NEW BOUNDS ON EXTREMAL NUMBERS IN ACYCLIC ORDERED GRAPHS

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Abstract. This paper is mainly concerned with the upper and lower bound of the number of edges an ordered graph can have avoiding a fixed forbidden ordered subgraph H. The only case where a sharp bound has not been discovered is when H has interval chromatic number 2, where H can be represented as a 0-1 matrix P. Let $ex_<(n, n, P)$ be the maximum weight of an n by n 0-1 matrix avoiding P. When P contains a cycle, the corresponding bound of $ex_<(n, n, P)$ is also known. Hence, the interesting case is when P is acyclic.

In this paper, we construct a family of patterns $\mathcal{P}$ such that for a positive integer m, there exists $P \in \mathcal{P}$ with $ex_<(n, n, P) = \Omega(n \log n \log\log n \cdots \underbrace{\log\log\cdots\log}_{m \text{ iterations}} n)$. This result suggests an improved lower bound for the least upper bound of extremal numbers in acyclic ordered graphs. In addition, we suggest a new method for attaining an upper bound of $ex_<(n, n, P)$ for a special set of patterns.

1. Introduction and Preliminaries

Turán-type extremal problems are well-studied for unordered graphs. However, in the relatively new area of ordered graphs, the case when the forbidden subgraph is acyclic is still not thoroughly understood and it will be the main focus of this paper.

In this paper, we will follow the notation and terminology used in Pach-Tardos [3].

A simple graph $G = (V, E)$ is an *ordered graph* if vertices in $V = V(G)$ are linearly ordered. An *underlying graph* of G is an unordered graph with the same sets of vertices and edges. A *subgraph* of G is a subgraph of the underlying graph of G whose vertices are order-isomorphic to the corresponding vertices in G. The *interval chromatic number* $\chi_<(H)$ of an ordered graph H is the minimum number of intervals the vertex set of H can be partitioned into so that no two vertices in the same interval are adjacent.

Let $G = (U, V, E)$ be an *ordered bipartite graph* with linearly ordered vertex sets $U = U(G)$ and $V = V(G)$ and edge set $E = E(G)$ contained in $U \times V$. An *underlying graph* of G is an unordered graph with the same sets of vertices and edges. A *subgraph* G' of G is a subgraph of the underlying graph of G with $U(G') \subset U(G)$ and $V(G') \subset V(G)$ preserving the order of the vertices in each set. For an ordered bipartite graph $G = (U, V, E)$, G is also considered an ordered graph where the orderings of vertices in $U(G)$ precedes those in $V(G)$.

For a fixed ordered (bipartite) graph H, we say H is *contained* in the ordered (bipartite) graph G if there exists a subgraph of G isomorphic to H. If H is not contained in G, we say G *avoids* H and that G is *H-free*.

Define $ex_<(n, H)$ as the maximum number of edges an ordered graph with n vertices can have avoiding an ordered (bipartite) graph H. Likewise, define $ex_<(n, m, H)$ as the maximum number of edges an ordered bipartite graph G with $|U(G)| = n$ and $|V(G)| = m$ can have avoiding an ordered bipartite graph H.

From previous results [3], $ex_<(n, H)$ is known when $\chi_<(H) > 2$:

$$ex_<(n, H) = \left(1 - \frac{1}{\chi_<(H) - 1}\right)\binom{n}{2} + o(n^2).$$

However, the extremal number of H when $\chi_<(H) = 2$ is unknown. In this case, we can write $H = (U, V, E)$ as an ordered bipartite graph. Note that if H has two adjacent consecutive vertices or has no isolated vertices, then the ordered bipartite notation of H is uniquely determined. Hence, we only consider ordered bipartite graphs in this paper.

For a 0-1 matrix A, a *submatrix* of A is obtained from A by deleting rows and columns without permuting rows and columns. The *weight* $w(A)$ of a matrix A is the number of 1s in A. A *pattern* P is a 0-1 matrix with weight of at least 1. For a 0-1 matrix B, we say B *represents* P if P is obtained by changing some 1 entries of B to 0. A *contains* a pattern P if there exists a submatrix of A that represents P. If P is not contained in A we say A *avoids* P.

For an ordered bipartite graph G with $U(G) = \{u_1, u_2, \cdots, u_n\}$ with $u_1 < u_2 < \cdots < u_n$ and $V(G) = \{v_1, v_2, \cdots, v_m\}$ with $v_1 < v_2 < \cdots < v_m$, we define an $n \times m$ matrix $A(G)$ whose row i corresponds to u_i and row j corresponds to v_j. The entry $a_{i,j} = 1$ if $(u_i, v_j) \in E(G)$ and $a_{i,j} = 0$ otherwise. Conversely, given an $n \times m$ 0-1 matrix A, we can define an ordered bipartite graph $G(A)$ whose vertices correspond to the rows and columns of A and edges correspond to the nonzero entries in A.

Define $ex_<(n,m,P)$ for pattern P as the maximum weight of an $n \times m$ 0-1 matrix avoiding P. Hence, $ex_<(n,m,A(H)) = ex_<(n,m,H)$ for every ordered bipartite graph H.

According to [3], we know that for an ordered bipartite graph H:

$$ex_<(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor, H) \le ex_<(n,H) = O(ex_<(n,n,H)\log n).$$

This result shows that $ex_<(n,H)$ can be roughly bounded by $ex_<(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor, H)$ and $ex_<(n,n,H)$ within a $(\log n)$-factor. In Section 2, we further show that for any ordered bipartite graph H, there exists an ordered bipartite graph H_1 such that $ex_<(n,H) \le ex_<(n,n,H_1)$. These results show that we are enough to consider the asymptotic bound of $ex_<(n,n,P)$ to find the asymptotic bound of $ex_<(n,H)$.

The case when the underlying graph of H contains a cycle has already been discussed in [3]. We will focus on the case when H is acyclic. For this special case, Füredi and Hajnal [6] conjectured that $ex_<(n,n,H) = O(n \log n)$ for any ordered bipartite graph H, but the conjecture was later refuted by Pettie [4], who constructed a pattern X such that $ex_<(n,n,X) = \Omega(n \log n \log\log n)$. Pach and Tardos prosposed a weaker upper bound of $n(\log n)^{O(1)}$ in [3], which is likely to be true.

In Section 3, we inductively construct a family of acyclic patterns $\mathcal{P}$ such that for any positive integer m, there exists $P \in \mathcal{P}$ satisfying:

$$ex_<(n,n,P) = \Omega(n \log n \log\log n \cdots \underbrace{\log\log\cdots\log}_{m \text{ iterations}} n).$$

In Section 4, we suggest a new method of partitioning a 0-1 matrix to obtain an upper bound of $ex_<(n,n,P)$ for a pattern P. This gives an explicit upper bound for certain cases. We expect that this method can be generalized.

For convenience of notation, every log in this paper is base 2. In addition, for an $n \times m$ matrix A, we define an $n \times m$ matrix $\overline{A}$ as $\overline{A}(i,j) = A(n-i,j)$ and denote I_n as an $n \times n$ identity matrix.

2. Relationship between $ex_<(n,H)$ and $ex_<(n,n,H)$

Pach and Tardos proved the following proposition about $ex_<(n,H)$ and $ex_<(n,n,A(H))$.

Proposition 2.1. (Pach and Tardos [3]) *Let H be an ordered graph with interval chromatic number 2 which has a unique decomposition into two intervals. Then we have*

$$ex_<(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor, A(H)) \le ex_<(n,H) = O(ex_<(n,n,A(H))\log n)$$

This results shows that $ex_<(n,H)$ can be roughly bounded by $ex_<(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor, A(H))$ and $ex_<(n,n,A(H))$ within a $(\log n)$-factor. In this section, we will show that for any ordered bipartite graph H, there exists an ordered bipartite graph H_1 such that $ex_<(n,H) \le ex_<(n,n,H_1)$.

We denote an ordered bipartite graph H as

$$\begin{aligned} U(H) &= \{u_1, u_2, \cdots, u_p\},\ u_1 < u_2 < \cdots < u_p \\ V(H) &= \{v_1, v_2, \cdots, v_q\},\ v_1 < v_2 < \cdots < v_q. \end{aligned}$$

Lemma 2.1. *Given a fixed ordered bipartite graph H, if the smallest vertex in $V(H)$ and the largest vertex in $U(H)$ are adjacent, then*

$$ex_<(n, H) \leq ex_<(n, n, H).$$

Proof. Let G be an ordered graph of n vertices with maximum number of edges while avoiding H. Enumerate the vertices of G as $k_1 < k_2 < \cdots < k_n$. Construct an ordered bipartite graph G' with $2n$ vertices enumerated as $k_1 < k_2 < \cdots < k_n$, $l_1 < l_2 < \cdots < l_n$ and $U(G') = \{k_1, k_2, \cdots, k_n\}$, $V(G') = \{l_1, l_2, \cdots, l_n\}$. For every $i, j \in [n]$, $(k_i, l_j) \in E(G')$ is equivalent to $j > i$ and $(k_i, k_j) \in E(G)$. Hence, $|E(G')| = |E(G)| = ex_<(n, H)$. Suppose H is contained in G'. Let k_{r_i} and l_{s_j} be the vertices corresponding to u_i and v_j. Since k_{r_p} and l_{s_1} are adjacent, $s_1 > r_p$. Therefore, there exists a subgraph of G isomorphic to H with vertices $k_{r_1}, k_{r_2}, \cdots, k_{r_p}, k_{s_1}, k_{s_2}, \cdots, k_{s_q}$, which is a contradiction. Therefore,

$$ex_<(n, H) = |E(G')| \leq ex_<(n, n, H). \quad \square$$

Define H_1 as an ordered bipartite graph derived from H by adding a leaf $v_0 \in V(H_1)$ adjacent to u_p with $v_0 < v_1 < v_2 < \cdots < v_q$.

$$\begin{aligned} U(H_1) &= U(H),\ V(H_1) = V(H) \cup \{v_0\}, \\ E(H_1) &= E(H) \cup \{(u_p, v_0)\}. \end{aligned}$$

In particular, when H is acyclic, H_1 is also acyclic.

The following theorem directly follows from Lemma 2.1 and the construction of H_1.

Theorem 1. *For a fixed ordered bipartite graph H,*

$$ex_<(n, H) \leq ex_<(n, n, H_1).$$

Proof. Since H_1 contains H, $ex_<(n, H) \leq ex_<(n, H_1)$. By Lemma 2.1, $ex_<(n, H_1) \leq ex_<(n, n, H_1)$. Thus,

$$ex_<(n, H) \leq ex_<(n, H_1) \leq ex_<(n, n, H_1). \quad \square$$

3. Construction of New Lower Bounds

In [4], two patterns shown in the following propositions are constructed to refute the upper bound conjectured by Füredi and Hajnal [6], which is $ex_<(n, n, P) = O(n \log n)$.

Proposition 3.1. *Define a pattern $X = \begin{pmatrix} & \bullet & \bullet \\ \bullet & & \bullet \end{pmatrix}$. Then $ex_<(n, n, X) = \Omega(n \log n)$.*

If we define a 0-1 matrix A as

$$A(i, j) = \begin{cases} 1 & \text{if } j - i = 2^t,\ t = 0, 1, 2, \cdots, \lfloor \log n \rfloor, \\ 0 & \text{otherwise,} \end{cases}$$

then $w(A) = \Omega(n \log n)$ and A does not contain X.

Proposition 3.2. (Pettie [4]) *There exists an acyclic forbidden matrix X for which $ex_<(n,n,X) = \omega(n \log n)$. Specifically, $ex_<(n,n,X) = \Omega(n \log n \log\log n)$ where*

$$X = \begin{pmatrix} & \bullet & & \bullet & \bullet \\ & & \bullet & & \bullet \\ & \bullet & & & \\ \bullet & & & & \bullet \end{pmatrix}.$$

In this section, we will constuct a family of acyclic patterns $\mathcal{P}$ and find the improved lower bound of $ex_<(n,n,P)$ when $P \in \mathcal{P}$. To demonstrate the motivation behind the proof, we include the corresponding ordered bipartite graphs of the 0-1 matrices.

Theorem 2. *Given an acyclic pattern P, there exists an acyclic pattern P' such that*

$$ex_<(n,n,P') = \Omega(n \cdot ex_<(\lceil \log n \rceil, \lceil \log n \rceil, P)).$$

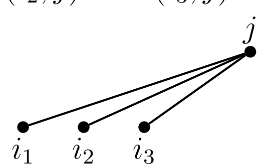
Let P be a $p \times q$ 0-1 matrix. If P is not connected, we can add more 1 entries so that P is still acyclic and connected. Since $ex_<(\cdot\,,\cdot\,,P)$ is non-decreasing when we add 1 entries to P, we can assume that P is connected. Define $k = \lfloor \frac{1}{4} \log n \rfloor$. Let A be the $k \times k$ 0-1 matrix with maximum weight avoiding P.

We construct an $n \times n$ 0-1 matrix A' such that

$$A'(i,j) = \begin{cases} 1 & \text{if } j - i = 4^{k+b} + 4^a,\ A(a+1,b+1) = 1,\ a,b \in \{0,1,\cdots,k-1\}, \\ 0 & \text{otherwise.} \end{cases}$$

For each entry $A(a+1,b+1) = 1$, there exist at least $n - 4^{2k-1} - 4^{k-1} > \frac{n}{2}$ pairs of (i,j) such that $j - i = 4^{k+b} + 4^a$. Hence, $w(A') > \frac{n}{2} \cdot w(A) = \frac{n}{2} \cdot ex_<(k,k,P)$.

Lemma 3.1. *Assume $A'(i_1,j) = A'(i_2,j) = A'(i_3,j) = 1$ and $i_1 < i_2 < i_3$.*



Suppose for $a_r, b_r \in \{0,1,\cdots,k-1\}$, $r \in \{1,2,3\}$,

$$j - i_1 = 4^{k+b_1} + 4^{a_1},\ j - i_2 = 4^{k+b_2} + 4^{a_2},\ j - i_3 = 4^{k+b_3} + 4^{a_3}.$$

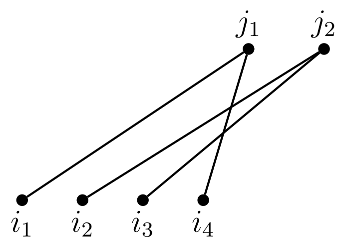
If $b_1 = b_3$, then $b_2 = b_1$ and $a_1 > a_2 > a_3$.

Proof. This follows directly from the inequality

$$j - i_3 < j - i_2 < j - i_1. \quad \square$$

Lemma 3.2. *Assume $A'(i_1,j_1) = A'(i_2,j_2) = A'(i_3,j_2) = A'(i_4,j_1) = 1$, $j_1 < j_2$ and $i_1 \le i_2 < i_3 \le i_4$.*

$$\begin{array}{cc} & \begin{matrix} j_1 & j_2 \end{matrix} \\ \begin{matrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{matrix} & \begin{pmatrix} \bullet & \\ & \bullet \\ & \bullet \\ \bullet & \end{pmatrix} \end{array}$$



Suppose for $a_{r,s}, b_{r,s} \in \{0, 1, \cdots, k-1\}$, $(r, s) \in \{(1, 1), (2, 2), (3, 2), (4, 1)\}$,

$$j_s - i_r = 4^{k+b_{r,s}} + 4^{a_{r,s}}.$$

If $b_{1,1} = b_{4,1}$, *then* $b_{2,2} = b_{3,2}$.

Proof. Since $j_2 - i_2 > j_2 - i_3$, we have $b_{2,2} \geq b_{3,2}$. If $b_{2,2} > b_{3,2}$, then

$$i_3 - i_2 = (j_2 - i_2) - (j_2 - i_3) > 4^{k+b_{3,2}}.$$

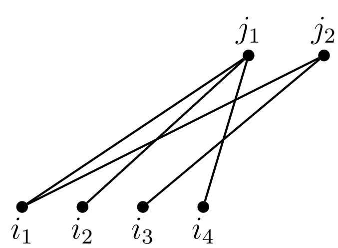
However,

$$i_4 - i_1 = (j_1 - i_1) - (j_1 - i_4) = 4^{a_{1,1}} - 4^{a_{4,1}} < 4^k.$$

From $i_4 - i_1 \geq i_3 - i_2$, this is a contradiction. Hence, $b_{2,2} = b_{3,2}$. □

Lemma 3.3. *Assume for* $j_1, j_2, i_1, i_2, i_3, i_4 \in \{1, 2, \cdots, n\}$, A' *contains a submatrix*

$$\begin{array}{cc} & \begin{array}{cc} j_1 & j_2 \end{array} \\ \begin{array}{c} i_1 \\ i_2 \\ i_3 \\ i_4 \end{array} & \begin{pmatrix} \bullet & \bullet \\ \bullet & \\ & \bullet \\ \bullet & \end{pmatrix} \end{array}$$



Suppose for $a_{r,s}, b_{r,s} \in \{0, 1, \cdots, k-1\}$, $(r, s) \in \{(1, 1), (1, 2), (2, 1), (3, 2), (4, 1)\}$,

$$j_s - i_r = 4^{k+b_{r,s}} + 4^{a_{r,s}}.$$

If $b_{1,1} = b_{4,1}$, *then* $a_{1,1} = a_{1,2}$.

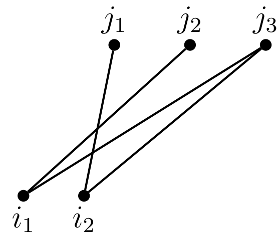
Proof. By Lemma 3.1, we have $b_{1,1} = b_{2,1} = b_{4,1}$ and by Lemma 3.2, we have $b_{1,2} = b_{3,2}$. From the inequality $i_2 - i_1 < i_3 - i_1 < i_4 - i_1$,

$$4^{a_{1,1}} - 4^{a_{2,1}} < 4^{a_{1,2}} - 4^{a_{3,2}} < 4^{a_{1,1}} - 4^{a_{4,1}}.$$

Therefore, $a_{1,1} = a_{1,2}$. □

Lemma 3.4. *Assume for $j_1, j_2, j_3, i_1, i_2 \in \{1, 2, \cdots, n\}$, A' contains a submatrix*

$$\begin{array}{c} \\ i_1 \\ i_2 \end{array} \begin{pmatrix} j_1 & j_2 & j_3 \\ & \bullet & \bullet \\ \bullet & & \bullet \end{pmatrix}$$



Suppose for $a_{r,s}, b_{r,s} \in \{0, 1, \cdots, k-1\}$, $(r, s) \in \{(1, 2), (1, 3), (2, 1), (2, 3)\}$,

$$j_s - i_r = 4^{k+b_{r,s}} + 4^{a_{r,s}}.$$

If $b_{1,3} \neq b_{2,3}$, then $b_{1,3} = b_{1,2}$.

Proof. Since $j_3 - i_1 > j_3 - i_2$ and $b_{1,3} \neq b_{2,3}$, we have $b_{1,3} > b_{2,3}$. From $j_3 - i_1 > j_2 - i_1$, we have $b_{1,3} \geq b_{1,2}$. $i_2 < j_1 < j_2$ implies that

$$(j_2 - i_1) + (j_3 - i_2) > j_3 - i_1.$$

Hence,

$$2(4^{k+b_{1,2}} + 4^{k+b_{2,3}}) > 4^{k+b_{1,3}}.$$

Therefore, $b_{1,3} = b_{1,2}$. $\square$

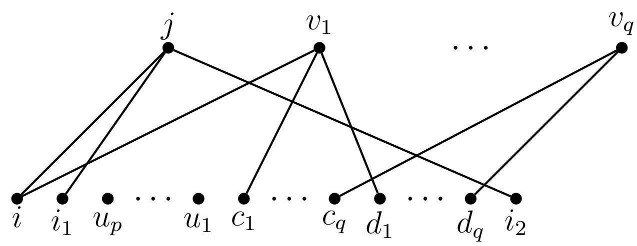
Let P'' be a $(p + 2q + 3) \times (q + 1)$ 0-1 matrix defined as

$$P'' = \left(\begin{array}{c|c} \bullet & \bullet \\ \bullet & \\ \hline & \overline{P} \\ \hline & I_q \\ \hline & I_q \\ \hline \bullet & \end{array}\right)$$

Remember that $\overline{P}$ is obtained by reversing the sequence of rows of P. Note that P'' is an acyclic connected pattern since P is acyclic and connected.

Lemma 3.5. *Assume P'' is contained in A'. Enumerate the corresponding rows and columns of the submatrix of A' which represents P'' as below.*

$$\left(\begin{array}{cc|ccc}
 & j & v_1 & \cdots & v_q \\
i & \bullet & \bullet & & \\
i_1 & \bullet & & & \\ \hline
u_p & & & & \\
\vdots & & & \overline{P} & \\
u_1 & & & & \\ \hline
c_1 & & & & \\
\vdots & & & I_q & \\
c_q & & & & \\ \hline
d_1 & & & & \\
\vdots & & & I_q & \\
d_q & & & & \\ \hline
i_2 & \bullet & & &
\end{array}\right)$$



Suppose for $a, a', b, b' \in \{0, 1, \cdots, k-1\}$,

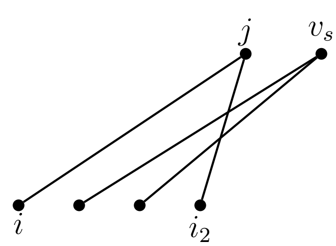
$$j - i = 4^{k+b} + 4^{a},\ j - i_2 = 4^{k+b'} + 4^{a'}.$$

If $b = b'$, then P is contained in A.

Proof. By Lemma 3.2, there exists $b_s \in \{0, 1, \cdots, k-1\}$ for all $s \in \{1, 2, \cdots, q\}$ such that

$$v_s - u = 4^{k+b_s} + 4^{a'_u},\ a'_u \in \{0, 1, \cdots, k-1\}$$

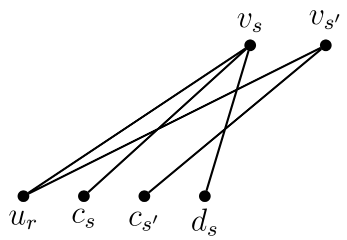
for all $u \in [i_1, i_3]$ whenever $A'(u, v_s) = 1$.



By Lemma 3.3, there exists $a_r \in \{0, 1, \cdots, k-1\}$ for all $r \in \{1, 2, \cdots, p\}$ such that

$$v_s - u_r = 4^{k+b_s} + 4^{a_r},\ \forall v_s \in \{v_1, v_2, \cdots, v_q\}$$

whenever $A'(u_r, v_s) = 1$. Similarly, $v_1 - i = 4^{k+b_1} + 4^{a}$.



Define $S = \{v_s \in \{v_1, v_2, \cdots, v_q\} | v_s - i = 4^{k+b_s} + 4^a\}$. Suppose $S \neq \{v_1, v_2 \cdots, v_q\}$. S is nonempty because $v_1 \in S$. Since P is connected, there exists $v_s, v_{s'} \in \{v_1, v_2, \cdots, v_q\}$ and $u_r \in \{u_1, u_2, \cdots, u_p\}$, such that $v_s \in S$, $v_{s'} \notin S$, and $A'(u_r, v_s) = A'(u_r, v_{s'}) = 1$. Hence, $v_{s'} - u_r = 4^{k+b_{s'}} + 4^{a_r}$, $v_s - u_r = 4^{k+b_s} + 4^{a_r}$ from the conclusion above, and $v_s - i = 4^{k+b_s} + 4^a$. Then, $v_{s'} - i = 4^{k+b_{s'}} + 4^a$ which implies $v_{s'} \in S$, a contradiction. Therefore, $S = \{v_1, v_2 \cdots, v_q\}$ which means that

$$v_s - i = 4^{k+b_s} + 4^a, \ \forall v_s \in \{v_1, v_2, \cdots, v_q\}.$$

Similarly, for $u_r \in \{u_1, u_2, \cdots, u_p\}$, there exists $v_s \in \{v_1, v_2, \cdots, v_q\}$ such that $A'(u_r, v_s) = 1$. Since $v_s - u_r = 4^{k+b_s} + 4^{a_r}$ and $v_s - i = 4^{k+b_s} + 4^a$, we have

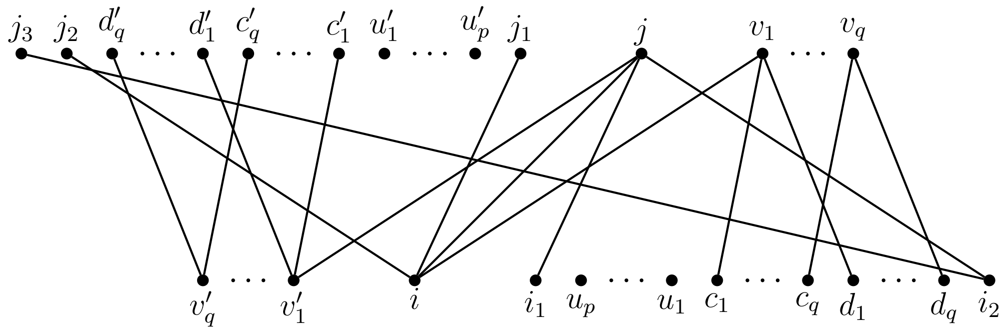
$$u_r - i = 4^a - 4^{a_r}, \ \forall u_r \in \{u_1, u_2, \cdots, u_p\}.$$

$u_p < u_{p-1} < \cdots < u_1$ and $v_1 < v_2 < \cdots < v_q$ implies that $a_1 < a_2 < \cdots < a_p$ and $b_1 < b_2 < \cdots < b_q$. In addition, if $A'(u_r, v_s) = 1$ then $A(a_r + 1, b_s + 1) = 1$. Therefore, P is contained in A. □

We now give a proof for Theorem 2.

Proof of Theorem 2. Let P' be a $(p+3q+3) \times (p+3q+4)$ matrix defined as below. Suppose P' is contained in A' and enumerate the corresponding rows and columns of the submatrix of A' which represents P'' as below.

$$P' = \begin{array}{cc}
 & \begin{array}{cc|ccc|ccc|ccc|cc|ccc} j_3 & j_2 & d'_q & \cdots & d'_1 & c'_q & \cdots & c'_1 & u'_1 & \cdots & u'_p & j_1 & j & v_1 & \cdots & v_q \end{array} \\
\begin{array}{c} v'_q \\ \vdots \\ v'_1 \\ i \\ i_1 \\ u_p \\ \vdots \\ u_1 \\ c_1 \\ \vdots \\ c_q \\ d_1 \\ \vdots \\ d_q \\ i_2 \end{array} &
\left(\begin{array}{cc|ccc|ccc|ccc|cc|ccc}
 & & & & & & & & & & & & & & & \\
 & & & I_q & & & I_q & & & \overline{P^T} & & & & & & \\
 & & & & & & & & & & & & \bullet & & & \\ \hline
 & \bullet & & & & & & & & & & \bullet & \bullet & \bullet & & \\
 & & & & & & & & & & & & \bullet & & & \\ \hline
 & & & & & & & & & & & & & & & \\
 & & & & & & & & & & & & & & \overline{P} & \\
 & & & & & & & & & & & & & & & \\ \hline
 & & & & & & & & & & & & & & & \\
 & & & & & & & & & & & & & & I_q & \\
 & & & & & & & & & & & & & & & \\ \hline
 & & & & & & & & & & & & & & & \\
 & & & & & & & & & & & & & & I_q & \\
 & & & & & & & & & & & & & & & \\ \hline
\bullet & & & & & & & & & & & & \bullet & & &
\end{array}\right)
\end{array}$$



Note that P' is an acyclic pattern since P'' is acyclic. Let

$$j - i = 4^{k+b} + 4^{a},\ j - i_2 = 4^{k+b'} + 4^{a'},\ j_2 - i = 4^{k+b''} + 4^{a''}$$

for $a, b, a', b', a'', b'' \in \{0, 1, \cdots, k-1\}$.

Case 1: $a = a'$

By Lemma 3.5, P is contained in A which is a contradiction.

Case 2: $a \neq a'$

By Lemma 3.4, $a = a''$. Since A' is symmetric with respect to the antidiagonal, antidiagonal symmetry of P' disregarding the first column implies that P is contained in A. Hence, it is a contradiction.

Thus, P' is not contained in A', which follows that

$$ex_<(n, n, P') \geq w(A') > \frac{n}{2} \cdot ex_<(k, k, P).$$

Because $\lceil \log n \rceil = O(k)$, we have $ex_<(\lceil \log n \rceil, \lceil \log n \rceil, P) = O(ex_<(k, k, P))$. Therefore,

$$ex_<(n, n, P') = \Omega(n \cdot ex_<(\lceil \log n \rceil, \lceil \log n \rceil, P)). \quad \square$$

Using the matrix X in Proposition 3.1 as the base case, we can inductively use Theorem 2 to prove the following corollary.

Corollary 3.1. *For any positive integer m, there exists an acyclic pattern P such that*

$$ex_<(n, n, P) = \Omega(n \log n \log\log n \cdots \underbrace{\log\log \cdots \log}_{m \text{ iterations}} n).$$

4. Upper Bound of $ex_<(n, n, P)$

Pach and Tardos have suggested an upper bound for small ordered forbidden graph.

Proposition 4.1. (Pach and Tardos [3]) *For any acyclic ordered forbidden graph H on at most 6 vertices with interval chromatic number 2, we have*

$$ex_<(n, H) \leq n(\log n)^{O(1)}.$$

In this section, we try new approaches of partitioning the matrix and focus on particular family of patterns P with $ex_<(n,n,P) = O(n \log n)$.

For nonnegative integers k and l, let P_1 be a $2 \times (k+l+1)$ acyclic pattern such that

$$P_1(i,j) = \begin{cases} 1 & (i,j) \in \{(1,j) | j \in [1, k+1]\} \cup \{(2,j) | j \in [k+1, k+l+1]\}, \\ 0 & \text{otherwise.} \end{cases}$$

$$P_1 = \begin{pmatrix} & 1 & 2 & \cdots & k+1 & k+2 & \cdots & k+l+1 \\ 1 & \bullet & \bullet & \cdots & \bullet & & & \\ 2 & & & & \bullet & \bullet & \cdots & \bullet \end{pmatrix}$$

If $k < l$, then rotate P_1 by 180 degrees. Without loss of generality, assume $k \geq l$.
For any $n \times m$ 0-1 matrix A, denote

$$a_j(A) = \sum_{i=1}^{\lfloor n/2 \rfloor} A(i,j),\ \ b_j(A) = \sum_{i=\lfloor n/2 \rfloor + 1}^{n} A(i,j).$$

and we decompose a matrix A into an $\lfloor n/2 \rfloor \times m$ 0-1 matrix A_1 which consists of the first $\lfloor n/2 \rfloor$ rows, and an $\lceil n/2 \rceil \times m$ 0-1 matrix A_2 which consists of the last $\lceil n/2 \rceil$ rows.

Lemma 4.1. *For an $n \times m$ 0-1 matrix A, suppose*

$$\sum_{j=1}^{m} \min(a_j(A), b_j(A)) > k\lfloor n/2 \rfloor + l\lceil n/2 \rceil.$$

Then P_1 is contained in A with the first row contained in A_1 and the second row contained in A_2.

Proof. Constuct an $n \times m$ 0-1 matrix A' from A by deleting the k leftmost nonzero entries for each row i of A with $1 \leq i \leq \lfloor n/2 \rfloor$ and the l rightmost nonzero entries for each row i of A with $\lfloor n/2 \rfloor + 1 \leq i \leq n$. If there are not enough nonzero entries in a row, delete all the nonzero entries. Then,

$$\sum_{j=1}^{m} \min(a_j(A'), b_j(A')) > 0$$

which implies the existence of column j with $A(i_1,j) = A(i_2,j) = 1$, $1 \leq i_1 \leq \lfloor n/2 \rfloor$, $\lfloor n/2 \rfloor + 1 \leq i_2 \leq n$ with row i_1 has k nonzero entries left to $A(i_1,j)$ and row i_2 has l nonzero entries right to $A(i_2,j)$. Hence, we are done. $\square$

For nonnegative integers p, q, k, l, let P_2 be a $(p+q) \times (k+l+1)$ acyclic pattern such that

$$P_2(i,j) = \begin{cases} 1 & (i,j) \in \{(1,j) | j \in [1, k+1]\} \cup \{(p+q, j) | j \in [k+1, k+l+1]\} \\ & \qquad \cup \{(i,1) | i \in [1,p]\} \cup \{(i, k+l+1) | i \in [p+1, p+q]\}, \\ 0 & \text{otherwise.} \end{cases}$$

$$P_2 = \begin{array}{cc}
 & \begin{array}{ccccccc} 1 & 2 & \cdots & k+1 & k+2 & \cdots & k+l+1 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ \vdots \\ p \\ p+1 \\ \vdots \\ p+q \end{array} &
\begin{pmatrix}
\bullet & \bullet & \cdots & \bullet & & & \\
\bullet & & & & & & \\
\vdots & & & & & & \\
\bullet & & & & & & \\
 & & & & & & \bullet \\
 & & & & & & \vdots \\
 & & & \bullet & \bullet & \cdots & \bullet
\end{pmatrix}
\end{array}$$

If $k < l$, then rotate P_2 by 180 degrees. Without loss of generality, assume $k \geq l$ through rotation.

Lemma 4.2. *For an $n \times m$ 0-1 matrix A, suppose*

$$\sum_{j=1}^{m} \min(a_j(A), b_j(A)) > k\lfloor n/2 \rfloor + l\lceil n/2 \rceil + \max(p, q) \cdot m.$$

Then P_2 is contained in A with the first p rows contained in A_1 and the last q rows contained in A_2.

Proof. Construct an $n \times m$ 0-1 matrix A' from A by deleting the last p nonzero entries in A_1 and the first q nonzero entries in A_2 for each column of A. If there are not enough nonzero entries in a column, delete all the nonzero entries. Then $a_j(A') \leq a_j(A) - p$ and $b_j(A') \leq b_j(A) - q$ for all $j \in [m]$. Hence,

$$\sum_{j=1}^{m} \min(a_j(A'), b_j(A')) > k\lfloor n/2 \rfloor + l\lceil n/2 \rceil.$$

By Lemma 4.1, P_1 is contained in A with the first row contained in A_1 and the second row contained in A_2. Therefore, P_2 is contained in A if we add back the deleted nonzero entries. □

Theorem 3. *Suppose for any $n \times m$ 0-1 matrix A, a given pattern P is contained in A whenever*

$$\sum_{i=1}^{m} \min(a_j(A), b_j(A)) > c_1 n + c_2 m$$

for some nonnegative real constants c_1, c_2. Then,

$$ex_<(n, m, P) \leq (c_1 n + c_2 m)\lceil \log n \rceil + m.$$

Proof. Let A be an $n \times m$ 0-1 matrix with maximum weight avoiding P. By the assumption,

$$\sum_{i=1}^{m} \min(a_j(A), b_j(A)) \leq c_1 n + c_2 m.$$

For each $j \in [m]$, we delete $a_j(A)$ nonzero entries in A_1 if $a_j(A) \leq b_j(A)$, and $b_j(A)$ nonzero entries in A_2 otherwise. Then, A has two $\lfloor n/2 \rfloor \times m_1$ and $\lceil n/2 \rceil \times m_2$ submatrices with $m_1 + m_2 = m$ and all the other entries not in the submatrices zero. Hence,

$$ex_<(n, m, P) \leq ex_<(\lfloor n/2 \rfloor, m_1, P) + ex_<(\lceil n/2 \rceil, m_2, P) + c_1 n + c_2 m.$$

Apply this process for $\lfloor n/2 \rfloor \times m_1$ and $\lceil n/2 \rceil \times m_2$ matrices again and continue. This process terminates in $\lceil \log n \rceil$ times and we are left with matrices with single row whose sum of the number of columns is equal to m. Therefore,

$$ex_<(n, m, P) \le (c_1 n + c_2 m)\lceil \log n \rceil + m. \quad \square$$

The following results are direct applications of Theorem 3 to Lemma 4.2.

Corollary 4.1.

$$ex_<(n, m, P_2) \le \left(\frac{k+l}{2} n + \max(p, q) m\right) \lceil \log n \rceil + m.$$

For the pattern $X = \begin{pmatrix} \bullet & \bullet & \\ \bullet & & \\ & & \bullet \\ & \bullet & \bullet \end{pmatrix}$, we can improve the bound of $ex_<(n, n, X)$ from $O(n(\log n)^2)$ proven in [3] to $O(n \log n)$ with a specific constant,

$$ex_<(n, n, X) \le 2n\lceil \log n \rceil + n.$$

We define a $p \times k$ pattern X a *walk pattern* if all the entries in X is zero except for a sequence of nonzero entries $x_1, x_2, \cdots, x_{p+k-1}$ with $x_1 = P(p, 1)$, $x_{p+k-1} = P(1, k)$, and two consecutive entries are adjacent. The following 4×4 pattern is an example of walk pattern.

$$\begin{pmatrix} & & & \bullet \\ & & & \bullet \\ & \bullet & \bullet & \bullet \\ \bullet & \bullet & & \end{pmatrix}$$

The following corollary extends our result.

Corollary 4.2. *Suppose there is a $(p+q) \times (k+l+1)$ pattern Z for nonnegative intergers p, q, k, l. If there exist $p \times (k+1)$ and $q \times (l+1)$ walk patterns X and Y such that $Z(i, j) = X(i, j)$ for $i \in [p], j \in [k+1]$, $Z(i+p, j+k) = Y(i, j)$ for $i \in [q], j \in [l+1]$, and $Z(i, j) = 0$ elsewhere, then*

$$ex_<(n, n, Z) = O(n \log n).$$

Proof. Let A be a 0-1 matrix. By Theorem 3, we are sufficient to prove that there exist nonnegative real constants c_1, c_2 such that whenever

$$\sum_{i=1}^{m} \min(a_j(A), b_j(A)) > c_1 n + c_2 m,$$

then Z is contained in A with the first p rows in A_1 and the last q rows in A_2. Apply induction on $p + q + k + l$. Delete the leftmost or rightmost nonzero entries for each row or delete the first nonzero entries in A_1 or the last nonzero entries in A_2. Then as in the proof of Lemma 4.1 and Lemma 4.2, we are done by the assumption. $\square$

5. Open Problems and Closing Remarks

Theorem 1 allows us to consider $ex_<(n,n,P)$ when studying the bound of $ex_<(n,H)$. By Corollary 3.1, we have found the pattern P with $ex_<(n,n,P) = \Omega(f(m,n))$ where $f(m,n) = n \log n \log\log n \cdots \underbrace{\log\log\cdots\log}_{m \text{ iterations}} n$. Inferring from the fact that sum of the reciprocals of $f(m,n)$ always diverges for every fixed positive integer m, we suggest the following conjecture.

Conjecture 1. *For an acyclic pattern P, the following sum of the reciprocals of extremal function diverges*

$$\sum_n \frac{1}{ex_<(n,P)} = \infty.$$

This conjecture guarantees a sharp bound for $ex_<(n,P)$. We suggest some open problems.

Problem 1. What is the family of acyclic patterns $\mathcal{P}$ such that $ex_<(n,n,P) = O(n\log n)$ for all $P \in \mathcal{P}$?

In section 4, we have observed some patterns P such that $ex_<(n,n,P) = O(n\log n)$. The method of partitioning the matrix seems likely to be generalized and find more patterns of P with such condition.

Problem 2. Is there an acyclic pattern P such that $ex_<(n,n,P) = \omega(f(m,n))$ for any positive integer m?

This problem is closely related to Conjecture 1. If such a P exists, then the conjecture is not true. In the proof of Theorem 2, we have used the sum of two powers of four for the construction of the 0-1 matrix. Using different set of positive integers such as the sum of several powers of four might increase the lower bound.

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