# The Fenchel–Nielsen Coordinates of Teichmüller Space

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**Abstract.** We develop the Fenchel–Nielsen coordinates for genus-$g$ Teichmüller space. Specifically, we prove the existence and uniqueness of pairs of pants with all possible boundary components. We then examine the Bers pants-decomposition of a Riemann surface, and deduce that the genus-$g$ Fenchel–Nielsen coordinates represent the points of genus-$g$ Teichmüller space.

**1. Introduction.** The Riemann Moduli Problem is to describe the isomorphism classes of all Riemann surfaces in a given topological class. Riemann solved the problem for simply connected Riemann surfaces: He found that the only possibilities are the disk, the plane, and the Riemann sphere. In the case of doubly connected Riemann surfaces, the problem is still not particularly difficult, although some technical complications arise, and they suggest that the problem should be clarified.

The moduli problem becomes interesting for the more complicated Riemann surfaces. According to the Uniformization Theorem, these Riemann surfaces are covered by the upper half-plane $H$, and therefore are hyperbolic. While studying the moduli problem, Otto Teichmüller first proposed a modification of the moduli problem that gives rise to what we now call Teichmüller space. The *moduli space* of a Riemann surface $R$ is the space of isomorphism classes of the complex structures on $R$; that is, the set of complex structures on $R$ modulo the orientation preserving homeomorphisms of $R$. The *Teichmüller space* of $R$ is a refinement of the moduli space, specifically the complex structures modulo homeomorphisms isotopic to the identity.

We examine Teichmüller space from a geometric viewpoint. In Section 2, we determine the Teichmüller space of the smallest and simplest hyperbolic Riemann surface, the *pair of pants*, which has Euler characteristic -1, and we see how the distinction between the moduli space and the Teichmüller space plays out in this special case. Then, in Section 3, we show how every compact hyperbolic Riemann surface may be decomposed into pairs of pants, and we use this decomposition to coordinatize genus-$g$ Teichmüller space.

**2. Existence and Uniqueness of Pairs of Pants.** As shown in Figure 2-1, a pair of pants is a genus-0 Riemann surface with three geodesic boundaries. In this section, we show that, given any three lengths of boundary components, a unique pair of pants is determined, and that all pairs of pants may be found in this manner.

*Definition* 2-1. A *Fuchsian model* of a hyperbolic Riemann surface $R$ is the group of Möbius transformations obtained from the quotient $H/R$ under a covering map $H \to R$.

**Lemma 2-2.** *Let $R$ be a hyperbolic Riemann surface, and let $\Gamma$ be a Fuchsian model of $R$. Let $\gamma(z)$ be a hyperbolic isometry in $\Gamma$, and let $l_\gamma$ be the length of the*

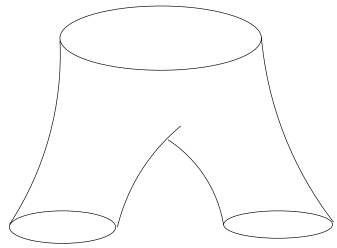

**Figure 2-1.** A pair of pants.

*closed geodesic in $R$ associated to $\gamma$. Then*

$$\operatorname{tr}^2(\gamma) = 4\cosh^2(l_\gamma/2)^2.$$

*Proof:* Conjugating by elements of $Aut(H)$, we may assume that $\gamma$ has the form $\gamma(z) = \lambda^2 z$ where $\lambda > 1$. So if $\gamma$ has the form $\frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbf{R}$, then we have $a = \lambda$ and $b = c = 0$ and $d = \frac{1}{\lambda}$. Hence $\operatorname{tr}(\gamma) = \gamma + 1/\gamma$ and

$$l_\gamma = \int_1^{\lambda^2} \frac{dy}{y} = 2\log\lambda. \qquad \square$$

**Proposition 2-3.** *The complex structure of a pair of pants $P$ is uniquely determined by the (hyperbolic) lengths of its ordered boundary components.*

*Proof:* (See [**2**, p. 53].) Let $P$ be a pair of pants with ordered boundary components $L_j$ of lengths $a_j$ for $j = 1, 2, 3$. Let $\Gamma$ be a Fuchsian model of $P$ acting on $H$ with generators $\gamma_1$ and $\gamma_2$. Now, $\Gamma$ is isomorphic to the fundamental group of $P$, which is generated by the isotopy classes of two of the boundary components of $P$. (Note that a loop in the isotopy class of the third boundary component is isotopic to the composition of the generators.) Thus $\gamma_1$ and $\gamma_2$ can then be associated to loops isotopic to $L_1$ and $L_2$, and similarly we have a $\gamma_3 = (\gamma_2 \circ \gamma_1)^{-1}$ associated to loops isotopic to $L_3$.

It is sufficient to show that $\gamma_1$ and $\gamma_2$ are uniquely determined by the $a_j$. Conjugating by elements of $Aut(H)$ if necessary, we assume that $\gamma_1$ and $\gamma_2$ have the forms

$$\gamma_1 = \lambda^2 z \text{ where } 0 < \lambda < 1,$$
$$\gamma_2 = \frac{az+b}{cz+d} \text{ where } a, b, c, d \in \mathbf{R}, \text{ and } ad - bc = 1, \text{ and } c > 0,$$

where $\gamma_2$ fixes 1. Then we have

$$\gamma_3 = \frac{a'z+b'}{c'z+d'} \text{ where } a' = a\lambda \text{ and } b' = \frac{b}{\lambda} \text{ and } c' = c\lambda \text{ and } d' = \frac{d}{\lambda}.$$

The lemma now yields the following equations:

$$\left(\lambda + \frac{1}{\lambda}\right)^2 = 4\cosh^2\left(\frac{a_1}{2}\right);$$
$$(a+d)^2 = 4\cosh^2\left(\frac{a_2}{2}\right);$$
$$(a'+d')^2 = 4\cosh^2\left(\frac{a_3}{2}\right).$$

Thus, assuming we've been keeping careful record of our signs, $\gamma_1$ and $\gamma_2$ are uniquely determined by the $a_j$. □

Pairs of pants are formed by gluing pairs of congruent hyperbolic hexagons along every other edge, as shown in Figure 2-2. As a matter of fact, for any three nonnegative real numbers we choose, there exists a unique hyperbolic hexagon with alternating sides of our chosen lengths. Thus, a unique pair of pants exists with boundary components of any given arbitrary lengths.

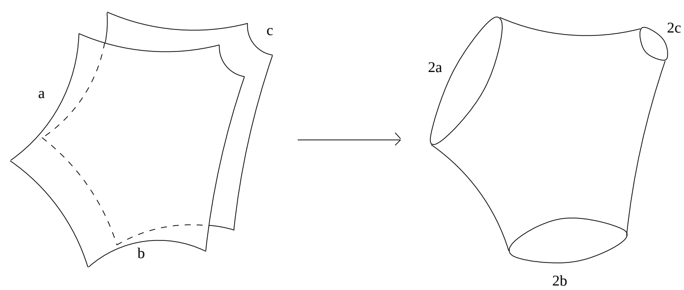


**Figure 2-2.** Constructing a pair of pants from two hyperbolic hexagons.

The orientation-preserving homeomorphisms of a pair of pants that are isotopic to the identity are just those that preserve individual boundary components. It follows from Proposition 2-3 that the Teichmüller space of a pair of pants is $\mathbf{R}^3_+$. However, the orientation-preserving homeomorphisms of $R$ also include homeomorphisms that permute boundary components. Hence the moduli space of a pair of pants is $\mathbf{R}^3_+/S_3$.

**3. The Bers Pants Decomposition of a Riemann Surface.** The Teichmüller space of a Riemann surface is in some ways much easier to study than the moduli space, in part due to the results of Teichmüller's Extremal Mapping Theorem. This theorem states that, among all homeomorphisms between compact hyperbolic Riemann surfaces in a given homotopy class, there exists a unique quasi-conformal mapping of minimal dialation. This fact means that Teichmüller space can be studied using the highly developed tools of complex analysis. We here develop a parameterization of $T_g$, the Teichmüller space of any genus $g$ Riemann surface, using a system known as the *Fenchel–Nielsen coordinates*.

In what follows, we assume that all of our Riemann surfaces are compact with (possibly empty) totally geodesic boundary.

We are going to use what we know about pairs of pants to characterize more complicated hyperbolic Riemann surfaces. Thus, we need to learn a number of things about the simple closed geodesics in Riemann surfaces, as they will form the boundary components of pairs of pants embedded in the surfaces. It turns out that the free homotopy class of any nontrivial simple loop in a Riemann surface $R$ contains a unique simple closed geodesic. To prove this result, we will need the following definition and lemma.

*Definition* 3-1. Let $c\colon (a,b) \to D$ be a curve in the Poincaré model of hyperbolic space. If $\lim_{t\to a} c(t)$ lies on the boundary of the disk, then this point is called an *endpoint at infinity* of $c$ (and similarly for the endpoint associated with $b$).

**Lemma 3-2.** *Let $R$ be a hyperbolic Riemann surface. Let $c$ be a loop in the free homotopy class of a simple closed geodesic $\gamma$ on $R$, and let $\tilde{c}$ and $\tilde{\gamma}$ be homotopic lifts to the universal cover $H$ of $R$. Then $\tilde{c}$ and $\tilde{\gamma}$ have the same endpoints at infinity.*

*Proof:* The cyclic subgroup of the deck transformations of $R$ that leaves $\tilde{\gamma}$ invariant also leaves $\tilde{c}$ invariant, so $\mathrm{dist}(\tilde{c}(t), \tilde{\gamma}(t))$ is bounded for all $t \in R$. □

**Proposition 3-3.** *Each nontrivial simple loop on a hyperbolic Riemann surface $R$ is isotopic to a unique simple closed geodesic on $R$.*

*Proof:* (Existence.) Again let $\tilde{\gamma}$ be the geodesic in $D$ with the same endpoints at infinity as $\tilde{c}$. Consider the simple closed geodesic $\gamma$ obtained by projecting $\tilde{\gamma}$ into $R$. Then $\gamma$ is isotopic to $c$. (Uniqueness.) Let $\delta$ be another simple closed geodesic in the free homotopy class of $\gamma$. Then $\tilde{\gamma}$ and $\tilde{\delta}$ must be goedesics in $D$ with the same endpoints, and so they must coincide. Similarly their projections down into $R$ must also coincide. □

The idea behind the Fenchel–Nielsen coordinates is that any hyperbolic Riemann surface may be decomposed into pairs of pants with distinguished boundaries, which then determine the complex structure of the entire surface. Thus if we parametrize the pants decomposition of a Riemann surface, we will have parametrized its Teichmüller space.

How exactly do we accomplish this decomposition? Cut a Riemann surface $R$ along a family of disjoint simple closed geodesics (according to the hyperbolic metric), as shown in Figure 3-1, until the resulting set contains no more simple closed geodesics that are not boundary elements. This set is then composed entirely of pairs of pants. Because

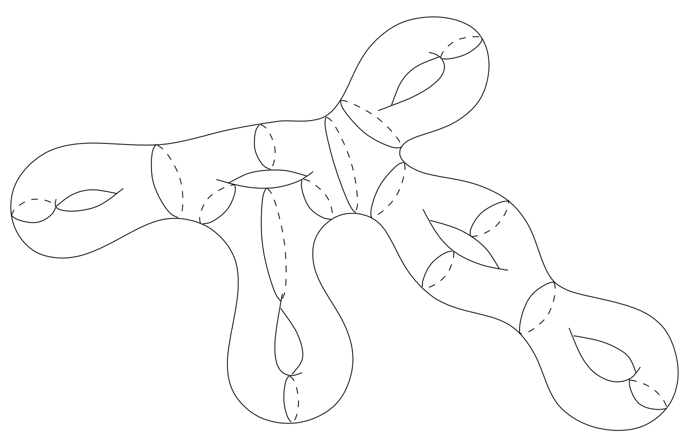

**Figure 3-1.** Decomposing a surface into pairs of pants.

we can perform this decomposition on any Riemann surface (by the classification of loops above), we can build all Riemann surfaces by gluing pairs of pants together. Therefore, all Riemann surfaces are uniquely determined by the lengths of the boundary components of their constituent pairs of pants, as well as the angles of "twist" between glued pairs of pants. Thus these lengths and angles parametrize the Teichmüller space of a Riemann surface; they are the Fenchel–Nielsen coordinates of Teichmüller space.

As shown in Figure 3-2, there are usually several ways to perform a pants decomposition on a hyperbolic Riemann surface. Each of these ways corresponds to loop *system*, a set of disjoint simple closed geodesics that is as large as possible.

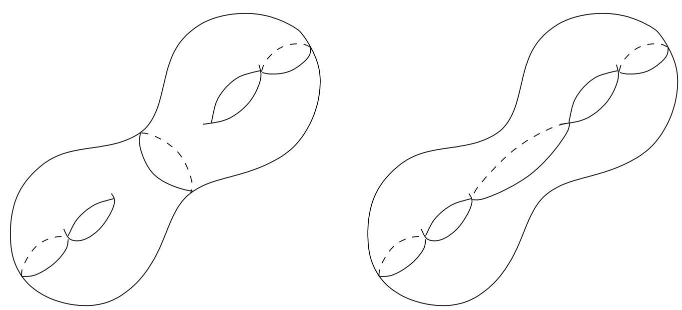

**Figure 3-2.** Alternative decompositions.

**Theorem 3-4.** *A hyperbolic Riemann surface $R$ of genus $g$ always contains a loop system of $3g-3$ disjoint simple closed geodesics. Regardless of which loop system we choose, cutting $R$ along the geodesics in the system always decomposes $R$ into $2g-2$ pairs of pants.*

*Proof:* Let $N$ be the number of disjoint simple closed geodesics in any such system on $R$, and let $M$ be the number of pairs of pants in a decomposition of $R$. Let $L$ be a loop in our set of simple closed geodesics. Let $n_1$ be the number of connected components of $R-L$, and let $g_1$ be the sum of the genera of the connected components of $R-L$. Then

$$g_1 - n_1 = (g-1) - 1,$$

and $R-L$ has two boundary components.

We proceed inductively, cutting $R$ successively along loops in our set of disjoint simple closed geodesics. Whenever we cut along a new loop, the resulting set has two more boundary components, and the sum of the genera less the number of connected components decreases by one. Thus we find

$$3M = 2N$$

and

$$M - 0 = N - (g-1),$$

and we are done. □

Intuitively we have just shown that any point in genus-$g$ Teichmüller space may be specified by $3g-3$ nonnegative real numbers (lengths of simple closed geodesics) and $3g-3$ real numbers (degree of twisting between glued pairs of pants). Thus $T_g$ is set-theoretically equivalent to $\mathbf{R}_+^{3g-3} \times \mathbf{R}^{3g-3}$.

**4. Remark about Twisting Coefficients.** It is not perfectly intuitive that the twisting coefficients specifying a Riemann surface should take on values outside $[0, 2\pi)$. Whereas twisting coefficients within this range may be visualized as simple rotation of glued pairs of pants, the "twist" that we keep referring to is actually a *Dehn twist*, akin to Dehn surgery. To perform a Dehn twist, we remove a collar around a simple closed geodesic that forms the boundary between two of our constituent pairs of pants. We then glue our collar back into our Riemann surface after rotating one of its boundary components by some angle $\theta$, so that curves that were originally geodesics orthogonal to our closed geodesic now wind around the collar by an amount $\theta$, as shown in Figure 4-1.

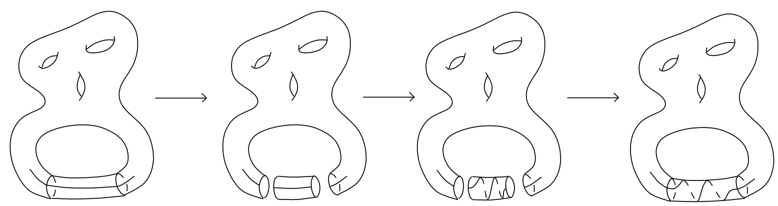

**Figure 4-1.** Dehn twisting.

These are the Fenchel-Nielsen coordinates of Teichmüller space, which allow us to uniquely specify any point in $T_g$. If we want to show that $T_g$ is homeomorphic to $\mathbf{R}_+^{3g-3} \times \mathbf{R}^{3g-3}$, then we would first have to specify a topology and metric on $T_g$.

## References


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