

## 1 Introduction

I am interested in questions about low-dimensional topology and geometric group theory centered around the mapping class group of a surface. Denoting a closed orientable genus  $g$  surface by  $\Sigma_g$ , the *mapping class group*  $\text{Mod}_g$  of  $\Sigma_g$  is the group of homotopy classes of orientation-preserving homeomorphisms of  $\Sigma_g$ . This group is one of the most ubiquitous objects in mathematics, playing an important role in areas ranging from the moduli space of algebraic curves to 3- and 4-dimensional topology and geometry. Because of this,  $\text{Mod}_g$  has been the object of intensive study over the last 100 years, beginning with the work of Dehn and Nielsen in the early 20<sup>th</sup> century and continuing with the work of Thurston, Birman, and many others.

My work uses techniques and ideas from combinatorial group theory, algebraic topology, group cohomology, and algebraic geometry. This last topic deserves special mention. The group  $\text{Mod}_g$  is the orbifold fundamental group of the moduli space  $\mathcal{M}_g$  of genus  $g$  algebraic curves, and one can often translate questions about  $\text{Mod}_g$  into questions about  $\mathcal{M}_g$  (and vice-versa). This interplay has long proved beneficial for both subjects. A second major theme which permeates much of my work is the study of various simplicial complexes upon which  $\text{Mod}_g$  acts. The most famous is the *complex of curves*, which was introduced by Harvey [22]. This is the simplicial complex whose  $k$ -simplices are sets  $\{\gamma_0, \dots, \gamma_k\}$  of homotopy classes of non-nullhomotopic simple closed curves on  $\Sigma_g$  that can be realized disjointly. It can be viewed as an analogue for  $\text{Mod}_g$  of the rational Tits building of an arithmetic subgroup of a semisimple Lie group.

My current research has three main directions. The first is the study of finite-index subgroups of  $\text{Mod}_g$ , or equivalently finite covers of  $\mathcal{M}_g$ . A guiding principle here is the much-studied analogy between  $\text{Mod}_g$  and lattices in semisimple Lie groups. One evident connection between these two areas is the representation  $\text{Mod}_g \rightarrow \text{Sp}_{2g}(\mathbb{Z})$  arising from the action of  $\text{Mod}_g$  on  $H_1(\Sigma_g; \mathbb{Z}) \cong \mathbb{Z}^{2g}$  (this representation lands in  $\text{Sp}_{2g}(\mathbb{Z})$  because the action preserves the algebraic intersection form). The second major focus of my research is the *Torelli group*  $\mathcal{I}_g$ , which is the kernel of this symplectic representation. Since  $\text{Sp}_{2g}(\mathbb{Z})$  is the orbifold fundamental group of the moduli space  $\mathcal{A}_g$  of principally polarized abelian varieties, one can view  $\mathcal{I}_g$  as measuring the difference between  $\mathcal{M}_g$  and  $\mathcal{A}_g$ . The third major focus of my research centers around the automorphism group of the free group  $F_n$  on  $n$  letters. There is a long-standing analogy between  $\text{Mod}_g$  and  $\text{Aut}(F_n)$ , and my research focuses on importing ideas centered around the Torelli group and the complex of curves to the setting of  $\text{Aut}(F_n)$ .

## 2 Finite covers of the moduli space of curves

We begin by discussing some results and questions about finite-index subgroups of  $\text{Mod}_g$ , or equivalently about finite covers of  $\mathcal{M}_g$ .

**Borel stability.** For  $L \geq 2$ , let  $\text{Sp}_{2g}(\mathbb{Z}, L)$  denote the *level  $L$  subgroup* of  $\text{Sp}_{2g}(\mathbb{Z})$ , i.e. the kernel of the natural map  $\text{Sp}_{2g}(\mathbb{Z}) \rightarrow \text{Sp}_{2g}(\mathbb{Z}/L\mathbb{Z})$ . A deep theorem of Borel [9] says that for a fixed  $k$ , we have

$$H_k(\text{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Q}) \cong H_k(\text{Sp}_{2g}(\mathbb{Z}); \mathbb{Q})$$

for  $g \gg 0$ . Denote by  $\text{Mod}_g(L)$  the kernel of the composition

$$\text{Mod}_g \rightarrow \text{Sp}_{2g}(\mathbb{Z}) \rightarrow \text{Sp}_{2g}(\mathbb{Z}/L\mathbb{Z}).$$

The group  $\text{Mod}_g(L)$  is the kernel of the action of  $\text{Mod}_g$  on  $H_1(\Sigma_g; \mathbb{Z}/L\mathbb{Z})$ . We make the following conjecture.

**Conjecture 2.1** (Borel stability for  $\text{Mod}_g$ ). *Fix  $k \geq 1$ . There exists some  $N \geq 1$  such that if  $g \geq N$  and  $L \geq 2$ , then  $H_k(\text{Mod}_g(L); \mathbb{Q}) \cong H_k(\text{Mod}_g; \mathbb{Q})$ .*

Two special cases of this conjecture are known. The case  $k = 1$  was proven by Hain in [19]. Answering a question of Hain, I proved the following.

**Theorem 2.2** (Putman, [46]). *We have  $H_2(\text{Mod}_g(L); \mathbb{Q}) \cong H_2(\text{Mod}_g; \mathbb{Q}) \cong \mathbb{Q}$  for  $g \geq 5$  and  $L \geq 2$ .*

In [46], I also showed that Conjecture 2.1 would follow from a weak form of homological stability for  $\text{Mod}_g(L)$ , similar to the well-known homological stability theorem for  $\text{Mod}_g$  due to Harer [21]. I hope to generalize the arguments of [46] to prove this weak homological stability result.

**Picard groups of moduli space.** Theorem 2.2 has strong implications for the geometry of moduli space. The finite cover of  $\mathcal{M}_g$  associated to  $\text{Mod}_g(L)$  is the moduli space  $\mathcal{M}_g(L)$  of Riemann surfaces with level  $L$  structures, i.e. fixed bases for  $H_1(\Sigma_g; \mathbb{Z}/L\mathbb{Z})$ . Recall that the Picard group  $\text{Pic}(X)$  of a quasiprojective variety  $X$  is the group of isomorphism classes of algebraic line bundles on  $X$ . Mumford [39] proved that the first Chern class map gives an isomorphism  $\text{Pic}(\mathcal{M}_g) \cong H^2(\text{Mod}_g; \mathbb{Z})$ , and later Harer [20] proved that

$$\text{Pic}(\mathcal{M}_g) \cong H^2(\text{Mod}_g; \mathbb{Z}) \cong \mathbb{Z}$$

for  $g \geq 5$ . In [19], Hain imitated Mumford's argument to show that there is an injection  $c_1 : \text{Pic}(\mathcal{M}_g(L)) \hookrightarrow H^2(\text{Mod}_g(L); \mathbb{Z})$ . Using Theorem 2.2, it is easy to see that in fact  $c_1$  is an isomorphism, so we obtain the following corollary.

**Corollary 2.3.** *The abelian group  $\text{Pic}(\mathcal{M}_g(L))$  has rank 1 for  $g \geq 5$  and  $L \geq 2$ .*

In fact, we can say more. The universal coefficients theorem for  $H^2(\text{Mod}_g(L); \mathbb{Z})$  gives a short exact sequence

$$0 \longrightarrow \text{Hom}(H_1(\text{Mod}_g(L); \mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \longrightarrow \text{Pic}(\mathcal{M}_g(L)) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

The line bundles associated to the torsion subgroup  $\text{Hom}(H_1(\text{Mod}_g(L); \mathbb{Z}), \mathbb{Q}/\mathbb{Z})$  are the flat line bundles, and in [50] I calculated this group under the assumption that  $4 \nmid L$  (partial previous results along these lines can be found in [40, 45, 52]). As for the  $\mathbb{Z}$  term, let  $\mathcal{L}_g \in \text{Pic}(\mathcal{M}_g) \cong \mathbb{Z}$  be a generator. A theorem of Arbarello-Cornalba [3] says that  $\mathcal{L}_g$  can be taken to be the determinantal bundle of the Hodge bundle. Let  $\mathcal{L}_g(L) \in \text{Pic}(\mathcal{M}_g(L))$  be the pullback of  $\mathcal{L}_g$ . Farkas [16] proved that if  $L$  is even, then  $\mathcal{L}_g(L)$  is divisible by 4. I proved that this is the best you can do if  $4 \nmid L$ .

**Theorem 2.4** (Putman, [50]). *For  $g \geq 5$  and  $L \geq 2$  such that  $4 \nmid L$ , the group  $\text{Pic}(\mathcal{M}_g(L))$  is generated modulo torsion by  $\frac{1}{n}\mathcal{L}_g(L)$ , where  $n = 1$  if  $L$  is odd and  $n = 4$  if  $L$  is even.*

*Remark.* In [50], I also proved an analogue of Theorem 2.4 for the moduli space  $\mathcal{A}_g$  of principally polarized abelian varieties. Letting  $\overline{\mathcal{L}}_g \in \text{Pic}(\mathcal{A}_g) \cong \mathbb{Z}$  be a generator, holomorphic sections of the pullback of  $k \cdot \overline{\mathcal{L}}_g$  to the moduli space of principally polarized abelian varieties with level  $L$  structures are Siegel modular forms of weight  $k$  and level  $L$ . In this context, my result strengthens a theorem of Deligne [13] (see below).

Proving the following conjecture would complete this picture.

**Conjecture 2.5.** *The condition  $4 \nmid L$  in Theorem 2.4 is unnecessary.*

In the proof of Theorem 2.4, the condition  $4 \nmid L$  is used in the proofs of several lemmas so that some known facts about Chevalley groups can be applied that are false in characteristic 2. For instance, the symplectic Lie algebra over  $\mathbb{Z}/p\mathbb{Z}$  is self-dual for odd primes  $p$ , but is not self-dual for  $p = 2$ . However, computer calculations indicate that the needed lemmas are still true for all  $L$ , though I suspect that the proofs will be quite a bit more subtle.

**Divisibility in higher covers.** In [13], Deligne proved that the universal central extension of  $\mathrm{Sp}_{2g}(\mathbb{Z})$  is not residually finite. By standard methods, this is equivalent to the existence of some  $N \geq 1$  with the following property. Recall that  $\mathcal{A}_g$  is the moduli space of principally polarized abelian varieties and  $\overline{\mathcal{L}}_g \in \mathrm{Pic}(\mathcal{A}_g) \cong \mathbb{Z}$  is a standard generator. Then the pullback of  $\overline{\mathcal{L}}_g$  to the Picard group of any finite cover of  $\mathcal{A}_g$  is divisible by at most  $N$ . In fact, Deligne showed that we can take  $N = 2$ .

Very recently, Funar [17] used quantum representations of  $\mathrm{Mod}_g$  to show that the universal central extension of  $\mathrm{Mod}_g$  is residually finite. Translating this into the language of moduli space, this implies that for all  $n \geq 1$ , there exists a finite cover  $X$  of  $\mathcal{M}_g$  such that the pullback of  $\mathcal{L}_g \in \mathrm{Pic}(\mathcal{M}_g)$  to  $\mathrm{Pic}(X)$  has an  $n^{\mathrm{th}}$  root. This suggests the following question.

**Question 2.6.** *Is there an algebro-geometric description of these finite covers and roots?*

As was discussed above, Farkas [16] showed that the pullback  $\mathcal{L}_g(L) \in \mathrm{Pic}(\mathcal{M}_g(L))$  of  $\mathcal{L}_g \in \mathrm{Pic}(\mathcal{M}_g)$  has a fourth root  $\omega$  if  $L$  is even. To indicate the sort of answer I am looking for in Question 2.6, I will describe Farkas's beautiful construction. Let  $S_g^+$  be the moduli space of curves with even spin structures, i.e. the set of pairs  $(\Sigma, \mu)$ , where  $\Sigma$  is a Riemann surface and  $\mu$  is an even theta characteristic. This is a finite cover of  $\mathcal{M}_g$  that is finitely covered by  $\mathcal{M}_g(L)$  for even  $L$ . Then  $\omega \in \mathrm{Pic}(\mathcal{M}_g(L))$  is the pullback of the line bundle in  $\mathrm{Pic}(S_g^+)$  associated to the divisor consisting of  $(\Sigma, \mu) \in S_g^+$  such that the even theta characteristic  $\mu$  has a non-zero holomorphic section.

**Virtual surjection onto  $\mathbb{Z}$ .** A standard first consequence of Kazhdan's property (T) is that lattices in higher rank semisimple Lie groups (like  $\mathrm{Sp}_{2g}(\mathbb{Z})$  for  $g \geq 2$ ) do not possess finite-index subgroups that surject onto  $\mathbb{Z}$  (see, e.g., [53, Theorems 7.1.4 and 7.1.7]). In other words, if  $\Gamma < \mathrm{Sp}_{2g}(\mathbb{Z})$  is a finite-index subgroup, then  $H_1(\Gamma; \mathbb{Q}) = 0$ . Though Anderson [1] has recently shown that  $\mathrm{Mod}_g$  does not satisfy property (T), the following conjecture of Ivanov [23] is still open.

**Conjecture 2.7.** *For  $g \geq 3$ , let  $\Gamma < \mathrm{Mod}_g$  be a finite-index subgroup. Then  $H_1(\Gamma; \mathbb{Q}) = 0$ .*

Only a few cases of Conjecture 2.7 are known. Hain [19] proved it for subgroups  $\Gamma$  that contain the Torelli group. Later, Boggi [8] and I [47] independently extended this to  $\Gamma$  that contain the *Johnson kernel*  $\mathcal{K}_g$ , which is the subgroup of  $\mathcal{S}_g$  generated by Dehn twists about separating curves. By work of Johnson [25], this is an infinite-index subgroup of  $\mathcal{S}_g$ . Finally, Bridson [10] and I [47] independently proved that powers of Dehn twists become torsion upon abelianizing finite-index subgroups of  $\mathrm{Mod}_g$ .

My strategy for attacking Conjecture 2.7 makes use of some remarkable work of Boggi [7] on the topology of the complex of curves. Combining Boggi's work with the trick I used to prove Theorem 2.2, one can reduce Conjecture 2.7 to a technical conjecture about certain linear representations of  $\mathrm{Mod}_g$ . Though this new conjecture seems difficult in general, I believe that it should be approachable in many specific cases. In particular, I hope to use it to prove Conjecture 2.7 for finite-index subgroups containing any term in the Johnson filtration (see §3 below for the definition).

### 3 The Torelli group

The next circle of ideas I wish to discuss concern the *Torelli group*. For technical reasons, we will restrict our attention to compact orientable genus  $g$  surfaces with 1 boundary component. Such a surface will be denoted  $\Sigma_{g,1}$  and its mapping class group will be denoted  $\mathrm{Mod}_{g,1}$ . The Torelli group  $\mathcal{I}_{g,1}$  is the kernel of the action of  $\mathrm{Mod}_{g,1}$  on  $H_1(\Sigma_{g,1}; \mathbb{Z})$ . We have a short exact sequence

$$1 \longrightarrow \mathcal{I}_{g,1} \longrightarrow \mathrm{Mod}_{g,1} \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1.$$

The group  $\mathcal{S}_{g,1}$  shows up frequently in the study of  $\text{Mod}_{g,1}$ , but many basic open questions about it remain unanswered.

**Combinatorial group theory of Torelli.** An infinite generating set for  $\mathcal{S}_{g,1}$  was constructed by Powell [41], following earlier work of Birman [6]. A new, modern proof of Powell’s result was given by me in [42]. Johnson [27] later showed that  $\mathcal{S}_{g,1}$  is finitely generated for  $g \geq 3$ , while McCullough-Miller [35] proved that  $\mathcal{S}_{2,1}$  is not finitely generated. An infinite presentation whose generators and relations have simple topological interpretations was constructed by me in [44]. However, the following basic question remains open.

**Question 3.1.** *For  $g \geq 3$ , is  $\mathcal{S}_{g,1}$  finitely presentable?*

I suspect that the answer is “no”, and in fact I conjecture that  $H_2(\mathcal{S}_{g,1}; \mathbb{Q})$  has infinite rank. The cohomology of  $\mathcal{S}_{g,1}$  remains very mysterious, and I have a long-term project to understand it via the action of the  $\mathcal{S}_{g,1}$  on the complex of curves. I will describe this a bit more below.

The finite generating set for  $\mathcal{S}_{g,1}$  constructed by Johnson is very large. Its size grows exponentially in  $g$ . Johnson [29] also calculated the abelianization of  $\mathcal{S}_{g,1}$ , and its rank grows cubically in  $g$ . This gives a cubic lower bound for the size of a generating set for  $\mathcal{S}_{g,1}$ . Johnson made the following conjecture.

**Conjecture 3.2.** *For  $g \geq 3$ , there exists a generating set  $S_g$  for  $\mathcal{S}_{g,1}$  whose size is bounded above by a polynomial in  $g$ .*

One motivation for this is that an analogous result was proved for the automorphism group of a free group by Magnus (see §4 for more details). My strategy for proving Conjecture 3.2 makes use of the following space. Fix a symplectic basis  $\{a_1, b_1, \dots, a_g, b_g\}$  for  $H_1(\Sigma_{g,1}; \mathbb{Z})$ . Define a graph  $X$  as follows.

- The vertices of  $X$  are isotopy classes of simple closed curves on  $\Sigma_{g,1}$  which are homologous to either  $a_1$  or  $b_1$ .
- Two vertices  $v$  and  $v'$  are joined by an edge if they can be represented by simple closed curves that intersect exactly once.

The group  $\mathcal{S}_{g,1}$  acts cocompactly on  $X$  (the quotient is a single edge), and I have recently proven that  $X$  is connected. This seems to be the first concrete space with nice topological properties upon which  $\mathcal{S}_{g,1}$  acts cocompactly. The vertex stabilizers are the stabilizers in  $\mathcal{S}_{g,1}$  of simple closed nonseparating curves, and by cutting along these curves we may regard these stabilizers as “Torelli groups on surfaces with multiple boundary components”. Some basic results about these stabilizers were worked out in my paper [42], but it is not yet known whether they are finitely generated. If I could prove that they were, then I suspect that I could prove Conjecture 3.2.

**The higher Torelli groups.** The Torelli group is the first term in the *Johnson filtration* of the mapping class group. This filtration, defined by Johnson in [26], plays an important role not only in the theory of the mapping class group but also in the theory of finite-type invariants of 3-manifolds (see, for instance, [18]).

Fix a surface  $\Sigma_{g,1}$  and let  $*$   $\in \partial \Sigma_{g,1}$  be a basepoint. Let  $\pi = \pi_1(\Sigma_{g,1}, *)$  and denote by  $\gamma_k(\pi)$  the  $k^{\text{th}}$  term of the lower central series of  $\pi$ . Recall that this is the inductively defined series

$$\gamma_0(\pi) = \pi \quad \text{and} \quad \gamma_{k+1}(\pi) = [\gamma_k(\pi), \pi].$$

The lower central series is preserved by the action of  $\text{Mod}_{g,1}$ , so we get an induced action of  $\text{Mod}_{g,1}$  on the quotients  $\pi / \gamma_k(\pi)$ .

**Definition.** The *Johnson filtration* of  $\text{Mod}_{g,1}$  is the normal series

$$\text{Mod}_{g,1} \triangleright \mathcal{I}_{g,1}(1) \triangleright \mathcal{I}_{g,1}(2) \triangleright \cdots$$

with  $\mathcal{I}_{g,1}(k)$  the kernel of the action of  $\text{Mod}_{g,1}$  on  $\pi/\gamma_k(\pi)$ .

Here are a few basic facts about the Johnson filtration.

- $\mathcal{I}_{g,1}(1)$  is the standard Torelli group.
- Johnson [26] proved that for  $k \geq 1$  the quotient  $\mathcal{I}_{g,1}(k)/\mathcal{I}_{g,1}(k+1)$  is a torsion-free abelian group. In fact, he constructed a sequence of homomorphisms

$$\tau(k) : \mathcal{I}_{g,1}(k) \rightarrow \text{Hom}(H, \mathcal{L}^k(H)),$$

where  $H = H_1(\Sigma_{g,1}; \mathbb{Z})$  and  $\mathcal{L}^k(H)$  is the  $k^{\text{th}}$  graded piece of the free Lie algebra on  $H$ . These homomorphisms, which have become known as the *Johnson homomorphisms*, have the property that  $\mathcal{I}_{g,1}(k+1) = \ker(\tau(k))$ . They occupy a central place in the subject and can be defined in an amazing number of different ways.

- Johnson [28] proved that  $\mathcal{I}_{g,1}(2) = \mathcal{K}_{g,1}$ , which we recall is the subgroup of  $\text{Mod}_{g,1}$  generated by separating twists.

In [49], I gave a new proof of Johnson's theorem asserting that  $\mathcal{I}_{g,1}(2) = \mathcal{K}_{g,1}$ . This proof uses topological properties of the complex of curves, and I believe that it should generalize to give generators for the higher Torelli groups as well.

**Project 3.3.** For  $k > 2$ , determine generators for  $\mathcal{I}_{g,1}(k)$ .

Closely related to this is the problem of determining the images of the Johnson homomorphisms.

**Project 3.4.** For  $k > 2$ , determine the image of the  $k^{\text{th}}$  Johnson homomorphism (at least in a stable range).

Finally, the following folk conjecture is still open. As was pointed out by Farb in [15], this conjecture implies that all that all the groups  $\mathcal{I}_{g,1}(k)$  are not finitely generated for  $k \geq 2$  (recall that Johnson proved that  $\mathcal{I}_{g,1}(1) = \mathcal{I}_{g,1}$  is finitely generated).

**Conjecture 3.5.** For  $g \geq 2$ , the group  $\mathcal{K}_{g,1}$  is not finitely generated.

In fact, I suspect that the group  $H_1(\mathcal{K}_{g,1}; \mathbb{Q})$  has infinite rank, which would immediately imply Conjecture 3.5. It seems to be very difficult to calculate  $H_1(\mathcal{K}_{g,1}; \mathbb{Q})$  directly, but I have a program for proving the analogous statement for  $\text{Aut}(F_n)$  (see §4 for more details). This program is designed to construct large families of abelian quotients of the analogous subgroup  $\text{IA}_n(2)$  of  $\text{Aut}(F_n)$ , and if it is successful then I believe that I should be able to prove Conjecture 3.5 by using the inclusion map  $\mathcal{K}_{g,1} \hookrightarrow \text{IA}_{2g}(2)$  to obtain new families of abelian quotients of  $\mathcal{K}_{g,1}$ .

**The cohomology of Torelli.** We finally turn to some questions about the cohomology groups of Torelli. The only specific homology group of  $\mathcal{S}_{g,1}$  that is known is the first, which was calculated by Johnson [29]. I gave a new proof of this result in [51]. Work of Bestvina-Bux-Margalit [5] shows that the cohomological dimension of  $\mathcal{S}_{g,1}$  is  $3g - 5$ . They also showed that  $H_{3g-5}(\mathcal{S}_{g,1}; \mathbb{Q})$  has infinite rank.

The questions we wish to consider center around homological stability. A theorem of Harer [21] says that for a fixed  $k$ , the groups  $H_k(\text{Mod}_{g,1}; \mathbb{Q})$  are independent of  $g$  when  $g \gg 0$ . Work of Johnson ([29]; see my paper [51] for a new proof) shows that for  $g \geq 3$  we have

$$H_1(\mathcal{S}_{g,1}; \mathbb{Q}) \cong \wedge^3 H_1(\Sigma_{g,1}; \mathbb{Q}).$$

Thus  $H_1(\mathcal{S}_{g,1}; \mathbb{Q})$  does not stabilize as a group. However, the conjugation action of  $\text{Mod}_{g,1}$  on  $\mathcal{S}_{g,1}$  induces an action of  $\text{Mod}_{g,1} / \mathcal{S}_{g,1} \cong \text{Sp}_{2g}(\mathbb{Z})$  on  $H_*(\mathcal{S}_{g,1}; \mathbb{Q})$ , and Johnson's result shows that  $H_1(\mathcal{S}_{g,1}; \mathbb{Q})$  stabilizes in some sense as an  $\text{Sp}_{2g}(\mathbb{Z})$ -module. I wish to investigate whether this holds for the higher homology groups.

**Question 3.6.** *Fix  $k$ . Do the  $\text{Sp}_{2g}(\mathbb{Z})$ -modules  $H_k(\mathcal{S}_{g,1}; \mathbb{Q})$  stabilize in some sense as  $g \rightarrow \infty$ ?*

Harer proved his stability result by applying standard technology (due originally to Quillen) for proving homological stability results to the action of  $\text{Mod}_g$  on the complex of curves. I hope to adapt this technology to the equivariant setting. A key tool will be some results I proved in [44] on the homotopy type of the quotient of the complex of curves by  $\mathcal{S}_g$ .

Morita has stated a specific conjecture in the direction of Question 3.6 which I hope to prove by these techniques. A deep theorem of Madsen–Weiss [33] says that

$$\lim_{g \rightarrow \infty} (H^*(\text{Mod}_{g,1}; \mathbb{Q})) \cong \mathbb{Q}[e_1, e_2, \dots],$$

where the  $e_i \in H^{2i}(\text{Mod}_{g,1}; \mathbb{Q})$  are the so-called Miller–Mumford–Morita classes (see [38] for an excellent survey of the basic properties of these classes). For odd  $i$ , the class  $e_i$  is the pullback to  $H^*(\text{Mod}_{g,1}; \mathbb{Q})$  of a class in  $H^*(\text{Sp}_{2g}(\mathbb{Z}); \mathbb{Q})$ , so  $e_i$  restricts to zero in  $H^*(\mathcal{S}_{g,1}; \mathbb{Q})$ . However, essentially nothing is known about the restriction of  $e_i$  to  $H^*(\mathcal{S}_{g,1}; \mathbb{Q})$  for even  $i$  (it is not even known whether or not the restriction of  $e_2$  is nonzero!). It is clear that this pullback is invariant under the action of  $\text{Sp}_{2g}(\mathbb{Z})$ . Morita has made the following conjecture. If  $G$  is a group and  $M$  a  $G$ -module, then denote by  $M^G$  the invariants of  $M$  under the action of  $G$ .

**Conjecture 3.7** (Morita, [37, Conjecture 3.4]). *For  $i \geq 1$ , the class  $e_{2i}$  restricts to a nontrivial element of  $H^{4i}(\mathcal{S}_{g,1}; \mathbb{Q})$  for  $g \gg 0$ . Moreover, for fixed  $k$  the groups  $(H^k(\mathcal{S}_{g,1}; \mathbb{Q}))^{\text{Sp}_{2g}(\mathbb{Z})}$  stabilize as  $g \rightarrow \infty$  and*

$$\lim_{g \rightarrow \infty} (H^*(\mathcal{S}_{g,1}; \mathbb{Q}))^{\text{Sp}_{2g}(\mathbb{Z})} \cong \mathbb{Q}[e_2, e_4, \dots].$$

Conjecture 3.7 is known for the first and second homology groups. The fact that  $(H^1(\mathcal{S}_{g,1}; \mathbb{Q}))^{\text{Sp}_{2g}(\mathbb{Z})} = 0$  for  $g \geq 3$  follows from Johnson's theorem [29] that says that for  $g \geq 3$  we have

$$H_1(\mathcal{S}_{g,1}; \mathbb{Q}) \cong \wedge^3 H_1(\Sigma_{g,1}; \mathbb{Q}).$$

The fact that  $(H^2(\mathcal{S}_{g,1}; \mathbb{Q}))^{\text{Sp}_{2g}(\mathbb{Z})} = 0$  for  $g \geq 6$  can be deduced from the Hochschild-Serre spectral sequence associated to the exact sequence

$$1 \longrightarrow \mathcal{S}_{g,1} \longrightarrow \text{Mod}_{g,1} \longrightarrow \text{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1.$$

The necessary ingredients are Johnson's calculations of  $H_1(\mathcal{S}_{g,1}; \mathbb{Q})$  and some facts about the cohomology of the symplectic group.

## 4 Automorphism groups of free groups

Let  $F_n$  be the free group on  $n$  letters. Just like for the mapping class group, one can define a Johnson filtration on  $\text{Aut}(F_n)$ .

**Definition.** For  $n, k \geq 1$ , define  $\text{IA}_n(k)$  to be the kernel of the action of  $\text{Aut}(F_n)$  on  $F_n/\gamma_k(F_n)$ .

This definition is due to Andreadakis [2] (in fact, it preceded the definition of the Johnson filtration on  $\text{Mod}_{g,1}$ ). The group  $\text{IA}_n(1)$ , which we will denote  $\text{IA}_n$ , is the kernel of the action of  $\text{Aut}(F_n)$  on  $H_1(F_n; \mathbb{Z}) \cong \mathbb{Z}^n$ . The serious study of  $\text{IA}_n$  goes back to work of Magnus [34] from 1935. Here are a few known facts.

- Magnus [34] proved that  $\text{IA}_n$  is finitely generated for all  $n$ . See [4] for a modern proof of Magnus's result (together with a calculation of the cohomological dimension of  $\text{IA}_n$ ).
- Krstić–McCool [32] proved that  $\text{IA}_3$  is not finitely presented. It is not known whether or not  $\text{IA}_n$  is finitely presented for  $n > 3$ .
- Cohen–Pakianathan [11], Farb [14], and Kawazumi [31] constructed Johnson homomorphisms  $\tau(k) : \text{IA}_n(k) \rightarrow \text{Hom}(H, \mathcal{L}^k(H))$  with the property that  $\text{IA}_n(k+1) \cong \ker(\tau(k))$ . They also proved that  $\tau(1)$  gives the abelianization of  $\text{IA}_n$ .

All the questions about the Torelli group in §3 have interesting analogues for  $\text{Aut}(F_n)$ . We will focus on the following versions of Question 3.1 and Conjecture 3.5.

**Question 4.1.** For  $n \geq 4$ , is  $\text{IA}_n$  finitely presentable?

**Conjecture 4.2.** For  $n \geq 3$ , the group  $\text{IA}_n(2)$  is not finitely presentable. In fact,  $H_1(\text{IA}_n(2); \mathbb{Q})$  has infinite rank.

Let me describe my basic strategy for addressing Question 4.1 and Conjecture 4.2. The generating set  $S_n$  Magnus [34] constructed for  $\text{IA}_n$  contains  $n \binom{n}{2}$  elements. We have a short exact sequence

$$1 \longrightarrow R_n \longrightarrow F(S_n) \longrightarrow \text{IA}_n \longrightarrow 1,$$

where  $F(S_n)$  is the free group on  $S_n$  and  $R_n$  is the normal closure of a complete set of relations for  $\text{IA}_n$ . The result of Cohen–Pakianathan [11], Farb [14], and Kawazumi [31] mentioned above implies that the abelianization of  $\text{IA}_n$  is isomorphic to  $\mathbb{Z}^{n \binom{n}{2}}$ ; the elements of  $S_n$  project to a basis for this abelianization. The upshot of this is that  $R_n \subset [F(S_n), F(S_n)]$ . Since  $\text{IA}_n(2) = [\text{IA}_n, \text{IA}_n]$ , we conclude that  $\text{IA}_n(2) \cong [F(S_n), F(S_n)]/R_n$ . In particular, if we can determine a presentation for  $\text{IA}_n$ , then we obtain a presentation for  $\text{IA}_n(2)$  for free! Such a presentation should allow the calculation of the abelianization of  $\text{IA}_n(2)$ , and thus the resolution of Conjecture 4.2.

Even more is true. The classical Hopf formula in group cohomology asserts that if  $G = \langle S \mid R \rangle$  is a group presentation, then  $H_2(G; \mathbb{Z}) \cong \frac{R \cap [S, S]}{[S, R]}$ . In the case of  $\text{IA}_n$  described above, we have  $R_n \cap [S_n, S_n] = R_n$ , so this simplifies to  $H_2(\text{IA}_n; \mathbb{Z}) \cong \frac{R_n}{[S_n, R_n]}$ . In this way, I expect a sufficiently explicit presentation for  $\text{IA}_n$  would allow me to calculate  $H_2(\text{IA}_n; \mathbb{Z})$ , and thus possibly resolve Question 4.1.

We are therefore led to the following project.

**Project 4.3.** Construct a natural presentation for  $\text{IA}_n$ .

For this project, I intend to imitate the argument of my paper [44], which constructed a natural infinite presentation for  $\mathcal{S}_{g,1}$ . The presentation in [44] was constructed using the main result of my paper [48], which gives a recipe for constructing presentations for groups acting on simplicial complexes without the need to identify a fundamental domain, like in most other theorems of this type. This result was applied to the action of  $\mathcal{S}_{g,1}$  on the complex of curves. To do this in the context of  $\text{IA}_n$ , I first need to construct an analogue of the complex of curves for  $\text{Aut}(F_n)$ . There are numerous possible definitions, all of which share some but not all the properties of the complex of curves (see [30] for a discussion). The version I will use is as follows.

**Definition.** A *partial basis* of  $F_n$  is a set  $\{x_1, \dots, x_k\} \subset F_n$  such that there exists  $x_{k+1}, \dots, x_n \in F_n$  so that  $\{x_1, \dots, x_n\}$  is a free basis for  $F_n$ . A *primitive element*  $x$  is a one-element partial basis. For  $x \in F_n$ , denote by  $\{x\}$  the conjugacy class of  $x$ . Define  $\mathcal{B}_n$  to be the simplicial complex whose simplices are the sets  $\{\{x_1\}, \dots, \{x_k\}\}$  for partial bases  $\{x_1, \dots, x_k\}$  of  $F_n$ .

Using a trick I introduced in my paper [43], I can prove that  $\mathcal{B}_n$  is connected for  $n \geq 2$  and simply connected for  $n \geq 3$ . Also, using an argument like one in my paper [44], I can prove that  $\mathcal{B}_n/\text{IA}_n$  is  $(n-1)$ -connected. Applying the main result of [48], I can conclude that  $\text{IA}_n$  is built in a simple way out of the stabilizers of vertices of  $\mathcal{B}_n$ . It remains to understand these vertex stabilizers and to figure out how to piece them together.

With Matt Day, I have proven some partial results in this direction. We begin with some notation. Observe that the stabilizers of vertices of  $\mathcal{B}_n$  are the stabilizers of the conjugacy classes of primitive elements of  $F_n$ .

**Definition.** Let  $\{x\}$  be a primitive element of  $F_n$ . Define  $\text{Aut}(F_n, \{x\})$  and  $\text{IA}_n(\{x\})$  to be the stabilizers of  $\{x\}$  in  $\text{Aut}(F_n)$  and  $\text{IA}_n$ , respectively.

One can view  $\text{Aut}(F_n, \{x\})$  as a sort of group of “automorphisms of a free group with boundary” (see [24]). The next step is to attempt to relate  $\text{IA}_n(\{x\})$  to  $\text{IA}_{n-1}$ . The model for this is the Birman exact sequence for the mapping class group. One version of this is of the form

$$1 \longrightarrow \pi_1(U\Sigma_g) \longrightarrow \text{Mod}_{g,1} \longrightarrow \text{Mod}_g \longrightarrow 1$$

with  $U\Sigma_g$  the unit tangent bundle of  $\Sigma_g$  (this version is due to Johnson [27]). Here the map  $\text{Mod}_{g,1} \rightarrow \text{Mod}_g$  comes from gluing a disc to the boundary component and the kernel  $\pi_1(U\Sigma_g)$  consists of mapping classes that “push” the boundary component around the surface while allowing it to rotate.

Define  $\mathbf{K}_{n,1}$  and  $\mathbf{IK}_{n,1}$  be the kernels of the natural surjections  $\text{Aut}(F_n, \{x\}) \rightarrow \text{Aut}(F_{n-1})$  and  $\text{IA}_n(\{x\}) \rightarrow \text{IA}_{n-1}$ . We have split exact sequences

$$1 \longrightarrow \mathbf{K}_{n,1} \longrightarrow \text{Aut}(F_n, \{x\}) \longrightarrow \text{Aut}(F_{n-1}) \longrightarrow 1$$

and

$$1 \longrightarrow \mathbf{IK}_{n,1}(k) \longrightarrow \text{IA}_n(\{x\}) \longrightarrow \text{IA}_{n-1} \longrightarrow 1.$$

These groups  $\mathbf{K}_{n,1}$  and  $\mathbf{IK}_{n,1}$  are much less well-behaved than the kernel  $\pi_1(U\Sigma_g)$  of the above exact sequence for  $\text{Mod}_{g,1}$ . We have proven the following result.

**Theorem 4.4** (Day–Putman, [12]). *For  $n \geq 3$ , the groups  $\mathbf{K}_{n,1}$  are finitely generated but not finitely presentable.*

We can also construct a natural infinite presentation of  $\mathbf{K}_{n,1}$  and determine its abelianization. To complete Project 4.3, the next step will be to determine a presentation for  $\mathbf{IK}_{n,1}$ .

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