Möbius inversion in homotopy thy.

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MIT topology seminar, Apr 5'th 2021.

Outline:

Ⓐ Classical Möbius inversion

Ⓑ A 'Space-level' lifting

Ⓒ Manifestations in familiar examples.

Ⓐ Classical Möbius inversion.

Problem: Compute Euler's Totient function

$\phi(n) = \#$ generators for $\mathbb{Z}/n\mathbb{Z}$.

Observation. (Gauss) Every $x \in \mathbb{Z}/n\mathbb{Z}$ generates some subgroup

$$\Longrightarrow \quad n = |\mathbb{Z}/n\mathbb{Z}| = \underbrace{\sum_{\substack{\text{subgroups} < \mathbb{Z}/n\mathbb{Z} \\ \sim d \mid n}} \phi(d)} \quad ⊛$$

Möbius ⇝ way to extract ϕ from relation ⊛.

General class of problems:

Want to understand (count) "configurations" subject to constraints [Hard!]

Know how understand general unconstrained configurations [Easy]

E.g. count proper colorings of graph $G = (V, E)$.

E.g. all (possibly non-proper) colorings $= |V|^{\text{colors}}$

⊛ Key relation: every general configuration satisfies the constraints of a smaller/easier problem.

E.g. by contracting monochromaic edges 

every non-proper coloring ⇝ proper coloring of a quotient.

Formally: $(I, \leq)$ a finite poset – "of problems" ordered by size/difficulty.

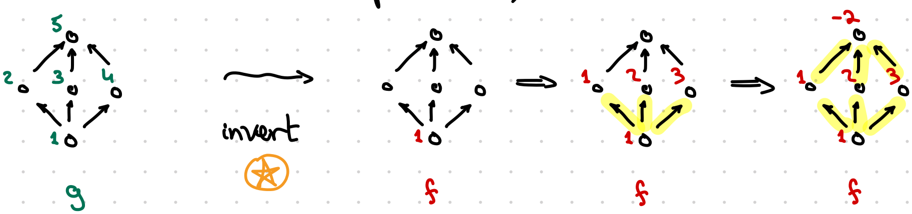
$g, f : I \to \mathbb{Z}$ functions.

Think $\begin{cases} g(x) = \# \text{ free configurations fitting problem "x".} & \text{know} \\ f(x) = \# \text{ constrained configurations fitting problem "x".} & \text{Want} \end{cases}$

satisfying relation

$$\underbrace{g(x) = \sum_{y \leq x} f(y).}_{\text{" every free configuration gives a constrained one " for a smaller problem.}} \quad (\star)$$

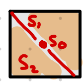
Want to invert relation $(\star)$, express f in terms of g.
(combinatorial problem)



- Topological example – stratified space $X = \bigcup_{\alpha \in I} S_\alpha$

$$\underbrace{\overline{S_\alpha} = \coprod_{\beta \leq \alpha} S_\beta}_{} \quad (\star)$$

closed $\to \overline{S_\alpha}$, $S_\beta \leftarrow$ open

$\overline{S_2} =$ 

Thm. (Möbius, ..., Rota, ...)
1832, 1964

$(I, \leq)$ locally finite poset = finite intervals (x, y)

There exists a "Möbius" function

$$\mu : I \times I \longrightarrow \mathbb{Z}$$

depending only on the order of I, that inverts $(\star)$ $\forall f, g$:

$$f(x) = \sum_{y \le x} g(y) \cdot \mu(y,x)$$

Many generalizations, e.g.
[Haigh, Leroux ~'80s extended to finite categories]

<u>Fact.</u> (P. Hall) $\boxed{\mu(y,x) = \tilde{\chi}(N(y,x))}$
reduced Euler characteristic.

This should be a theorem in homotopy theory!

- 5 years ago 2 papers appeared independently: giving a homological construction of this for stratified spaces.

– D. Petersen "A spectral sequence for stratified spaces..." arXiv:1603.01137

– P. Tosteson "Lattice spectral sequence..." arXiv:1612.06034

Let's make this about homotopy.

(B) A 'Space-level' lifting

<u>Setup:</u>

$$\begin{array}{ccc} (I, \le) & & I \\ g \downarrow & \rightsquigarrow & \downarrow G \\ \mathbb{Z} & & \mathcal{M} \end{array}$$

- I = diagram shape ~ small (∞-) category*
  G = diagram ~ functor.
- $\mathcal{M}$ = homotopical category / ∞-category**

** <u>Assumptions:</u> $\mathcal{M}$ has weak equivalences, and is
- <u>pointed</u> (= has zero object)
- <u>cocomplete</u> (= has <u>homotopy</u> colimits) / simplicial model structure ⇒ can form geometric realization & Bar const.

- really what we need: $\mathrm{hocolim} : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}$

  s.t. 1) homotopy invariant – $\forall\, G \xrightarrow{\sim} G'$ natural trans. that is pointwise an equivalence $\Downarrow$ $\mathrm{hocolim}\, G \xrightarrow{\sim} \mathrm{hocolim}\, G'$ equiv.

  2) agrees with colim – $\exists\, G \xrightarrow{\sim} G'$ s.t. $\mathrm{hocolim}\, G' = \mathrm{colim}\, G'$.

! We omit the "ho" – colim means homotopy invariant.

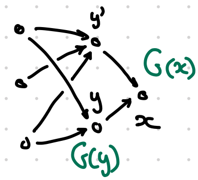
- $G : I \to \mathcal{M}$ a functor (analog of func. $g : I \to \mathbb{Z}$)

  where $g(x) = \sum_{y \leq x} f(y)$ ⊛

  (what should play the role of f ?
  must remove all $g(y)$ with $y < x$ ...)

Definition. The Margin of G is the diagram

$$\Delta G : \underset{x \in I}{x} \longmapsto G(x) \Big/ \underset{y \overset{\neq}{\to} x}{\mathrm{colim}}\, G(y)$$



The total homotopy cofiber of G restricted to $I_{/x}$.

The relation $G(x) = \sum_{y \leq x} \Delta G(y)$ will hold

Up to extensions.

Need an assumption on I.

* **Assumption:** I is a <u>relatively EI</u> - category.

<u>Definition.</u> $\mathcal{C}$ is **EI** if every **E**ndomorphism is an **I**somorphism.

I is **relatively EI** if every slice

I/x = (category of arrows $y \to x$)

is an EI category.

Equivalently, for every triangle
$$\begin{tikzcd} y \arrow[rr, "\sim"] \arrow[dr, "f"'] & & y \arrow[dl, "f"] \\ & x & \end{tikzcd}$$
must be invertible.

(monomorphisms, posets, ...)

$\Downarrow$

Isomorphism classes of arrows y/x → a poset:

$[y/x] \leq [z/x] \iff \exists$ arrow $y/x \to z/x$.

(since $[a] \leq [b]$ + $[b] \leq [a]$ $\Rightarrow$ loop $a \circlearrowleft$ ~ iso.)

Denote $(\Lambda_x, \leq)$, poset with maximum $\hat{1} = [x/x]$.

<u>Definition.</u> a <u>Λ_x-shaped filtration</u> on object $X \in \mathcal{M}$ is a diagram

$$\begin{tikzcd} & & & X_{<\lambda} \arrow[r] & X_{\leq\lambda} \arrow[dr] & & & \\ \cdots & X_{<\nu} \arrow[r] & X_{\leq\nu} \arrow[ur] \arrow[dr] & & & X_{<\hat{1}} \arrow[r] & X_{\leq\hat{1}} \arrow[r, "\sim"] & X \\ & & & X_{<\mu} \arrow[r] & X_{\leq\mu} \arrow[ur] & & & \end{tikzcd}$$

for all $\nu < \lambda, \mu < \hat{1} \in \Lambda_x$.

- graded quotients : $gr_\lambda X := X_{\leq\lambda} / X_{<\lambda}$

Works like ordinary filtrations -

Can get a spectral sequence using any $\Lambda_x \hookrightarrow \mathbb{Z}$.

(non-canonical.)

or, a "spectral system" - due to Matschke.

$\Delta G(x) := G(x) / \operatorname{colim}_{y \lneq x} G(y)$
total cofib.
"$G(x) = \sum_{y \le x} \Delta G(y)$" ⊛

Thm. (G) $\forall x \in I$, the value $G(x)$ has natural Λ_x-shaped filtration, such that

$$\underbrace{\operatorname{gr} G(x) \simeq \bigvee_{[y/x] \in \Lambda_x} \Delta G(y) / \operatorname{Aut}(y/x)}$$ ⊛

Cor. When I a finite poset, $G(y)$ finite type spaces/complexes

$$\tilde{\chi}(G(x)) = \sum_{y \to x} \tilde{\chi}(\Delta G(y))$$ – recovering ⊛.

Also, can pick linear extension $I \overset{rk}{\hookrightarrow} \mathbb{Z}$

$\Longrightarrow$ spectral sequence in E-homology / cohomology

$$E^1_{p,q} \cong \bigoplus_{\substack{y \to x \\ rk = p}} E_{p+q}(\Delta G(y)) \Longrightarrow E_{p+q}(G(x)).$$

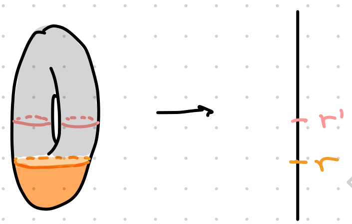
(But now don't need I to be finite -

Example. (Morse) $M \longrightarrow \mathbb{R}$
smooth closed manifold, Morse function

$\Longrightarrow$ diagram

$G : \mathbb{R} \to \mathrm{Top}_*$

$r \longmapsto (M_{\le r})_+ = f^{-1}((-\infty, r])_+$
sublevel sets

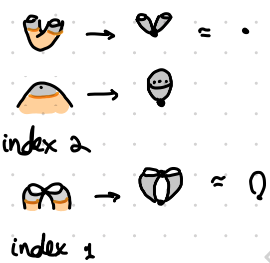


Then $\Delta G(r) \simeq \begin{cases} * & r \text{ regular value} \\ \bigvee_{\substack{p \in crit(f) \\ p \mapsto r}} S^{i(p)} & \text{else} \end{cases}$

$\Downarrow$

⊛ spectral sequence

~ handle complex.



We are really after the inverse ~ "$\Delta G(x) = \sum_{y \le x} G(y) \cdot \mu(y,x)$"

Thm. (G) The margin $\Delta G(x)$ has a natural Λ_x^{op}-shaped filtration (descending)

$$\cdots \to \Delta G(x)^{\geq \mu} \to \Delta G(x)^{>\lambda} \to \Delta G(x)^{\geq \lambda} \to \cdots$$

with associated graded

$$\mathrm{gr}\, \Delta G(x) \simeq \underbrace{\bigvee_{\hat{1} \neq [y/x] \in \Lambda_x} G(y) \wedge \Sigma \Sigma' N(I_{y/x}) \big/ \mathrm{Aut}(y/x)}_{\vee\, G(x)}$$

- $I_{y/x}$ = category of <u>strict</u> factorizations of y/x:

$$\begin{tikzcd} y \arrow[rr] \arrow[dr, "\neq"'] & & x \\ & z \arrow[ur, "\neq"'] & \end{tikzcd}$$

- Σ reduced suspension
- Σ' <u>unreduced</u> suspension, with canonical basepoint

Cone (X)  $\Sigma' X$

- Product with nerves exists in $\mathcal{M}$, as constant colimits

$$\operatorname*{colim}_{S} (\mathrm{const}\, X) = X \wedge N(S)_+$$

- $\mathrm{Aut}(y/x) \circlearrowright I_{y/x}$ by precomposition $\overset{\varphi^{-1}}{\circlearrowright} y \to z \to x$.

Cor. When I a finite poset, $G(y)$ finite type

$$\tilde{\chi}(\Delta G(x)) = \sum_{y \to x} \tilde{\chi}(G(y)) \cdot \underbrace{\tilde{\chi}(N(I_{y/x}))}_{M(y,x)}$$

$M(y,x)$ - classical Möbius function.

Also, pick $I \xrightarrow{rk} \mathbb{Z}$

$\Rightarrow$ spectral sequence, e.g. (Petersen, Tosteson)

$$E^1_{p,q} = \bigoplus_{rk\, y/x = p} \bigoplus_{i+j = p+q} \tilde{H}^i(G(y), \tilde{H}^{j-2}(N(I_{y/x}))) \Rightarrow \tilde{H}^{p+q}(\Delta G(x))$$

! The formula $\Leftarrow$ more fundamental fact about colim over EI:

Thm. (G) $\exists$ natural filtration on "(colim_{EI})" with

$$\mathrm{gr}\left(\mathrm{colim}_{J}\, G\right) \simeq \bigvee_{[a]\in \mathrm{Iso}(J)} G(a) \wedge \Sigma' N(J_{a/\!/}) \big/ \mathrm{Aut}(a)$$

Slogan: separate the topology from the combinatorics.

colimits built from $\begin{cases} \bullet \text{ combinatorics of } I \\ \bullet \text{ topology of values } G(x). \end{cases}$

The spectral sequence above lets us deal with each separately, then diff'ls reassemble.

Functoriality, monoidality, duality.

All constructions and proof use only formal properties of colimits.

$\Longrightarrow$ natural in
- $I \to I'$ reflecting $\simeq$
- $G \to G'$
- $M \to M'$ preserving equiv. colim's & zero.

In particular, respects Day $\otimes$ under mild hyp.

+ Dual theory for hocolim $\rightsquigarrow$ holim.

Note: sometimes can define Euler char even when $|\mathrm{Aut}(a)| = \infty$.

E.g. $\pi = \underbrace{\pi_1(\vee_n S^1)}_{F_n}, \pi_1(S_g), \underbrace{\pi_1(\mathrm{Conf}_n \mathbb{R}^2)}_{Br_n}, \ldots$

then $\pi \circlearrowright \underset{\text{finite type}}{F} \rightsquigarrow F/_\pi$ is flat F-bundle over finite-type base

$\Longrightarrow$ Has Euler number.

Q. Can this extend Leinster's definition of Euler char. for categories?

(c) Manifestations in familiar examples.

Many ways to apply Möbius to configuration spaces.

Example. $I = \mathrm{FinSurj}^{op}$, "$[n] \to [n+k]$" $:=$ $[n] \twoheadleftarrow [n+k]$

all monomorphisms, small-to-big.

X - CW complex.

Define a diagram

$$G : I \to \mathrm{Top}_*$$

$$[n] \mapsto \left(X^{[n]}\right)^+.$$

- Over $[3]$ -

$$\begin{tikzcd}
 & X^2 \leftrightarrow X^2 \arrow[dr, "\Delta_{ij}"] & \\
X^+ \arrow[ur, "\Delta"] \arrow[r] \arrow[dr] & X^2 \leftrightarrow X^2 \arrow[r] & X^{3\,+} \\
 & \underset{\text{swap}}{X^2 \leftrightarrow X^2} \arrow[ur] &
\end{tikzcd}$$

all diagonals.

$\longrightarrow$ $\underset{\circlearrowleft \Sigma_n}{\Delta G([n])} \simeq X^n/\mathit{diags} \simeq \underset{\circlearrowleft \Sigma_n}{\mathrm{Conf}_n(X)^+}$ - the ordered configuration spaces.

Cor. 1) $(X^n)^+$ is filtered by partitions of $[n]$ (ordered by refinement)

with $\underbrace{\mathrm{gr}\,(X^n)^+ \simeq \bigvee_{B_1 \amalg \dots \amalg B_k = [n]} \mathrm{Conf}_k(X)^+}$

Conversely,

2) $\mathrm{Conf}_n(X)^+$ is filtered by partitions of $[n]$ (with opposite order)

and $\underbrace{\mathrm{gr}\,\mathrm{Conf}_n(X)^+ \simeq \bigvee_{\underbrace{B_1 \amalg \dots \amalg B_k}_{\beta} = [n]} (X^k)^+ \wedge \Sigma\Sigma^? N(\underbrace{\Pi_n^{<\beta}}_{\text{the partition poset of } [n].})}$

$\hookrightarrow$ relates to operad composition.

Thm. (Bibby-G) As a symmetric sequence, $\mathrm{Conf}_\bullet(X)^+$ is an algebra (for $\otimes_{\mathrm{Day}}$) compatibly with the grading,

and $\mathrm{gr}\,\mathrm{Conf}_\bullet(X)^+ \simeq \mathrm{Comm} \underset{\uparrow}{\circ} \big(X^+ \wedge (\Sigma\Sigma' \underbrace{N(\Pi_\bullet)}_{\text{partition posets}})\big)$ ⊛

operad composition

Recall: $\Sigma\Sigma' N(\Pi_n) \simeq \bigvee_{(n-1)!} S^{n-1} \circlearrowleft \Sigma_n$

Representation on $H^* \cong \mathrm{Lie}(n) \otimes \mathrm{sgn}$ - Lie operad.

⊛ is the Koszul resolution for the Comm-coalg. X^+.

Q. How to relate this to Knudsen's work on E_n-enveloping algebras & $\mathrm{Conf}_\bullet(M^n)$?

Q. Relation to Goodwillie derivatives of Id ?

Problem: reproduce the argument in motivic spaces

(need comparison $U(\text{diags}) \simeq \mathrm{hocolim}(\text{diags})$ ... cdh topology?)

[A different construction for Conf_n works for schemes, with gr ~ Thom spaces for diagonals $\Delta \hookrightarrow X^k$]

Problem: Enriched version?

- I graded abelian category, ΔG ~ indecomposables Möbius should give standard Koszul resolution.
- How will Möbius play with orthogonal functor calculus?

A word about the proof: basechange

2 ingredients -

1) relative colims -

$I \xrightarrow{f} J$

$$\Longrightarrow \quad \operatorname*{colim}_J f_! F \simeq \operatorname*{colim}_I F$$

2) Beck-Chevalley / basechange -

$$\begin{tikzcd} I \vec{\times}_K J = \{(i, j, f(i) \to g(j))\} \arrow[r, "\tilde{g}"] \arrow[d, "\tilde{f}"'] \arrow[dr, phantom, "\swarrow"] & I \arrow[d, "f"] \arrow[r, "F"] & M \\ J \arrow[r, "g"'] & K & \end{tikzcd}$$

$$\Longrightarrow \quad \tilde{f}_! \tilde{g}^* \xrightarrow{\sim} g^* f_! \quad \text{equiv.}$$

Then, taking $\operatorname*{colim}_J g^* f_! F \simeq \operatorname*{colim}_I \underbrace{\tilde{g}_! \tilde{g}^*}_{} F$

introduces a nerve if I a groupoid.

Another example: smooth hypersurfaces in $\mathbb{P}^n$

✪ Vassiliev studies $U_{n,d}$ = smooth hypersurfaces $\subset \mathbb{P}^n$ degree d

Poset: possible singularity types λ = pt,
2pts,
⋮
line,
line + point,
⋮

✪ $V(\lambda) = \bigcup U(\mu)$

all hypersurfaces with singularity at least λ — singular locus type exactly $\mu \leq \lambda$

& have a fibration $\mathbb{A}^N \to V(\lambda)$
$\downarrow$
Moduli space of all $\lambda \subset \mathbb{P}^n$ — understood

Möbius

$$\mathrm{gr}\, U_{n,d} \simeq \bigvee_{\lambda \text{ sing. types}} V(\lambda) \wedge \Sigma\Sigma^1 N(\underbrace{\text{singularities} < \lambda}_{\text{poset}})$$

[Das] computes cohomology of
- smooth cubic surf.
- smooth cubic surf. equipped with a line.

Thank

you!