18.700 PROBLEM SET 5 SOLUTIONS

(1) 1.4.4: Recall that a basis is a linearly independent set of vectors that spans the space in question. Thus, for each of these problems, one uses row reductions to find the rank of the matrix. If said rank is $r$, a basis for the rowspace and columnspace is simply a set of $r$ linearly independent vectors over the respective spaces. The dimension of the nullspace is found from the rank + nullity theorem, and so $n - r$ linearly independent vectors in the nullspace will form a basis for the nullspace.

(a) After row reduction, our matrix is \[
\begin{pmatrix}
1 & 5 & 2 \\
2 & 3 & 1 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & -\frac{1}{7} \\
0 & 1 & \frac{3}{7} \\
\end{pmatrix}.
\]
This matrix has rank 2, so a basis for the rowspace is \[
(1, 0, -\frac{1}{7}), (0, 1, \frac{3}{7})
\]
A basis for the column space is \[
\begin{pmatrix}
1 \\
2 \\
\end{pmatrix},
\begin{pmatrix}
5 \\
3 \\
\end{pmatrix}
\]
A basis for the nullspace is:
\[
\begin{pmatrix}
\frac{1}{7} \\
-\frac{3}{7} \\
1 \\
\end{pmatrix}
\]

(b) After row reduction, the matrix is
\[
\begin{pmatrix}
1 & 4 & 5 & 8 \\
5 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
3 & 1 & 2 & 0 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 2 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}.
\]
The matrix has rank 2, and we can find appropriate basis vectors. Rowspace basis: \{
(1, 0, 0), (0, 1, 2)\}

Columnspace basis:
\[
\begin{pmatrix}
1 \\
5 \\
1 \\
3 \\
\end{pmatrix},
\begin{pmatrix}
4 \\
0 \\
0 \\
1 \\
\end{pmatrix}
\]

Nullspace basis:
\[
\begin{pmatrix}
0 \\
-2 \\
1 \\
\end{pmatrix}
\]

(c) After row reduction, the matrix is
\[
\begin{pmatrix}
2 & 1 & 1 \\
3 & 1 & 2 \\
5 & 2 & 3 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & -1 \\
0 & 0 & 0 \\
\end{pmatrix}.
\]
The matrix has rank 2, so the basis vectors are: Rowspace basis: \{(1, 0, 1), (0, 1, -1)\}

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Columnspace basis:
\[
\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}
\]

Nullspace basis:
\[
\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}
\]

(d) After row reduction, the matrix is:
\[
\begin{pmatrix} 2 & 1 \\ 0 & 0 \\ 1 & 1 \\ 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}
\]

This matrix has rank 2, and the basis vectors are:

Rowspace basis: \[(0, 1), (1, 0)\]

Columnspace basis:
\[
\begin{pmatrix} 2 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}
\]

Nullspace basis: the nullspace is the trivial one.

(2) 1.4.10: \(V\) is a finite-dimensional vector space - say it has dimension \(n\). Because \(H\) is a subspace of \(V\), it has a basis. Suppose that \(\text{Dim}(H) > \text{Dim}(V)\). Then, it has \(m\) basis vectors \(h_1, h_2, \ldots, h_m\), where \(m > n\). By definition, all the \(h_i\) are linearly independent from each other. But then, let us choose the first \(n\) of these vectors. They are \(n\) linearly independent vectors in \(V\); thus, they form a basis, and span \(V\). But then, because \(H \subset V\), the first \(n\) basis vectors also contain all the \(h_i\) where \(i > n\). This gives a contradiction, because we assumed all the \(h_i\) to be linearly independent. So the dimension of \(H\) can be at most \(\text{Dim}(V)\).

(3) 1.4.12:

(a) If we consider \(\mathbb{C}\) as a real vector space, it has dimension 2. Possible basis vectors are 1 and \(i\).

(b) \(V\) is a vector space over the field of complex numbers. Say it has basis \(v_1, v_2, \ldots, v_n\). Then, we can change it into a real vector space with basis vectors \(v_1, iv_1, v_2, iv_2, \ldots, v_n, iv_n\). This is because any vector \(v \in V\) can be written as
\[
v = \sum_{i=1}^{n} z_i v_i
\]
for \(z \in \mathbb{C}\). What we are doing is decomposing the coefficients \(z_i\) into their real and imaginary parts (because our coefficients in this case are limited to the reals).

(4) 1.4.18: \(U\) and \(W\) are subspaces of a space \(V\). Say \(\text{dim}(U) = j, \text{dim}(W) = k, \text{and dim}(V) = l\). By assumption we have that \(j + k > l\). Suppose for contradiction that \(U \cap W = \{0\}\). Then, take a basis \(\{u_1, u_2, \ldots, u_j\}\) of \(U\) and a basis \(\{w_1, w_2, \ldots, w_k\}\) of \(W\). They are each linearly independent, and because \(U \cap W\) is trivial, the union of these two basis is linearly independent as well. But now, the space spanned by this is a subspace of \(V\) (because all
the basis vectors of $U$ and $V$ are in $V$), with dimension greater than $l$ (because $j + k > l$). This is a contradiction from problem 1.4.10, so $U$ and $W$ could not have been disjoint.

(5) 1.5.2

This is a reordering of the first and third coordinates: we have

$$Bu_B = v \rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} v_B = v$$

so $v_b = \begin{pmatrix} c \\ b \\ a \end{pmatrix}$.

(6) 1.5.6 The matrix of the basis is \( \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \). The matrix of transforming into this basis is simply the inverse of this matrix, which is:

$$\frac{1}{3} \begin{pmatrix} 1 & 1 & -1 \\ -2 & 1 & 2 \\ 2 & -1 & 1 \end{pmatrix}$$

(7) 1.5.10

There does exist such a matrix. From 1.5.4, given an ordered basis $B'$, there exists an invertible matrix $P$ such that for any vector $v$, $(v)_{B'} = P^{-1}v$. Now, we have two ordered basis $B_1$ and $B_2$; let their respective matrices of transformation of bases be $P_1$ and $P_2$ respectively. Then, we have $P_1(v)_{B_1} = v = P_2(v)_{B_2}$, and so $P_2^{-1}P_1(v)_{B_1} = (v)_{B_2}$. So $P_2^{-1}P_1$ is the required matrix.

(8) 2.1.2(ace): For these problems, we are simply applying row reductions again, and then factoring out the constant multiples on the diagonal entries.

(a) 2.1.2.a: The determinant is 2

(b) 2.1.2.c: The determinant is 0

(c) 2.1.2.e: The determinant is 0

(9) 2.1.4 An easy example is $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Note that $A$ and $B$ are both of determinant 0, whereas $A + B = I_2$, which has determinant 1.

(10) 2.1.8

The general equation of a line is $\alpha x + \beta y = 1$. Thus, these three points are collinear iff there exists some $(\alpha, \beta)$ for which this relation holds true for all of them. We can rewrite this as: $(a,b), (c,d), (e,f)$ are collinear iff

$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

has a solution.
Now, consider our original matrix. What our above condition is stating is whether the columns in our original matrix were linearly independent or not. If the determinant is 0, then the rank of the matrix is less than 3, meaning that the columns are not linearly independent, and so there do exist $\alpha, \beta$ for which our condition is true (and a line passing through the points). If there is a line passing through the points, then the last column is a linear combination of the first two, making the rank of the matrix less than 3, and making our matrix have determinant zero.