mid v \in \text{H}_1(\Sigma_g; \mathbb{Z}_L),\ s_1, s_2, s_3, s_4 \in B \text{ distinct},\ e, e' \in \{-1, 1\}, \\ &\qquad\qquad i(s_1, s_3) = 1,\ i(s_1 + es_2, s_3 + e's_4) = 0\}.\end{aligned}$$

We then have the following lemma.

Lemma 7.7. *The set $\phi(V)$ spans the $\mathbb{Q}$-vector space Q_g^L.*

Proof. Set $B' = \{s_1 + es_2 \mid s_1, s_2 \in B, -1 \leq e \leq 1\}$. Observe first that it is enough to show that Q_g^L is spanned by the set

$$V' = \{\mathbf{X}(v,x,y) \mid v \in \mathrm{H}_1(\Sigma_g;\mathbb{Z}_L),\, x,y \in B' \text{ distinct},\, i(x,y) = 0\}.$$

Indeed, it is easy to see that using Lemma 7.5 we can write every element of V' as a sum of elements of $\phi(V)$. For example, we have

$$\begin{aligned}\mathbf{X}(v,a_1,a_2-b_4) &= \mathbf{X}(v,a_2-b_4,a_1) = \mathbf{X}(v,a_2,a_1) + \mathbf{X}(v+a_2,-b_4,a_1)\\ &= \mathbf{X}(v,a_2,a_1) - \mathbf{X}(v+a_2-b_4,b_4,a_1).\end{aligned}$$

Call a pair of elements $\{a,b\} \subset \pi_1(\Sigma_g)$ a *toral pair* if a and b can be realized by simple closed curves A and B that only intersect at the basepoint such that a regular neighborhood of $A \cup B$ is homeomorphic to $\Sigma_{1,1}$. Observe that if $\{a,b\}$ is a toral pair, then $[a,b]$ is a simple closed genus 1 separating curve. Also, if $\{a,b\}$ is a toral pair, then $\{a^f, b^f\}$ is also a toral pair for all $f \in \pi_1(\Sigma_g)$. Indeed, the action of Mod_g^1 on $\pi_1(\Sigma_g)$ takes toral pairs to toral pairs, and by using the "point-pushing subgroup" of Mod_g^1, we can obtain any inner automorphism of $\pi_1(\Sigma_g)$.

Define

$$W = \{[a,b] \mid \{a,b\} \subset \pi_1(\Sigma_g) \text{ is a toral pair},\, \overline{a} \in B,\, \overline{b} \in B'\}.$$

We claim that W generates $[\pi_1(\Sigma_g), \pi_1(\Sigma_g)]$. Indeed, let $\overline{W} < [\pi_1(\Sigma_g), \pi_1(\Sigma_g)]$ be the subgroup generated by W. Let $X = \{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ be a standard basis for $\pi_1(\Sigma_g)$ (like in Figure 7.a) such that $\overline{\alpha}_i = a_i$ and $\overline{\beta}_i = b_i$ for $1 \leq i \leq g$. Consider $x, y \in X$ and $f \in \pi_1(\Sigma_g)$ such that $x \neq y$. It is enough to prove that $[x,y]^f \in \overline{W}$. This is trivially true if $\{x,y\} = \{\alpha_i, \beta_i\}$ for some $1 \leq i \leq g$. Now assume that $\{x,y\} \neq \{\alpha_i, \beta_i\}$ for any $1 \leq i \leq g$. There are several cases, all of which are handled similarly. We will do the case that $x = \alpha_i$ and $y = \alpha_j$ for $i < j$ and leave the others to the reader. Observe that $\{\alpha_i \beta_j^{-1}, \alpha_j\} \in W$. Since

$$[\alpha_i, \alpha_j]^f = [\alpha_i \beta_j^{-1}, \alpha_j]^{\beta_j f} [\beta_j, \alpha_j]^f,$$

we conclude that $[\alpha_i, \alpha_j]^f \in \overline{W}$, as desired.

Now consider $x \in W$. The curve x can be realized by a simple closed genus 1 separating curve. Let $S_{x,1}$ be the subsurface of Σ_g to the left of a realization of x and $S_{x,2}$ be the subsurface to the right of the same realization of x. Let i be such that the genus of $S_{x,i}$ is 1. By construction, $\mathrm{Image}(\mathrm{H}_1(S_{x,i};\mathbb{Z}_L) \to \mathrm{H}_1(\Sigma_g;\mathbb{Z}_L))$ has a symplectic basis $\{a,b\}$ with $a \in B$ and $b \in B'$. It is easy to check that this implies that if $j = i+1$ modulo 2, then $\mathrm{Image}(\mathrm{H}_1(S_{x,j};\mathbb{Z}_L) \to \mathrm{H}_1(\Sigma_g;\mathbb{Z}_L))$ has a symplectic basis consisting of elements of B'. From these facts, we can deduce that there is a basis $W_{x,1} \cup W_{x,2}$ for $\pi_1(\Sigma_g)$ with the following two properties. First, for $1 \leq j \leq 2$, if $y \in W_{x,j}$, then $\overline{y} \in B'$ and the curve y is a simple closed curve lying in $S_{x,j}$. Second, if $y \in W_{x,1}$ and $z \in W_{x,2}$, then the set $\{x,y,z\}$ has the same intersection pattern as the curves in Figure 3.c.

We conclude that $C_g^L(\mathbb{Z}) \otimes C_g^L$ is generated by the set $W_1' \cup W_2'$, where the W_i' are as follows.

$$\begin{aligned} W_1' &= \{\langle\!\langle x \rangle\!\rangle \otimes \langle\!\langle y,z \rangle\!\rangle^f \mid x \in W,\, y,z \in W_{x,j} \text{ for some } 1 \leq j \leq 2,\, f \in \mathrm{H}_1(\Sigma_g;\mathbb{Z}_L)\},\\ W_2' &= \{\langle\!\langle x \rangle\!\rangle \otimes \langle\!\langle y,z \rangle\!\rangle^f \mid x \in W,\, y \in W_{x,1},\, z \in W_{x,2},\, f \in \mathrm{H}_1(\Sigma_g;\mathbb{Z}_L)\}.\end{aligned}$$

It is clear that $W_1' \subset \mathscr{S}_g^L(1)$, so $\rho(W_1') = 0$. Also, by construction we have $\rho(w) \in V'$ for $w \in W_2'$. The lemma therefore follows from Lemma 7.1. $\square$

Let $\psi' : \mathscr{A}_g \to I_g^L$ be the composition of the projection $\phi : \mathscr{A}_g \to Q_g^L$ with the map $Q_g^L \to I_g^L$ induced by ∂. By Lemma 6.2, for a generator $X(v, w_1, w_2)$ of $\mathscr{A}_g$, we have

$$\psi'(X(v, w_1, w_2)) = [\![v]\!] - [\![v + w_1]\!] - [\![v + w_2]\!] + [\![v + w_1 + w_2]\!].$$

Now define $\mathscr{B}_g$ to be the $\mathbb{Q}$-vector space whose basis is the set of formal symbols $[\![v]\!]$ for $v \in \mathrm{H}_1(\Sigma_g; \mathbb{Z}_L)$. The vector space $\mathscr{B}_g$ depends on L, but we will suppress this to simplify our notation. We have a surjection $\mathscr{B}_g \to I_g^L$ whose kernel is the 1-dimensional subspace spanned by $\sum_{v \in \mathrm{H}_1(\Sigma_g; \mathbb{Z}_L)} [\![v]\!]$. Examining our relations, there is an induced map $\psi : \mathscr{A}_g \to \mathscr{B}_g$ with

$$\psi(X(v, w_1, w_2)) = [\![v]\!] - [\![v + w_1]\!] - [\![v + w_2]\!] + [\![v + w_1 + w_2]\!]$$

such that the diagram

$$\begin{array}{ccc} \mathscr{A}_g & \xrightarrow{\psi} & \mathscr{B}_g \\ {\scriptstyle \phi}\downarrow & & \downarrow \\ Q_g^L & \longrightarrow & I_g^L \end{array}$$

commutes. Lemma 7.7 implies that the restriction of ϕ to $\langle V \rangle$ is surjective. Lemma 4.6 asserts that the map $Q_g^L \to I_g^L$ is isomorphism. To prove this, it is enough to prove the following lemma.

Lemma 7.8. *For $g \geq 4$ and $L \geq 2$, the restriction of the map $\psi : \mathscr{A}_g \to \mathscr{B}_g$ to $\langle V \rangle$ is injective and $\mathscr{B}_g = \psi(\langle V \rangle) \oplus \langle [\![0]\!] \rangle$.*

The proof of Lemma 7.8 is contained in §8.

8 A linear algebraic dénouement

Let $B = \{a_1, b_1, \ldots, a_g, b_g\}$ and $V = V_1 \cup V_2 \cup V_3$ and $\mathscr{A}_g$ and $\mathscr{B}_g$ and ψ be as in §7. In this section, we prove Lemma 7.8. Our proof is lengthy, but the basic idea is as follows.

- Using the relations in $\mathscr{A}_g$, we will show that the set $\langle V \rangle$ is generated by a set containing $\dim \mathscr{B}_g - 1$ elements.
- By carefully examining the image of ψ, we will show that $\mathscr{B}_g = \psi(\langle V \rangle) + \langle [\![0]\!] \rangle$.

A simple dimension count will then establish the lemma.

To make the calculations a bit more palatable, we will break this down into several steps. We start with the following relations in $\mathscr{A}_g$.

Lemma 8.1. *Fix $g \geq 1$, and let $\{s_1, s_2, s_3\} \subset \mathrm{H}_1(\Sigma_g; \mathbb{Z}_L)$ be a unimodular set such that $i(s_1, s_3) = i(s_2, s_3) = 0$ and $-1 \leq i(s_1, s_2) \leq 1$. Then for all $v \in \mathrm{H}_1(\Sigma_g; \mathbb{Z}_L)$ we have the following two relations.*

$$X(v, s_2, s_3) = X(v + s_1, s_2, s_3) + X(v, s_1, s_3) - X(v + s_2, s_1, s_3),$$
$$X(v - s_1, s_2, s_3) = X(v, s_2, s_3) + X(v - s_1, s_1, s_3) - X(v - s_1 + s_2, s_1, s_3).$$

Proof. The first relation follows from the fact that $X(v, s_1 + s_2, s_3) = X(v, s_1, s_3) + X(v + s_1, s_2, s_3)$ and $X(v, s_1 + s_2, s_3) = X(v, s_2, s_3) + X(v + s_2, s_1, s_3)$. The second follows from the first via the substitution $v \mapsto v - s_1$. $\square$

Next, we eliminate the need for V_3.

Lemma 8.2. *For $g \geq 1$, we have $\langle V \rangle = \langle V_1, V_2 \rangle$.*

Proof. For $x, y \in \mathscr{A}_g$, write $x \equiv y$ if x and y are equal modulo $\langle V_1, V_2 \rangle$. Consider $X(v, w_1, w_2) \in V_3$. Our goal is to show that $X(v, w_1, w_2) \equiv 0$. For concreteness, we will do the case $X(v, w_1, w_2) = X(v, a_i + b_j, b_i + a_j)$ for some $1 \leq i, j \leq g$ with $i \neq j$; the other cases are similar.

Observe first that Lemma 8.1 implies that

$$X(v, b_i + a_j, a_i + b_j) = X(v + a_i, b_i + a_j, a_i + b_j) + X(v, a_i, a_i + b_j) - X(v + b_i + a_j, a_i, a_i + b_j).$$

Since $X(v, a_i, a_i + b_j), X(v + b_i + a_j, a_i, a_i + b_j) \in V_2$, we deduce that $X(v, a_i + b_j, b_i + a_j) \equiv X(v + a_i, a_i + b_j, b_i + a_j)$. In a similar manner, we have $X(v + a_i, a_i + b_j, b_i + a_j) \equiv X(v + (a_i + b_j), a_i + b_j, b_i + a_j)$. Iterating this, we obtain $X(v, a_i + b_j, b_i + a_j) \equiv X(v + k(a_i + b_j), a_i + b_j, b_i + a_j)$ for all $k \in \mathbb{Z}$. But this implies that

$$X(v, a_i + b_j, b_i + a_j) \equiv \frac{1}{L} \sum_{k=0}^{L-1} X(v + k(a_i + b_j), a_i + b_j, b_i + a_j) = 0,$$

as desired. □

We now determine what ψ does to V_1.

Lemma 8.3. *Fix $g \geq 1$. Set $\mathscr{B}_g^1 = \langle \{ [\![ca_i + db_i]\!] \mid c, d \in \mathbb{Z}_L,\ 1 \leq i \leq g \} \rangle$. Then the map $\psi|_{\langle V_1 \rangle}$ is injective and $\mathscr{B}_g = \psi(\langle V_1 \rangle) \oplus \mathscr{B}_g^1$.*

Proof. The proof will by induction on g. For the base case $g = 1$, the set V_1 is empty and the assertion is trivial. Assume now that $g \geq 2$ and that the lemma is true for all smaller g. Define

$$\begin{aligned} V_1^I &= \{ X(v, s_1, s_2) \in V_1 \mid s_1, s_2 \notin \{a_g, b_g\} \}, \\ V_1^A &= \{ X(v, a_g, s) \mid X(v, a_g, s) \in V_1 \}, \\ V_1^B &= \{ X(v, b_g, s) \mid X(v, b_g, s) \in V_1 \}, \end{aligned}$$

so $V = V_1^I \cup V_1^A \cup V_1^B$. The proof will consist of three steps.

Step 1. *Set $\mathscr{B}_g^2 = \langle \{ [\![ca_i + db_i + ea_g + fb_g]\!] \mid c, d, e, f \in \mathbb{Z}_L,\ 1 \leq i \leq g-1 \} \rangle$. Then the map $\psi|_{\langle V_1^I \rangle}$ is injective and $\mathscr{B}_g = \psi(\langle V_1^I \rangle) \oplus \mathscr{B}_g^2$.*

For $e, f \in \mathbb{Z}_L$, define

$$\begin{aligned} \mathscr{B}_g(e, f) &= \langle \{ [\![v + ea_g + fb_g]\!] \mid v \in \langle a_1, b_1, \ldots, a_{g-1}, b_{g-1} \rangle \} \rangle, \\ \mathscr{B}_g^2(e, f) &= \langle \{ [\![ca_i + db_i + ea_g + fb_g]\!] \mid c, d \in \mathbb{Z}_L, 1 \leq i \leq g-1 \} \rangle, \\ V_1^I(e, f) &= \{ X(v + ea_g + fb_g, s_1, s_2) \in V_1 \mid s_1, s_2 \notin \{a_g, b_g\},\ v \in \langle a_1, b_1, \ldots, a_{g-1}, b_{g-1} \rangle \}. \end{aligned}$$

Observe that

$$\mathscr{B}_g = \bigoplus_{e,f \in \mathbb{Z}_L} \mathscr{B}_g(e, f) \quad \text{and} \quad \mathscr{B}_g^2 = \bigoplus_{e,f \in \mathbb{Z}_L} \mathscr{B}_g^2(e, f) \quad \text{and} \quad V_1^I = \bigsqcup_{e,f \in \mathbb{Z}_L} V_1^I(e, f).$$

Moreover, for all $e,f\in\mathbb{Z}_L$ we have $\psi(V_1^I(e,f))\subset\mathscr{B}_g(e,f)$. We conclude that it is enough to prove that $\psi|_{\psi(\langle V_1^I(e,f)\rangle)}$ is injective and $\mathscr{B}_g(e,f)=\psi(\langle V_1^I(e,f)\rangle)\oplus\mathscr{B}_g^2(e,f)$ for all $e,f\in\mathbb{Z}_L$.

Consider $e,f\in\mathbb{Z}_L$. The map $[\![v]\!]\mapsto[\![v+ea_g+fb_g]\!]$ induces an isomorphism $\rho:\mathscr{B}_{g-1}\to\mathscr{B}_g(e,f)$ that restricts to an isomorphism $\mathscr{B}_{g-1}^1\cong\mathscr{B}_g^2(e,f)$. Let $V_{1,g-1}\subset\mathscr{A}_{g-1}$ be the $(g-1)$-dimensional analogue of V_1 and $\psi_{g-1}:\mathscr{A}_{g-1}\to\mathscr{B}_{g-1}$ be the $(g-1)$-dimensional analogue of ψ. The map $X(v,w_1,w_2)\mapsto X(v+ea_g+fb_g,w_1,w_2)$ induces a homomorphism $\mathscr{A}_{g-1}\to\mathscr{A}_g$ that restricts to a surjection $\rho':\langle V_{1,g-1}\rangle\to\langle V_1^I(e,f)\rangle$. Observe that the diagram

$$\begin{array}{ccc}\langle V_{1,g-1}\rangle & \xrightarrow{\psi_{g-1}} & \mathscr{B}_{g-1}\\ \rho'\big\downarrow & & \rho\big\downarrow\\ \langle V_1^I(e,f)\rangle & \xrightarrow{\psi} & \mathscr{B}_g(e,f)\end{array}$$

commutes. By the induction hypothesis, $\psi_{g-1}|_{\langle V_{1,g-1}\rangle}$ is injective and $\mathscr{B}_{g-1}=\psi_{g-1}(\langle V_{1,g-1}\rangle)\oplus\mathscr{B}_{g-1}^1$. We conclude that ρ' is injective and thus an isomorphism. Moreover, $\psi|_{\langle V_1^I(e,f)\rangle}$ is injective and $\mathscr{B}_g(e,f)=\psi(\langle V_1^I(e,f)\rangle)\oplus\mathscr{B}_g^2(e,f)$, as desired.

Step 2. *Set $\mathscr{B}_g^3=\langle\{[\![ca_i+db_i+fb_g]\!],[\![ea_g+fb_g]\!]\mid c,d,e,f\in\mathbb{Z}_L,\ 1\le i\le g-1\}\rangle$. Then the map $\psi|_{\langle V_1^I,V_1^A\rangle}$ is injective and $\mathscr{B}_g=\psi(\langle V_1^I,V_1^A\rangle)\oplus\mathscr{B}_g^3$.*

Define

$$V_1^{A,1}=\{X(ca_i+db_i+ea_g+fb_g,a_g,s)\mid 1\le i\le g-1,\ c,d,e,f\in\mathbb{Z}_L,\ s\in\{a_i,b_i\}\}\subset V_1^A.$$

We claim that $\langle V_1^I,V_1^A\rangle=\langle V_1^I,V_1^{A,1}\rangle$. Indeed, consider $X(w,a_g,s)\in V_1^A$. By Lemma 8.1, for any $s'\in\{a_1,b_1,\ldots,a_{g-1},b_{g-1}\}$ with $s'\neq s$ and $i(s,s')=0$, we have

$$X(w-s',a_g,s)=X(w,a_g,s)+X(w-s',s',s)-X(w-s'+a_g,s',s).$$

Hence modulo $\langle V_1^I\rangle$, we have $X(w,a_g,s)$ equal to $X(w-s',a_g,s)$. Iterating this, modulo $\langle V_1^I\rangle$ we have $X(w,a_g,s)$ equal to an element of $V_1^{A,1}$, as desired.

For $x\in\mathbb{Z}_L$, we will denote by $|x|$ the unique integer representing x with $0\le|x|<L-1$. Noting that $\psi(V_1^{A,1})\subset\mathscr{B}_g^2$, we claim that $\mathscr{B}_g^2=\psi(\langle V_1^{A,1}\rangle)+\mathscr{B}_g^3$. Indeed, assume that there is some $[\![ca_i+db_i+ea_g+fb_g]\!]\in\mathscr{B}_g^2$ that is not in $\psi(\langle V_1^{A,1}\rangle)+\mathscr{B}_g^3$. Choose $[\![ca_i+db_i+ea_g+fb_g]\!]$ such that $|c|+|d|+|e|$ is minimal among elements with this property. By assumption we must have $|e|$ and one of $|c|$ or $|d|$ (say $|c|$) nonzero. Setting $w'=ca_i+db_i+ea_g+fb_g$, we then have $X(w'-a_i-a_g,a_g,a_i)\in V_1^{A,1}$ and

$$[\![w']\!]-\psi(X(w'-a_i-a_g,a_g,a_i))=[\![w'-a_i]\!]+[\![w'-a_g]\!]-[\![w'-a_i-a_g]\!].$$

We conclude that one of $[\![w'-a_i]\!]$, $[\![w'-a_g]\!]$, or $[\![w'-a_i-a_g]\!]$ is not in $\psi(\langle V_1^{A,1}\rangle)+\mathscr{B}_g^3$, contradicting the minimality of $|c|+|d|+|e|$.

Now define

$$\begin{aligned}V_1^{A,2}=&\{X(ca_i+db_i+ea_g+fb_g,a_g,a_i)\mid 1\le i\le g-1,\ c,d,e,f\in\mathbb{Z}_L\}\\ &\cup\{X(db_i+ea_g+fb_g,a_g,b_i)\mid 1\le i\le g-1,\ d,e,f\in\mathbb{Z}_L\}\subset V_1^{A,1},\\ V_1^{A,3}=&\{X(v,a_g,s)\in V_1^{A,2}\mid \text{the } s \text{ and } a_g\text{-coordinates of } v \text{ do not equal } L-1\}\subset V_1^{A,2}.\end{aligned}$$

We will prove that $\langle V_1^{A,3}\rangle = \langle V_1^{A,1}\rangle$. Using the third relation in the definition of $\mathscr{A}_g$, we see that $\langle V_1^{A,3}\rangle = \langle V_1^{A,2}\rangle$. It is thus enough to prove that $\langle V_1^{A,1}\rangle = \langle V_1^{A,2}\rangle$. An element of $V_1^{A,1} \setminus V_1^{A,2}$ is of the form $X(w'', a_g, b_i)$. Lemma 8.1 says that

$$X(w''-a_i, b_i, a_g) = X(w'', b_i, a_g) + X(w''-a_i, a_i, a_g) - X(w''-a_i+b_i, a_i, a_g).$$

Iterating this, we conclude that modulo $\langle V_1^{A,2}\rangle$, we have $X(w'', a_g, b_i)$ equal to an element of $V_1^{A,2}$, as desired.

We deduce from the above two paragraphs that $\mathscr{B}_g^2 = \psi(\langle V_1^{A,3}\rangle) + \mathscr{B}_g^3$. Since $V_1^{A,3}$ contains

$$\begin{aligned}(g-1)(L^2(L-1)^2 + L(L-1)^2) &= ((g-1)(L^2-1)L^2 + L^2) - ((g-1)(L^2-1)L + L^2) \\ &= \dim(\mathscr{B}_g^2) - \dim(\mathscr{B}_g^3)\end{aligned}$$

elements, we obtain that $\psi|_{\langle V_1^{A,3}\rangle}$ is injective and $\mathscr{B}_g^2 = \psi(\langle V_1^{A,3}\rangle) \oplus \mathscr{B}_g^3$. By Step 1, the fact that $\langle V_1^I, V_1^A\rangle = \langle V_1^I, V_1^{A,1}\rangle$, and the fact that $\langle V_1^{A,3}\rangle = \langle V_1^{A,1}\rangle$, we conclude that $\psi|_{\langle V_1^I, V_1^A\rangle}$ is injective and $\mathscr{B}_g = \psi(\langle V_1^I, V_1^A\rangle) \oplus \mathscr{B}_g^3$, as desired.

Step 3. *Recall that $\mathscr{B}_g^1 = \langle\{[\![ca_i + db_i]\!] \mid c,d \in \mathbb{Z}_L,\ 1 \le i \le g\}\rangle$. The map $\psi|_{\langle V_1^I, V_1^A, V_1^B\rangle}$ is injective and $\mathscr{B}_g = \psi(\langle V_1^I, V_1^A, V_1^B\rangle) \oplus \mathscr{B}_g^1$.*

The argument for this step is very similar to the argument in Step 2, so we only sketch it. Define

$$\begin{aligned}
V_1^{B,1} =& \{X(ca_i + db_i + fb_g, b_g, s) \mid 1 \le i \le g-1,\ c,d,f \in \mathbb{Z}_L,\ s \in \{a_i, b_i\}\} \subset V_1^B, \\
V_1^{B,2} =& \{X(ca_i + db_i + fb_g, b_g, a_i) \mid 1 \le i \le g-1,\ c,d,f \in \mathbb{Z}_L\} \\
&\cup \{X(db_i + fb_g, b_g, b_i) \mid 1 \le i \le g-1,\ d,f \in \mathbb{Z}_L\} \subset V_1^{B,1}, \\
V_1^{B,3} =& \{X(v, b_g, s) \in V_1^{B,2} \mid \text{the } s \text{ and } b_g\text{-coordinates of } v \text{ do not equal } L-1\} \subset V_1^{B,2}.
\end{aligned}$$

Noting that $\psi(V_1^{B,1}) \subset \mathscr{B}_g^3$, arguments similar to the arguments in Step 2 show that $\langle V_1^I, V_1^A, V_1^B\rangle = \langle V_1^I, V_1^A, V_1^{B,1}\rangle$, that $\mathscr{B}_g^3 = \psi(\langle V_1^{B,1}\rangle) + \mathscr{B}_g^1$, that $\langle V_1^{B,1}\rangle = \langle V_1^{B,2}\rangle$, and that $\langle V_1^{B,2}\rangle = \langle V_1^{B,3}\rangle$.

We deduce that $\mathscr{B}_g^3 = \psi(\langle V_1^{B,3}\rangle) + \mathscr{B}_g^1$. Since $V_1^{B,3}$ contains

$$\begin{aligned}(g-1)((L-1)^2 L + (L-1)^2) &= ((g-1)(L^2-1)L + L^2) - (g(L^2-1)+1) \\ &= \dim(\mathscr{B}_g^3) - \dim(\mathscr{B}_g^1)\end{aligned}$$

elements, we obtain that $\psi|_{\langle V_1^{B,3}\rangle}$ is injective and $\mathscr{B}_g^3 = \psi(\langle V_1^{B,3}\rangle) \oplus \mathscr{B}_g^1$. By Step 2, the fact that $\langle V_1^I, V_1^A, V_1^B\rangle = \langle V_1^I, V_1^A, V_1^{B,1}\rangle$, and the fact that $\langle V_1^{B,3}\rangle = \langle V_1^{B_1}\rangle$, we conclude that $\psi|_{\langle V_1^I, V_1^A, V_1^B\rangle}$ is injective and $\mathscr{B}_g = \psi(\langle V_1^I, V_1^A, V_1^B\rangle) \oplus \mathscr{B}_g^1$, as desired. □

Our final lemma is a further relation in $\mathscr{A}_g$.

Lemma 8.4. *Let $\{a_1', b_1', a_2', b_2'\}$ be a unimodular subset of $\mathrm{H}_1(\Sigma_g;\mathbb{Z}_L)$ with $i(a_1', b_1') = i(a_2', b_2') = 1$ and $i(a_1', a_2') = i(a_1', b_2') = i(b_1', a_2') = i(b_1', b_2') = 0$. Then for all $v \in \mathrm{H}_1(\Sigma_g;\mathbb{Z}_L)$ we have*

$$\begin{aligned}&X(v, a_1', a_2') - X(v+b_1', a_1', a_2') - X(v+b_2', a_1', a_2') + X(v+b_1'+b_2', a_1', a_2') \\ &\quad = X(v, b_1', b_2') - X(v+a_1', b_1', b_2') - X(v+a_2', b_1', b_2') + X(v+a_1'+a_2', b_1', b_2')\end{aligned}$$

Proof. The group $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts on $\mathscr{A}_g$, and there exists some $f \in \mathrm{Sp}_{2g}(\mathbb{Z})$ such that $f(a_i') = a_i$ and $f(b_i') = b_i$ for $i = 1,2$. We can therefore assume that $a_i' = a_i$ and $b_i' = b_i$ for $i = 1,2$. But an easy calculation shows that ψ takes both sides of our relation to the same element of $\mathscr{B}_g$, so the lemma follows from Lemma 8.3. □

We can now prove Lemma 7.8.

Proof of Lemma 7.8. Define $\mathscr{A}_g' = \mathscr{A}_g/\langle V_1 \rangle$ and $\mathscr{B}_g' = \mathscr{B}_g/\psi(\langle V_1 \rangle)$. We have an induced map $\psi' : \mathscr{A}_g' \to \mathscr{B}_g'$. Using the direct sum decomposition of Lemma 8.3, we will identify $\mathscr{B}_g'$ with the subspace $\langle \{ [\![ca_i + db_i]\!] \mid c,d \in \mathbb{Z}_L,\, 1 \leq i \leq g\} \rangle$ of $\mathscr{B}_g$. Letting $V_2' \subset \mathscr{A}_g'$ be the image of $V_2 \subset \mathscr{A}_g$, Lemmas 8.2 and 8.3 say that it is enough to prove that $\psi'|_{\langle V_2' \rangle}$ is injective and $\mathscr{B}_g' = \psi'(\langle V_2' \rangle) \oplus \langle [\![0]\!] \rangle$.

Let $\phi : \mathscr{A}_g \to \mathscr{A}_g'$ be the projection. The proof will require seven claims. It follows the same pattern as Steps 2 and 3 of the proof of Lemma 8.3. In Claims 1–3 and 5–6, we will obtain a "minimal" size generating set for $\langle V_2' \rangle$. In Claims 4 and 7, we will show that $\mathscr{B}_g' = \psi'(\langle V_2' \rangle) + \langle [\![0]\!] \rangle$. A dimension count will then establish the lemma.

Claim 1. *Let $s, s_1, s_2 \in B$ satisfy $s \neq s_1, s_2$ and $i(s,s_1) = i(s,s_2) = 0$. Then for all $v \in \mathrm{H}_1(\Sigma_g;\mathbb{Z}_L)$ and $e_1, e_2 \in \{-1,1\}$, we have $\phi(X(v,s,s+e_1s_1)) = \phi(X(v,s,s+e_2s_2))$.*

Proof of Claim. For $1 \leq i \leq 2$, using the second relation in the definition of $\mathscr{A}_g$ we get that

$$X(v+e_1s_1+s, e_2s_2, s), X(v+e_2s_2+s, e_1s_1, s) \in \langle V_1 \rangle.$$

Hence

$$\phi(X(v, e_1s_1+e_2s_2+s, s)) = \phi(X(v, e_1s_1+s, s) + X(v+e_1s_1+s, e_2s_2, s)) = \phi(X(v,s,s+e_1s_1))$$

and

$$\phi(X(v, e_1s_1+e_2s_2+s, s)) = \phi(X(v, e_2s_2+s, s) + X(v+e_2s_2+s, e_1s_1, s)) = \phi(X(v,s,s+e_2s_2)).$$

The claim follows. □

In light of Claim 1, we will denote by $Y(v,s)$ the image in V_2' of $X(v,s,s+e's')$, where $e' \in \{-1,1\}$ and $s' \in B$ are arbitrary elements such that $X(v,s,s+e's') \in V_2$.

Claim 2. *Consider $Y(v,s) \in V_2'$. Pick $1 \leq i \leq g$ such that $s \in \{a_i, b_i\}$. Write $v = v_1 + v_2$ with $v_1 \in \langle a_i, b_i \rangle$ and $v_2 \in \langle \{a_j,\, b_j \mid j \neq i\} \rangle$. Then $Y(v,s) = Y(v_1,s)$.*

Proof of Claim. Consider $s' \in \{a_j,\, b_j \mid j \neq i\}$. It is enough to show that $Y(v-s',s) = Y(v,s)$. Pick $s'' \in \{a_j,\, b_j \mid j \neq i\}$ such that $s'' \neq s'$ and $i(s',s'') = 0$. Observe that $Y(v,s) = \phi(X(v,s,s+s''))$ and $Y(v-s',s) = \phi(X(v-s',s,s+s''))$. Lemma 8.1 says that

$$X(v-s', s, s+s'') = X(v,s,s+s'') + X(v-s', s', s+s'') - X(v-s'+s, s', s+s''). \tag{9}$$

For w equal to $v-s'$ or $v-s'+s$, we have

$$X(w, s', s+s'') = X(w, s+s'', s') = X(w,s,s') + X(w+s, s'', s') \in \langle V_1 \rangle.$$

Applying ϕ to both sides of (9), we thus obtain that $Y(v,s) = Y(v-s',s)$, as desired. □

Claim 3. *For all $1 \le i \le g$ and $v \in \langle a_i, b_i \rangle$, we have*

$$Y(v,a_i) - 2Y(v+b_i,a_i) + Y(v+2b_i,a_i) = Y(v,b_i) - 2Y(v+a_i,b_i) + Y(v+2a_i,b_i).$$

Proof of Claim. Pick $1 \le j < k \le g$ such that $i \ne j,k$. Lemma 8.4 applied with $(a'_1,b'_1,a'_2,b'_2) = (a_i+a_k, b_i-b_j, a_i+a_j, b_i-b_k)$ says that

$$\begin{aligned} &X(v,a_i+a_k,a_i+a_j) - X(v+b_i-b_j,a_i+a_k,a_i+a_j) - X(v+b_i-b_k,a_i+a_k,a_i+a_j) \\ &\quad + X(v+b_i-b_j+b_i-b_k,a_i+a_k,a_i+a_j) \\ =&X(v,b_i-b_j,b_i-b_k) - X(v+a_i+a_k,b_i-b_j,b_i-b_k) - X(v+a_i+a_j,b_i-b_j,b_i-b_k) \\ &\quad + X(v+a_i+a_k+a_i+a_j,b_i-b_j,b_i-b_k). \end{aligned} \tag{10}$$

Since $X(v+a_i,a_k,a_i+a_j) \in \langle V_1 \rangle$, we have that

$$\phi(X(v,a_i+a_k,a_i+a_j)) = \phi(X(v,a_i,a_i+a_j) + X(v+a_i,a_k,a_i+a_j)) = Y(v,a_i).$$

Similarly, we have $\phi(X(v+b_i-b_j,a_i+a_k,a_i+a_j)) = Y(v+b_i-b_j,a_i)$. By Claim 2, this equals $Y(v+b_i,a_i)$. Continuing in this manner, we deduce that ϕ maps (10) to the desired relation between the $Y(\cdot,\cdot)$. □

For the next claim, recall that we are using Lemma 8.3 to identify $\mathscr{B}'_g = \mathscr{B}_g/\psi(\langle V_1 \rangle)$ with the subspace $\langle \{ [\![ca_i+db_i]\!] \mid c,d \in \mathbb{Z}_L, 1 \le i \le g\} \rangle$ of $\mathscr{B}_g$.

Claim 4. *For some $1 \le i \le g$, let $s \in \{a_i,b_i\}$ and $v \in \langle a_i,b_i \rangle$. Then $\psi'(Y(v,s)) = [\![v]\!] - 2[\![v+s]\!] + [\![v+2s]\!]$.*

Proof of Claim. Let $\rho : \mathscr{B}_g \to \mathscr{B}'_g$ be the projection. Pick $1 \le j \le g$ such that $j \ne i$. Observe that $Y(v,s) = \phi(X(v,s,s-a_j))$ and $X(v+s-a_j,a_j,s) \in V_1$. Thus $\psi'(Y(v,s))$ equals

$$\begin{aligned} \rho(\psi(X(v,s-a_j,s))) &= \rho(\psi(X(v,s-a_j,s) + X(v+s-a_j,a_j,s))) \\ &= \rho(([\![v]\!] - [\![v+s-a_j]\!] - [\![v+s]\!] + [\![v+2s-a_j]\!]) \\ &\qquad + ([\![v+s-a_j]\!] - [\![v+s]\!] - [\![v+2s-a_j]\!] + [\![v+2s]\!]) \\ &= \rho([\![v]\!] - 2[\![v+s]\!] + [\![v+2s]\!]). \end{aligned}$$

Since $v, v+s, v+2s \in \langle a_i,b_i \rangle$, this equals $[\![v]\!] - 2[\![v+2]\!] + [\![v+2s]\!]$, as desired. □

For some $1 \le i \le g$, consider $s \in \{a_i,b_i\}$ and $v \in \langle a_i,b_i \rangle$. Making use of Claim 4, an easy induction establishes that for $n \ge 1$, we have

$$\psi'(\sum_{k=1}^{n} k \cdot Y(v+(k-1)s,s)) = [\![v]\!] - (n+1)[\![v+ns]\!] + n[\![v+(n+1)s]\!].$$

In particular, setting $Z(v,s) = \sum_{k=1}^{L} k \cdot Y(v+(k-1)s,s)$, we have

$$\psi'(Z(v,s)) = L[\![v+s]\!] - L[\![v]\!]. \tag{11}$$

We now prove the following.

Claim 5. *For all* $1 \le i \le g$ *and* $v \in \langle a_i, b_i \rangle$, *we have*

$$Z(v,a_i) - 2Z(v+b_i,a_i) + Z(v+2b_i,a_i) = L \cdot Y(v+a_i,b_i) - L \cdot Y(v,b_i).$$

Proof of Claim. Observe that $Z(v,a_i) - 2Z(v+b_i,a_i) + Z(v+2b_i,a_i)$ equals

$$\sum_{k=1}^{L} k \cdot (Y(v+(k-1)a_i,a_i) - 2Y(v+(k-1)a_i+b_i,a_i) + Y(v+(k-1)a_i+2b_i,a_i)).$$

By Claim 3, this equals

$$\sum_{k=1}^{L} k \cdot (Y(v+(k-1)a_i,b_i) - 2Y(v+ka_i,b_i) + Y(v+(k+1)a_i,b_i)).$$

An argument similar to the argument used to calculate ψ' of $Z(\cdot,\cdot)$ then shows that this equals $L \cdot Y(v+a_i,b_i) - L \cdot Y(v,b_i)$, and we are done. $\square$

Claim 6. *We have*

$$\begin{aligned}\langle V_2' \rangle = &\langle \{Y(ca_i+db_i,a_i) \mid 1 \le i \le g,\ c,d \in \mathbb{Z}_L,\ c \ne L-1\} \\ &\cup \{Y(db_i,b_i) \mid 1 \le i \le g,\ d \in \mathbb{Z}_L,\ d \ne L-1\} \rangle.\end{aligned}$$

Proof of Claim. Claim 2 implies that

$$\langle V_2' \rangle = \{Y(v,s) \mid v \in \langle a_i,b_i \rangle \text{ and } s \in \{a_i,b_i\} \text{ for some } 1 \le i \le g\}. \tag{12}$$

If $v \in \{a_i,b_i\}$ for some $1 \le i \le g$, then

$$Z(v,a_i) \in \langle \{Y(ca_i+db_i,a_i) \mid 1 \le i \le g,\ c,d \in \mathbb{Z}_L\} \rangle.$$

We can therefore use the relation in Claim 5 to reduce (12) to

$$\begin{aligned}\langle V_2' \rangle = &\langle \{Y(ca_i+db_i,a_i) \mid 1 \le i \le g,\ c,d \in \mathbb{Z}_L\} \\ &\cup \{Y(db_i,b_i) \mid 1 \le i \le g,\ d \in \mathbb{Z}_L\} \rangle.\end{aligned} \tag{13}$$

Finally, if $v \in \langle a_i,b_i \rangle$ and $s \in \{a_i,b_i\}$ for some $1 \le i \le g$, then from the third relation in the definition of $\mathscr{A}_g$, we obtain the relation $\sum_{k=0}^{L-1} Y(v+k \cdot s,s) = 0$ in $\mathscr{A}_g'$. This allows us to reduce (13) to the claimed generating set, and we are done. $\square$

Claim 7. $\mathscr{B}_g' = \langle \psi'(V_2') \rangle + \langle [\![0]\!] \rangle$.

Proof of Claim. Consider $c,d \ge 0$ with $(c,d) \ne 0$. With the convention that an empty sum of abelian group elements is the zero element, we can use (11) to get that

$$\begin{aligned}\frac{1}{L}\psi'(\sum_{j=1}^{c} Z(ja_i,a_i) + \sum_{k=1}^{d} Z(ca_i+kb_i,b_i)) &= ([\![ca_i]\!] - [\![0]\!]) + ([\![ca_i+db_i]\!] - [\![ca_i]\!]) \\ &= [\![ca_i+db_i]\!] - [\![0]\!].\end{aligned}$$

Here the first equality follows from the fact that the indicated sums become telescoping sums after applying ψ'. The claim follows. $\square$

Observe now that the generating set for $\langle V_2' \rangle$ given by Claim 6 has

$$g(L(L-1)+L-1) = g(L^2-1) = \dim(\mathscr{B}_g') - 1$$

elements, so by Claim 7 we have that $\psi'|_{\langle V_2' \rangle}$ is injective and $\mathscr{B}_g' = \langle \psi'(V_2') \rangle \oplus \langle [\![0]\!] \rangle$, as desired. $\square$

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