Trace Methods for algebraic stacks.

MIT topology

"Nilpotent extensions of ∞-categories and the cyclotomic trace"

jt w/ V. Sosnilo.

X = scheme

$$\rightsquigarrow \underset{\substack{\cap \\ \mathrm{Sp.}}}{K(X)} := K(\overset{\substack{\text{perf cpx's.} \\ \wr}}{\mathrm{Perf}_X})$$

$$\pi_0 K(X) = K_0(X) \cong \frac{\mathbb{Z}\{\mathcal{E} \in \mathrm{Perf}(X)\}}{\{\mathcal{E}' \to \mathcal{E} \to \mathcal{E}''\}}$$

$$\mathcal{E} = [\mathcal{E}'] + [\mathcal{E}'']$$

X reasonable;

"resolution property"

$$\left(\cong \frac{\mathbb{Z}\{V \in \mathrm{Vect}(X)\}}{\{0 \to V' \to V \to V'' \to 0\}} \right)$$

Separated.

$X/\mathbb{C}$, finite type.

topological K-thy spectrum.

$\rightsquigarrow \underline{KU}(X^{an})$ analytic space underlying X

$X^{an} = X(\mathbb{C})$.

- $KU_0(X) = \left\{\begin{matrix}\text{topological} \\ \mathbb{C}\text{-vector} \\ \text{bundles} \\ \text{on } X.\end{matrix}\right\}$
- Bott periodic

$\mathcal{C} \in \mathrm{Cat}^{\mathrm{Perf}} \leftarrow$ ∞-category of small stable idempotent cplt ∞-cat's + exact functors.

$\rightsquigarrow K(\mathcal{C})$. (e.g. $\mathrm{Perf}(X)$, $\mathrm{Perf}(\mathrm{RMod}_A)$).

$A = \mathbb{E}_1$-algebra

If $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}_{\mathbb{C}}$ $\rightsquigarrow$ $\mathbb{C}$-linear version of $\mathrm{Cat}^{\mathrm{perf}}$.

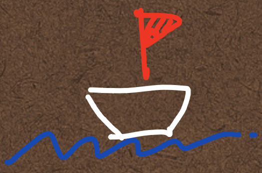
$\rightsquigarrow$ top. version of K-theory ?

examples

- geometric
- $A = \mathbb{C}$-dga $\rightsquigarrow \mathrm{Perf}(\mathrm{RMod}_A)$.

A. Blanc '15.

$$\begin{tikzcd}
 & \mathrm{Mod}_{KU} \arrow[d] \\
\mathrm{Cat}^{\mathrm{perf}}_{\mathbb{C}} \arrow[ur, dashed, "K^{\mathrm{top.}}"] \arrow[r, "K^{\mathrm{top}}"] & \mathrm{Sp}
\end{tikzcd}$$



A. Blanc '15 X sep. finite type $\mathbb{C}$-scheme
A. Perry IHC.

- $K^{top}(\mathrm{Perf}_X) \simeq KU(X^{an})$
- $\mathcal{C} \hookrightarrow \mathrm{Perf}_X$ summand [localizing inv.].

then $\underbrace{K^{top}(\mathcal{C}) \hookrightarrow KU(X^{an})}_{\text{summand.}}$

A. Blanc. '15 [Chern character.]

$$\begin{tikzcd}
K^{top}(\mathcal{C}) \arrow[r] \arrow[d] & HP(\mathcal{C}) \\
K^{top}(\mathcal{C}) \otimes \mathbb{C} \arrow[ur, dashed, "\text{Chern character.}"'] &
\end{tikzcd}$$

periodic cyclic homology.

- Recall :

  - $KU_{\mathbb{Q}} \simeq H\mathbb{Q}[u^{\pm 1}]$.

    periodized rational coh. $|u|=2$.

  - $X \in \mathrm{SmProp}_{\mathbb{C}}$

$$C^\bullet(X;\mathbb{C}) \xrightarrow{\sim} R\Gamma(X^{an}, \Omega^\bullet_{an}) \xleftarrow{\sim} R\Gamma(X, \Omega^\bullet)$$

analytic de Rham cplx.

algebraic dR coh.

Poincaré, de Rham

Grothendieck.

- $HP(X/\mathbb{C}) \xrightarrow{\sim} R\Gamma(X, \Omega^\bullet)[u^{\pm 1}]$

avatar of de Rham cohomology. Weibel. (see Antieau's "HP and dR")

So, if X is $\underline{\mathrm{SmProp}}_{\mathbb{C}}$ then

$$\mathrm{Ch} : \underset{\otimes \mathbb{C}}{K^{top}(X^{an})} \xrightarrow[\text{iso}]{\sim} HP(X/\mathbb{C})$$

[Blanc.]

Question:

For which $\mathcal{C}$ is the map

$$K^{top}(\mathcal{C}) \otimes \mathbb{C} \xrightarrow{ch} HP(\mathcal{C})$$

an equivalence?

$\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}_{\mathbb{C}} \rightsquigarrow \mathrm{Ind}(\mathcal{C})$. take on all filt colims formally.

- [smooth.] $\mathrm{Mod}_{\mathbb{C}} \xrightarrow{\mathrm{coev}} \mathrm{Ind}(\mathcal{C}) \otimes \mathrm{Ind}(\mathcal{C}^{op})$

  compact, i.e. pres. compact objects.

- [proper.] Maps: $\mathrm{Ind}(\mathcal{C}^{op}) \otimes \mathrm{Ind}(\mathcal{C}) \xrightarrow{\mathrm{ev.}} \mathrm{Mod}_{\mathbb{C}}$

  compact.

Example [Serre] $R = \mathbb{E}_\infty$, $A = \mathbb{E}_1$-R-alg.

then, tfae:

1) $\mathrm{Perf}(\mathrm{RMod}_A)$ smooth.

2) A is compact in A-A^{op}-bimod(Mod_R).
$\in \mathrm{Mod}_R$

$$\mathrm{Mod}_R \longrightarrow \mathrm{LMod}_A \otimes \mathrm{RMod}_A \simeq A\text{-}A\text{-bimod}(\mathrm{Mod}_R)$$

$$R \longmapsto A.$$

Conjecture (Lattice conjecture)

If $\mathcal{C}$ = smooth proper $\mathbb{C}$-linear cat, Ch is an equivalence.

$\Longrightarrow \quad HP(\mathcal{C}) \cong K^{top}(\mathcal{C}) \otimes \mathbb{C}$

on.

Conjugation

Thm (Kaledin, Mathew) [analog of Hdg - dR degeneration.]

$$HH_*(\mathcal{C})[u^{\pm 1}] \Longrightarrow HP(\mathcal{C})$$

if $\mathcal{C}$ sm proper then this S.S. collapses.

$\Longrightarrow$

Status

- $\mathcal{C} \simeq \mathrm{Perf}(X)$ $X \in \mathrm{SmProp}_{\mathbb{C}}$ [Blanc '15].
- $\mathcal{C} \simeq \mathrm{Perf}(B)$ B = finite-dim $\mathbb{C}$-algebra   – " –
- $\mathcal{C} \simeq \mathrm{Perf}(B)$ B = smooth, proper, "connective" $\mathbb{C}$-dga [Raedschelders-Stevenson '19].

$\left(\begin{array}{c} \mathrm{Perf}(B) \hookrightarrow \mathrm{Perf}(X) \\ X \text{ Sm Prop} \end{array} \right)$.

- OK if $\mathcal{C} \hookrightarrow \mathrm{Perf}(X)$ [Orlov's geometricity conjecture]

- $\mathcal{C} \simeq \mathrm{Perf}(\text{Stack})$ [Halpern-Leistner, Pomerleano '16].

- Stack $\rightsquigarrow$ DM
  - $\rightsquigarrow [X/G]$ $G =$ algebraic group
  - $X =$ smooth, quasi projective (not necc. proj!)

- "Hidden properness" $\rightsquigarrow$ $\mathbb{A}^{n+1} \setminus 0$
  - $\rightsquigarrow [\mathbb{A}^{n+1} \setminus 0 / \mathbb{G}_m] = \mathbb{P}^n$

Konovalov ['21]

$\sim$ $\pi_0 A \to \pi_0 B$ surj, kernel is nilpotent.

- $A \longrightarrow B$ nilpotent ext of connective $\mathbb{C}$-dga. Then if lattice true for $B \Rightarrow$ lattice true for A

$A \to \pi_0 A$.

- X = derived scheme, if X^{cl} has lattice then X has lattice.

Thm (E.-Sosnilo) $X/\mathbb{C}$ reasonably nice derived Artin stack. Then, if X^{cl} has lattice, X has lattice.

- any derived enhancement of HLP.

How we stumbled upon this result.

<u>Thm</u>. (Dundas-Goodwillie-McCarthy).

$R \to S$ nilpotent extension of

conn. $\mathbb{E}_1$-rings then

$$\begin{tikzcd} K(R) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pull}"] & K(S) \arrow[d] \\ TC(R) \arrow[r] & TC(S) \end{tikzcd}$$

$$TC : \mathcal{C}at^{perf} \longrightarrow Sp$$

built from $THH(\mathcal{C}) \circlearrowleft \mathbb{T}$

top. Hochschild inv.

circle grp.

$$(THH(\mathcal{C}))^{h\mathbb{T}} \simeq TC^{-} \underset{\varphi}{\overset{can}{\rightrightarrows}} TP^{\wedge} \simeq (THH(\mathcal{C}))^{t\mathbb{T}}$$

(cyclotomic Frob)

- a milder invariant than K-theory.

Nikolaus – Scholze.

$$\mathrm{cof}\left(\Sigma THH(\mathcal{C})_{h\mathbb{T}} \to THH(\mathcal{C})^{t\mathbb{T}} \right).$$

Question (Hesselholt): $\mathrm{tr}: K \longrightarrow TC$.

- $K^{\mathrm{inv}} := \mathrm{Fib}(K \longrightarrow TC)$.

$\mathcal{C} \xrightarrow{F} \mathcal{D}$ exact of small stable idem. cpl. ∞-cat.

when is $K^{\mathrm{inv}}(F)$ an $\simeq$

- Ask more generally for a truncating inv. (Land-Tamme) $E(R) \xrightarrow{\simeq} E(\pi_0 R)$

$R = \mathbb{E}_1$-ring, conn

DGM:

$$\begin{array}{ccc} \mathcal{C} & & \mathcal{D} \\ \shortparallel & & \shortparallel \\ \mathrm{Perf}_R & \longrightarrow & \mathrm{Perf}_S \end{array}$$

$R \longrightarrow S$ nilp ext.

What's a good notion of a "nilpotent extension" of stable ∞-cat's?

Schwede - Shipley / derived Morita theory.

⚠ R is often non conn.

$R = \mathbb{E}_1$-ring

$\mathcal{C} \overset{?}{\simeq} \mathrm{RMod}_R$. presentable, stable

$\iff \exists\, X \in \mathcal{C}$ satisfying : if $Y \in \mathcal{C}$

$\mathrm{Ext}^n(X, Y) = 0 \quad \forall n \implies Y = 0.$

. $R = \mathrm{End}(X)$, $\mathcal{C}^\omega \simeq \mathrm{Perf}_R.$

. this constrains $\mathcal{C}$.

Categories with many generators.

- k = field. R = coefficient ring
- $DM(k;R)$ = Voevodsky's derived cat. of motives
- generators = $\left\{ M(X)(q) : \begin{array}{l} X \in Sm_k \\ q \in \mathbb{Z} \end{array} \right\}$.

- G = affine algebraic group $/k$ (k ring, or scheme).

- $D_{qc}(\underline{B_k G})$ = derived ∞-cat of quasicoh. sheaves

- generators = $\left\{ \begin{array}{l} \text{irrep's of} \\ G. \end{array} \right\}$.   [Hall-Rydh]

- more generally $D_{qc}([X/G])$ X = affine.

Key idea: these categories are determined by additive subcat's !

- $$\mathrm{DM}(k;\mathbb{Z}) \xrightarrow{\sim} \underbrace{\mathcal{P}_\Sigma(\mathrm{Chow})}_{\text{adding sifted colimits}} \otimes \mathrm{Spt}$$

- $$\mathrm{Perf}(BG) \xrightarrow{\sim} \mathcal{P}_\Sigma(\mathrm{Irreps}(G)) \otimes \mathrm{Spt}.$$

Idea:
- prove a version of DGM for additive cat's (or $\mathbb{E}_1$-rings w/ many obj's).
- find a way to recognize when a stable cat is determined by a additive one as above.

Defn (Bondarko.) A weight structure on $\mathcal{C}$ consists of

- $(\mathcal{C}_{w\le 0}, \mathcal{C}_{w\ge 0})$ retract closed.

such that A.) $\Sigma\mathcal{C}_{w\ge 0} \subseteq \mathcal{C}_{w\ge 0}$

$\Sigma^{-1}\mathcal{C}_{w\le 0} \subseteq \mathcal{C}_{w\le 0}$

B.) $\pi_0 \mathrm{Maps}(\mathcal{C}_{w\le 0}, \mathcal{C}_{w\ge 1})$

$\simeq 0$ "no map $\to$"

c.t.a.-

such that A.) $\Sigma \mathcal{C}_{w\geq 0} \subseteq \mathcal{C}_{w\geq 0}$

$\Sigma^{-1} \mathcal{C}_{w\leq 0} \subseteq \mathcal{C}_{w\leq 0}$

B.) $\pi_0 \mathrm{Maps}(\mathcal{C}_{w\leq 0}, \mathcal{C}_{w\geq 1})$

$\simeq 0$

C.) $X \in \mathcal{C}$

$$X_{\leq 0} \longrightarrow X \longrightarrow X_{\cancel{\geq 0}.\ \geq 1.}$$

$$X \longmapsto X_{\leq 0} \longrightarrow X \longrightarrow X_{\geq 1}$$

not functorial ! (compare w t-str.)

Postnikov

wt -str.

<u>Example</u> $\mathcal{C} = Sp.$

$$Sp_{w\geq 0} = \{ E : \pi_* E = 0 , * < n \} , \quad Sp_{w \leq 0} = \left\{ E : \begin{array}{c} H\mathbb{Z}_* E = 0 \\ * > 0 , \\ H\mathbb{Z}_0 E \text{ free} \end{array} \right\}$$

$\mathcal{C}_{w \leq n}$ = object made of cells dim $\leq n$.

$\mathcal{C}_{w \geq n}$ = — " — $\geq n$.

$$\mathcal{C}^{\heartsuit_w} := Sp_{w \geq 0} \cap Sp_{w \leq 0}$$

" ex. **Bondarko.**

$$\left\{ \bigoplus_{\text{finite}} \mathbb{S}^0 \right\}$$

Example. $Ch^b(A)$

$$
\begin{tikzcd}
C_{w \le n} \arrow[d] & \cdots & C_{n-1} \arrow[l] \arrow[d, dash] & C_n \arrow[l] \arrow[d, equal] & 0 \arrow[l] \arrow[d, dash] & \cdots \arrow[l] \\
C \arrow[d] & & C_{n-1} \arrow[l] \arrow[d, dash] & C_n \arrow[l] \arrow[d, dash] & C_{n+1} \arrow[l] \arrow[d, equal] & \\
C_{w \ge n+1} & & 0 & 0 \arrow[l, dash] & C_{n+1} \arrow[l, dash] & \cdots \arrow[l]
\end{tikzcd}
$$

brutal / stupid truncation.

Fact :

$$\mathcal{C}^{\heartsuit_w} = \text{additive } \infty\text{-category.}$$

$\mathcal{A}$ additive ∞-cat.

$$\mathcal{A}^{fin} \hookrightarrow \mathrm{Fun}^{\times}(\mathcal{A}^{op}, Sp)$$

$\shortparallel$

· finite cell $\mathcal{A}$-modules. := Yoneda img + retracts.

has wt-str.

· $\mathcal{A} \longrightarrow \mathcal{A}^{fin}$ gen $\mathcal{C}_{w\geq 0}$ part.

<u>Thm</u> (Sosnilo 19) all are small cats

$$((-)^{fin}, w) : Cat^{add} \rightleftarrows WCat^{st,b} : (-)^{\heartsuit_w}$$

- $(-)^{\heartsuit_w}$ is fully faithful
- restr. to an equiv on idempotent complete

$(\mathcal{C}, w)$ bounded weight

str. then recognize

$$\mathcal{C} \simeq P_\Sigma(\mathcal{C}^{\heartsuit_w}).$$

$\underline{\text{Def}^{n}}$ [E.-Sosnilo] $A \xrightarrow{f} B$ $\otimes$-functor

of additive ∞-cat's. is nilpotent

extension if

I) f is ess. surj

II) $\pi_0 \mathrm{Maps}(x, y) \to \pi_0 \mathrm{Maps}(fx, fy)$

surj (f)

composition

III) $\exists\, n$ such that $f_1 \ldots f_n \in A$ w/

$f(f_1 \circ \ldots f_n) \simeq 0 \Rightarrow f_1 \circ \ldots f_n \simeq 0.$

$\underline{\text{Def}^{\underline{n}}}$. [E.-Sosnilo]. $\mathcal{C}, \mathcal{D}$ weighted, F wt-exact.

A functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is a nilpotent extension of weighted stable ∞-categories if

1) $\mathcal{C}, \mathcal{D}$ are weighted, F wt exact

2) $\mathcal{C}^{\heartsuit_w} \to \mathcal{D}^{\heartsuit_w}$ is nilpotent ext.

Thm (E.-Sosnilo) $(\mathcal{C}, w) \xrightarrow{F} (D, w')$

nilpotent extension, $E : \underset{(K^{\text{inv.}})}{\mathrm{Cat}^{\mathrm{perf}}} \longrightarrow \mathrm{Sp}$

truncating then : $E(F)$ is an

equivalence.

Corollary k = base discrete ring,

G = linearly reductive, embeddable, group scheme.

$R \to S$, $\mathbb{E}_\infty$-k-alg's nilpotent ext's.

then

$$\begin{tikzcd} K^G(R) \arrow[r] \arrow[d] & K^G(S) \arrow[d] \\ TC^G(R) \arrow[r] & TC^G(S) \end{tikzcd}$$

Examples

- finite constant G, $\frac{1}{|G|} \in k(p)$

- $k = \mathbb{Q}$-alg, G = reductive w/ geom. conn. fibres. (classically)

- k = char $p > 0$. $\mathbb{G}_m^{\times n}$, μ_{p^r}, extensions.

Pf of lattice for derived stacks.

- $\underset{\text{tr.}}{\underline{\mathcal{Y}}} \longrightarrow K^{top} \otimes \mathbb{C} \longrightarrow \underset{\text{tr.}}{HP}$

  Konarski, tr.

- K^{top} truncating, HP truncating

- Alper-Hall-Rydh $\Rightarrow$ X is Nis

locally quotient stack. $[X/G]$ X = affine, derived.

use our main thm for stacks.

<u>Bonus thm</u> (E.-Sosnilo) E truncating, X good Artin stacks, $E : \mathrm{Stk}_X^{op} \longrightarrow \mathrm{Sp}$ has cdh-descent.

- generalizes Hoyois-Krishna (uses motivic homotopy theory) for $E = KH$.

Corollary. $\mathrm{Chow} \hookrightarrow \mathrm{DM}$

$$\mathrm{DM} \longrightarrow \mathrm{Ch}^b(\mathrm{Chow})$$

· char $k=0$ or invert char.

$$\begin{tikzcd}
K(\mathrm{DM}) \arrow[r] \arrow[d] & K(\mathrm{Chow}) \arrow[d] \\
TC(\mathrm{DM}) \arrow[r] & TC(\mathrm{Chow}).
\end{tikzcd}$$

<u>Corollary</u> G = finite group. $\mathrm{Span}(\mathrm{Fin}_G)$

$$\begin{tikzcd}
K(Sp^G) \arrow[r] \arrow[d] & K(\mathrm{Span}(\mathrm{Fin}_G)^{gp}) \arrow[d] \\
TC(Sp^G) \arrow[r] & TC(\mathrm{Span}(\mathrm{Fin}_G)^{gp})
\end{tikzcd}$$

Understand $K(Sp^G)$?

- Kubrak - Prikhodko: "p-adic Hodge theory for stacks" ⟵ follows Bhatt - Scholze.
- Want: follow Bhatt - Morrow - Scholze / Clausen - Mathew - Morrow. (w/ Sosnilo)

- (R, I) henselian pair, G-action on (R, I), G = finite.

$$K^{\mathrm{inv}, G}(R)/n \simeq K^{\mathrm{inv}, G}(R/I)/n.$$

- $K^{\text{ét}} \longrightarrow TC$ on $\mathbb{F}_p$-stacks. ; Atiyah-Segal style results

Thanks for listening !