3.1.2 We can represent linear transformations $T, U : \mathbb{R}^3 \to \mathbb{R}^3$ as multiplying by 3×3 matrices $[T], [U]$. Then we only need to find $[T], [U]$ such that the two matrices do not commute.

One example would be

$$[T] = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

and

$$[U] = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

i.e. $T(x, y, z) = (2x, x + y, y + z)$, and $U(x, y, z) = (y, x, z)$

3.1.4 (a) $(g \circ f)(x, y, z) = g(z - x, z - y) = (y - x, 0)$. Therefore,

$$[g \circ f] = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(b) $[f]$ is the zero matrix, so $[g \circ f]$ is also a zero matrix.

$$[g \circ f] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(c)

$$[f] = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

so

$$\begin{aligned} [g \circ f] &= [g][f] \\ &= \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 0 & 0 \\ 5 & 0 & 0 \end{pmatrix} \end{aligned}$$

3.1.6 For $u, v \in V$,

$$\begin{aligned} G(u + v) &= (T(u + v), U(u + v)) \\ &= (T(u) + T(v), U(u) + U(v)) \\ &= (T(u), U(u)) + (T(v), U(v)) \\ &= G(u) + G(v) \end{aligned}$$

and for $k \in \mathbb{R}$

$$\begin{aligned} G(kv) &= (T(kv), U(kv)) \\ &= (kT(v), kU(v)) \\ &= k(T(v), U(v)) \\ &= kG(v) \end{aligned}$$

So by definition, G is a linear transformation.

3.1.8 Assume $f(X), g(X) \in P^n$, and $k \in \mathbb{R}$.

(a) S is not linear.
S is linear if $S(f(X)+g(X)) = S(f(X)) + S(g(X))$, and $S(kf(X)) = kS(f(X))$.

$$\begin{aligned} S(kf(X)) - kS(f(X)) &= (kf(X))^2 - kf(X) \\ &= (k^2 - k)f(X) \end{aligned}$$

So $S(kf(X)) = kS(f(X))$ if and only if $k = 0$ or $k = 1$. So the equality does not hold for all k. Therefore, S is not a linear transformation.

(b) G is linear.
G is linear if $G(f(X)+g(X)) = G(f(X))+G(g(X))$, and $G(kf(X)) = kG(f(X))$.

$$G(f(X) + g(X)) = f(X+1) + g(X+1) = G(f(X)) + G(g(X))$$

and

$$G(kf(X)) = kf(X+1) = kG(f(X))$$

3.2.2 If T_A is represented by a $m \times n$ matrix A, then T_A is a linear mapping $\mathbb{R}^n \to \mathbb{R}^m$, so $rk(T_A) = rk(A)$, and $null(T_A) = n - rk(T_A) = m - rk(A)$. The range of T_A is the column space of A, and the kernel of T_A is the kernel of A.

(a) $rk(T_A) = rk(A) = 2$
$nul(T_A) = 4 - rk(A) = 4 - 2 = 2$
Range of T_A is

$$span \left\{ \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Kernel is

$$span\left\{\begin{pmatrix} -4 \\ 4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{7}{2} \\ \frac{5}{2} \\ 0 \\ 1 \end{pmatrix}\right\}$$

(b) $rk(T_A) = rk(A) = 2$
$nul(T_A) = 2 - rk(A) = 0$
Range of T_A is

$$span\left\{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 9 \end{pmatrix}\right\}$$

Kernel is

$$\left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$$

(c) $rk(T_A) = rk(A) = 3$
$nul(T_A) = 2 - rk(A) = 0$
Range of T_A is

$$span\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}$$

Kernel is

$$\left\{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right\}$$

(d) $rk(T_A) = rk(A) = 1$
$nul(T_A) = 1 - rk(A) = 0$
Range of T_A is

$$span\left\{\begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \\ 0 \end{pmatrix}\right\}$$

Kernel is

$$\left\{\begin{pmatrix} 0 \end{pmatrix}\right\}$$

(e) $rk(T_A) = rk(A) = 1$
$nul(T_A) = 5 - rk(A) = 4$
Range of T_A is

$$span\left\{\left(\ 1\ \right)\right\}$$

Kernel is

$$span\left\{\begin{pmatrix}-3\\1\\0\\0\\0\end{pmatrix},\begin{pmatrix}-3\\0\\1\\0\\0\end{pmatrix},\begin{pmatrix}-1\\0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}\right\}$$

(f) $rk(T_A) = rk(A) = 1$
$nul(T_A) = 2 - rk(A) = 1$
Range of T_A is

$$span\left\{\begin{pmatrix}2\\3\end{pmatrix}\right\}$$

Kernel is

$$span\left\{\begin{pmatrix}-2\\1\end{pmatrix}\right\}$$

3.2.6 Define $T' : V \to im(T)$, such that $T'(v) = T(v)$ for all $v \in V$.

First of all, we already know that T is one-one, so T' is also one-one, since it is an identical map as T.

Also, $im(T') = im(T)$. So T' is also onto. Therefore, T' is an isomorphism, V and $im(T)$ are isomorphic.

3.2.10 (a) Both T and U are one-one, so

$$v_1 = v_2 \Longleftrightarrow T(v_1) = T(v_2), U(v_1) = U(v_2)$$

.

Then

$$\begin{aligned} & T(U(v_1)) = (T \circ U)(v_1) = (T \circ U)(v_2) = T(U(v_2)) \\ \Longleftrightarrow\ & U(v_1) = U(v_2) \\ \Longleftrightarrow\ & v_1 = v_2 \end{aligned}$$

Therefore, by definition, $T \circ U$ is one-one.

(b) Both T and U are onto, so $T(V) = V$ and $U(V) = V$. Here $T(V)$ refers to the set of images when T operates on all elements in set V, i.e $im(T)$.

Then $(T \circ U)(V) = T(U(V)) = T(V) = V$, thus $im(T \circ U) = V$.

Therefore, $T \circ U$ is onto.

3.2.12 (a) If T has a right inverse $R : W \to V$, $(T \circ R)(w) = w$ for all $w \in W$. So

$$im(T \circ R) = \{(T \circ R)(w) \mid w \in W\} = W$$

$im(T \circ R) = T(im(R)) \subseteq T(V) = im(T)$, so $W \subseteq im(T)$.

We also know that $im(T)$ is a subset of W, so $im(T) = W$. T is onto.

(b) If T has a left inverse $L : W \to V$, $L \circ T(v) = v$ for all $v \in V$. So if $T(v_1) = T(v_2)$, we have

$$\begin{aligned} T(v_1) &= T(v_2) \\ (L \circ T)(v_1) &= (L \circ T)(v_2) \\ v_1 &= v_2 \end{aligned}$$

$T(v_1) = T(v_2)$ implies $v_1 = v_2$. T is one-one.

3.2.16 Prove by contradiction:

Assume that there exists i such that $rk(T^i) = rk(T^{i+1}) > 0$.

First note that if $T^n = 0$, then for all $m > n$. $T^m = T^{m-n} \circ T^n = 0$, so we must have $i < n$.

$$\begin{aligned} im(T^{i+1}) &= im(T^i \circ T) \\ &= T^i(im(T)) \\ &\subseteq T^i(V) \\ &= im(T^i) \end{aligned}$$

So $im(T^{i+1})$ is a subspace of $im(T^i)$. Since $rk(T^i) = rk(T^{i+1})$, so we must have $im(T^i) = im(T^{i+1})$.

For all $j > i$,

$$\begin{aligned} im(T^j) &= im(T^{j-i} \circ T^i) \\ &= T^{j-i}(im(T^i)) \\ &= T^{j-i}(im(T^{i+1})) \\ &= im(T^{j-i} \circ Ti + 1) \\ &= im(T^{j+1}) \end{aligned}$$

Thus, $im(T^i) = im(T^{i+1}) = im(T^{i+2}) = \cdots = im(T^n)$

However, we have assumed that $rk(T^i) > 0$, so $rk(T^n) > 0$, which contradicts with the fact that $T^n = 0$. Therefore, we must have $rk(T^i) > rk(T^{i+1})$ for all i such that $rk(T^i) > 0$.

3.2.26 Assume $dim(W) = k$, and $dim(V) = n$, $k \leq n$.

Let $(w_1, w_2, \ldots, w_k)$ be a basis for W, and $(w_1, w_2, \ldots, w_k; v_1, v_2, \ldots, v_{n-k})$ a basis of V obtained by expansion. Note that all of w_i and v_j are linearly independent.

There exists a unique linear map $T : V \to V$ such that $T(w_i) = 0$, and $T(v_j) = v_j$, for all i, j.

Let $v = a_1w_1 + \cdots + a_kw_k + b_1v_1 + \cdots + b_{n-k}v_{n-k}$, then

$$\begin{aligned}
& v \in ker(T) \\
\iff & T(v) = 0 \\
\iff & a_1T(w_1) + \cdots + a_kT(w_k) + b_1T(v_1) + \cdots + b_{n-k}T(v_{n-k}) = 0 \\
\iff & b_1v_1 + \cdots + b_{n-k}v_{n-k} = 0 \\
\iff & b_1 = \cdots = b_{n-k} = 0 \\
\iff & v = a_1w_1 + \cdots + a_kw_k \\
\iff & v \in W
\end{aligned}$$

Therefore T is a linear transformation such that $ker(T) = W$.

Let S be the unique linear map such that $S(w_i) = w_i$, and $S(v_j) = 0$, for all i, j. Then

$$\begin{aligned}
S(v) &= a_1S(w_1) + \cdots + a_kS(w_k) + b_1S(v_1) + \cdots + b_{n-k}S(v_{n-k}) \\
&= a_1w_1 + \cdots + a_kw_k
\end{aligned}$$

This is true for all $a_i \in \mathbb{F}$, so $im(S) = span(w_1, w_2, \ldots, w_k) = W$.

Therefore S is a linear transformation such that $im(S) = W$.