18.700 PROBLEM SET 2 SOLUTIONS

Note: the algebra for row reductions is not shown, because it is assumed that the students know how to perform row reductions.

(1) 0.4.3:

(a) After row reductions (that you should be familiar with from the previous problem set),
we find
\[
\begin{pmatrix}
1 & 0 & 0 & 5/3 \\
0 & 1 & 0 & -1/3 \\
0 & 0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
23/3 \\
8/3 \\
5
\end{pmatrix}
\]
From this, the solutions are: \(x = \frac{23}{3} - \frac{5}{3}t, y = \frac{8}{3} + \frac{1}{3}t, z = 5 - t, w = t\)

(b) After elimination, we have
\[
\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
And the solutions are \(x = 1, y = -t, z = t\)

(c) After elimination, we have
\[
\begin{pmatrix}
1 & 0 & -1 & 0 & 1 \\
0 & 1 & 0 & -5 & -1
\end{pmatrix}
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\]
And the solutions are
\(x_1 = t_3 - t_5, x_2 = 5t_4 + t_5, x_3 = t_3, x_4 = t_4, x_5 = t_5\)

(d) After elimination, we have
\[
\begin{pmatrix}
1 & 0 & 3/2 \\
0 & -2 & -1 \\
0 & 0 & -1/2
\end{pmatrix}
\]
Which is inconsistent, so there are no solutions.

(2) 0.4.4: After reduction, the system becomes
\[
\begin{pmatrix}
1 & 2 & 1 \\
0 & -1 & -1 \\
0 & 0 & 2
\end{pmatrix}
\]
Which is inconsistent.

(3) 0.4.9: We can show that the two matrices are not row equivalent by finding the row-reduced echelon form of both. The left matrix becomes:
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
Whereas the second matrix reduces to

\(\text{Date: October 3, 2009.}\)
\[
\begin{pmatrix}
1 & 1 & 2 \\
0 & 1 & 3/2 \\
0 & 0 & 0
\end{pmatrix}
\]. These are not the same (and they have different rank), so the two matrices are not row-equivalent.

(4) 0.4.10: Say that we have two solutions \( Ax = b \) and \( Ax' = b \). Then, because of linearity, we have \( A(x - x') = 0 \). Because \( x \neq x' \) by assumption, take any scalar \( c \in \mathbb{R} \). Let \( v = x - x' \). Then, it is easy to check that \( A(x + cv) = b \). We can let \( c \) vary over all of \( \mathbb{R} \), and so this creates an infinite set of solutions.

(5) 0.5.2:

For all of the following, the general solution is the particular solution + the homogeneous solution.

(a) After row reduction, we find the homogeneous solution is \( x = 0, y = 0, z = 0 \) (the matrix had full rank). So there is only one solution: \( x = 3/4, y = -1, z = 1/4 \)

(b) After row reduction, we find that the homogeneous solution is \( x = -3t - 6s, y = t/3 + 2s, z = t, w = x \), and a particular solution is \( x = -1, y = 1, z = 0, w = 0 \).

(c) After row reduction, you will find that the homogeneous equation only yields \( x = 0, y = 0 \). Thus, there is only the particular solution \( x = 1, y = 1 \).

(d) The homogeneous solution yields \( x = 0, y = 0, z = 0, w = 0 \), and the equation to the particular case gives an inconsistency - so there is no solution to this system of equations.

(6) 0.5.3: After row reduction, we have

\[
\begin{pmatrix}
1 & 1 & 2 \\
0 & 1 & 1 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= 
\begin{pmatrix}
a \\
b - 3a \\
c + 2a - b
\end{pmatrix}
\]

In order for this to have a unique solution, we need that the system of equation be consistent. This means that \( 0 = 0 \) in the last row, and so \( c + 2a - b = 0 \).

(7) 0.5.4: An upper triangular matrix has rank \( n \) iff all its diagonal entries are nonzero. An easy way to see why this is is by going through row reduction. Each row can reduce the rows above it because of the triangular structure of the matrix - as a result, the whole matrix can be reduced to the identity.

(8) 0.5.6:

(a) False. An easy counterexample is if \( A \) were the matrix of all zeroes, and \( b \) were any nontrivial vector.

(b) False. An easy example is again if \( A \) were the zero matrix, and \( b \) is this time the zero vector.

(c) False. \( x = 0 \) is always a solution of the homogeneous equatoin.

(d) False. We can easy append extra variables through more columns. For example, \( A = [1000] \) is a rank \( 1 \times 4 \) matrix, and \( Ax = 0 \) obviously has infinite solutions.

(9) 0.6.3: Given invertible matrix \( A \) and its inverse \( A^{-1} \), we know that the inverse \( B \) of \( A^{-1} \) is such that \( BA^{-1} = A^{-1}B = I \). But because \( A^{-1} \) is the inverse of \( A \), \( AA^{-1} = A^{-1}A = I \). So \( A \) is the inverse of \( A^{-1} \) too, and \( A = (A^{-1})^{-1} \)
(10) 0.6.7: If $C$ were invertible, and $A = CBC^{-1}$, we have $A^{-1} = CB^{-1}C^{-1}$. This is easy to check: $AA^{-1} = CBC^{-1}CB^{-1}C^{-1} = BB^{-1}C^{-1} = CC^{-1} = I$. Thus, $A$ is invertible if $B$ is invertible. The other direction is trivial - because $A = CBC^{-1}$, $B = C^{-1}AC$, which is equivalent to the first case. So $B$ is invertible if $A$ is invertible, too.

(11) 0.6.9: Suppose $A$ had a left inverse. Call it $B$. Then, $BA = I$. We are given that $AC = 0$ for $C$ nontrivial. Multiplying on the left by $B$, we find that $BAC = C = 0$. But that gives us a contradiction, so such a $B$ cannot exist.