18.901 (Spring 2009) : Solns to HW 6

Andrew Putman

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Problem 21.3. Let X_n be a metric space with metric d_n for $n \in \mathbb{Z}_+$.
a)Show that

$$\rho(x, y) = \max_{1 \leq i \leq n}(d_i(x_i, y_i))$$

is a metric for the product space $X_1 \times \cdots \times X_n$.

Solution : The only nontrivial fact to check is the triangle inequality. Consider $x, y, z \in X_1 \times \cdots \times X_n$. For each $1 \leq i \leq n$, we have

$$d_i(x_i, y_i) \leq d_i(x_i, z_i) + d_i(y_i, z_i).$$

Taking the maximum of both sides as i varies, we get that

$$\rho(x, y) = \max_{1 \leq i \leq n}(d_i(x_i, y_i)) \leq \max_{1 \leq i \leq n}(d_i(x_i, z_i) + d_i(z_i, y_i)).$$

Since the $d_i(\cdot, \cdot)$ are nonnegative, this is less than or equal to

$$\max_{1 \leq i \leq n}(d_i(x_i, z_i)) + \max_{1 \leq i \leq n}(d_i(z_i, y_i)) = \rho(x, z) + \rho(z, y),$$

as desired. □

b) Let $\overline{d}_i = \min\{d_i, 1\}$. Show that

$$D(x, y) = \sup_{i \geq 1}(\frac{\overline{d}_i(x_i, y_i)}{i})$$

is a metric for the product space $X_1 \times X_2 \times \cdots$.

Solution : Observe first that since $\frac{\overline{d}_i}{i}$ is bounded above by $1/i$ for each i, this supremum is always at most 1. In particular, it is never infinity. The only other nontrivial fact to check is the triangle inequality. Consider $x, y, z \in X_1 \times X_2 \times \cdots$. For each $i \geq 1$, we have

$$d_i(x_i, y_i) \leq d_i(x_i, z_i) + d_i(y_i, z_i).$$

Multiplying this by $1/i$, we get that

$$\frac{d_i(x_i, y_i)}{i} \leq \frac{d_i(x_i, z_i)}{i} + \frac{d_i(y_i, z_i)}{i}.$$

Taking the supremum of this over $i \geq 1$, we get that

$$\rho(x, y) = \sup_{i\geq 1}(\frac{d_i(x_i, y_i)}{i}) \leq \sup_{i\geq 1}(\frac{d_i(x_i, z_i)}{i} + \frac{d_i(z_i, y_i)}{i}).$$

Since the $d_i(\cdot, \cdot)$ are nonnegative, this is less than or equal to

$$\sup_{i\geq 1}(\frac{d_i(x_i, z_i)}{i}) + \sup_{i\geq 1}(\frac{d_i(z_i, y_i)}{i}) = \rho(x, z) + \rho(z, y),$$

as desired. □

Problem 21.4. Show that $\mathbb{R}_\ell$ and the ordered square satisfy the first countability axiom.

Solution : We start with $\mathbb{R}_\ell$. Consider $x \in \mathbb{R}_\ell$. For $i \geq 1$, set

$$U_i = [x, x + \frac{1}{i}).$$

If U is a neighborhood of x, then there is some interval $[a, b) \subset U$ such that $x \in [a, b)$. In particular, $x < b$. If $i \geq 1$ satisfies $\frac{1}{i} < b - x$, then $U_i \subset [a, b) \subset U$. Hence U_i is a countable basis at x.

We now deal with the ordered square $I \times I$. Consider $x \in I \times I$. Set

$$A_x = \{a \in I \times I \mid a < x \text{ and both coordinates of } a \text{ are rational}\}.$$

and

$$B_x = \{b \in I \times I \mid x < b \text{ and both coordinates of } b \text{ are rational}\}.$$

Since $A_x, B_x \subset \mathbb{Q} \times \mathbb{Q}$, we have that A_x and B_x are countable. Hence $A_x \times B_x$ is countable. For $(a, b) \in A_x \times B_x$, define

$$U_{a,b} = (a, b).$$

For every interval (a', b') in $I \times I$ with $x \in (a', b')$, we can find $a \in A_x$ and $b \in B_x$ such that $a' < a < b < b'$, so $U_{a,b} \subset (a', b')$. Hence the $U_{a,b}$ form a countable basis at x. □

Problem 21.6. Define $f_n : [0,1] \to \mathbb{R}$ by the equation $f_n(x) = x^n$. Show that the sequence $\{f_n(x)\}$ converges for each $x \in [0,1]$, but that the sequence of functions $\{f_n\}$ does not converge uniformly.

Solution : Consider $x \in [0,1]$. If $0 \leq x < 1$, then it is clear that $\lim_{n\to\infty} f_n(x) = 0$. If $x = 1$, then $f_n(x) = 1$ for all n, so $\lim_{n\to\infty} f_n(x) = 1$. If the sequence of functions $\{f_n\}$ converged uniformly, then the limit would be continuous; however, we just saw that the limit is

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1 \end{cases}$$

□

Problem 21.7. This is just an easy unwrapping of the definitions, so I'll omit it.

Problem 23.2. Let $\{A_n\}$ be a sequence of connected subspaces of X such that $A_n \cap A_{n+1} \neq \emptyset$ for all n. Show that $\cup A_n$ is connected.

Solution : Define

$$B_n = \cup_{i=1}^n A_i.$$

We will prove that B_n is connected by induction on n. The case $n = 1$ is trivial. Now assume that $n > 1$ and that B_{n-1} is connected. We have $B_n = B_{n-1} \cup A_n$. Moreover, both B_{n-1} and A_n are connected and

$$B_{n-1} \cap A_n \subset A_{n-1} \cap A_n \neq \emptyset.$$

Thus by a theorem from the lectures, B_n is connected, as desired.

Pick some arbitrary $x \in A_1$. Observe now that for all $n, m \geq 1$, we have

$$x \in A_1 \subset B_n \cap B_m.$$

Thus by the same theorem from the lectures, $\cup A_n = \cup B_n$ is connected. □

Problem 23.3. Let $\{A_\alpha\}_{\alpha \in J}$ be a collection of connected subspaces of X; let A be a connected subspace of X. Show that if $A \cap A_\alpha \neq \emptyset$ for all $\alpha \in J$, then $A \cup (\cup_{\alpha \in J} A_\alpha)$ is connected.

Solution : For $\alpha \in J$, set $B_\alpha = A \cup A_\alpha$. Since $A \cap A_\alpha$ is nonempty and both A and A_α are connected, B_α is connected. Pick some $x \in A$. Observe that for all $\alpha \in J$,

$$x \in A \subset A \cup A_\alpha = B_\alpha.$$

Thus since all the B_α are connected, we can conclude that

$$\cup_{\alpha \in J} B_\alpha = A \cup (\cup_{\alpha \in J} A_\alpha)$$

is connected, as desired. □

Problem 23.5. A space is *totally disconnected* if its only connected subspaces are one-point sets. Show that if X has the discrete topology, then X is totally disconnected. Does the converse hold?

Solution : Assume that X has the discrete topology. If $A \subset X$ contains more than one point and $x \in A$, then $\{x\}$ and $A \setminus \{x\}$ are open (since X is discrete) nonempty disjoint sets whose union is A, so A is no connected.

The converse is false. A counterexample is $\mathbb{Q}$. □

Problem 23.7. Is the space $\mathbb{R}_\ell$ connected?

Solution : No. We have a separation

$$\mathbb{R}_\ell = (-\infty, 0) \cup [1, \infty).$$

□

Problem 23.8. Determine whether or not $\mathbb{R}^\omega$ is connected in the uniform topology.

Solution : It is not connected. The key observation is that if $\{x_i\}$ and $\{y_i\}$ are sequences whose distance in the uniform metric is at most, say, $1/2$, then $|x_i - y_i| < 1/2$ for all i. In particular, $\{x_i\}$ is bounded if and only if $\{y_i\}$ is bounded. Thus the sets

$$U = \{\{x_i\} \in \mathbb{R}^\omega \mid x_i \text{ is unbounded}\}$$

and

$$V = \{\{x_i\} \in \mathbb{R}^\omega \mid x_i \text{ is bounded}\}$$

are open, as each contains the ball of radius $1/2$ around each of its points. It is also clear that $U \cap V = \emptyset$ and that $U \cup V = \mathbb{R}^\omega$, so $U \cup V$ is a separation of $\mathbb{R}^\omega$. □