Second Differences by a Second Route

The point of Problem 8 (Section 1.2) is that we do not want to "square" the centered first difference, because the result stretches from $x - 2\Delta x$ to $x + 2\Delta x$. The first difference is $(u_{j+1} - u_{j-1})/2h$, and once more produces $(u_{j+2} - 2u_j + u_{j-2})/(2h)^2$. Involving u_{j-2} and u_{j+2} is not necessary and not convenient, when second order accuracy is our goal.

New point This squaring will succeed when we start with $\Delta u_j = (u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}})/h$. Repeating that centered first difference gives the second difference we want, and the half steps disappear. Now $\Delta^2 u_j$ reaches only to u_{j+1} and u_{j-1}:

$$\Delta(\Delta u_j) = \frac{1}{h}[(u_{j+1} - u_j)/h - (u_j - u_{j-1})/h] = (u_{j+1} - 2u_j + u_{j-1})/h^2.$$

Second point A good way to see the algebra is to apply these differences to exponential functions e^{ikx}. Then $u_j = e^{ikjh}$. The first difference is:

$$\frac{1}{h}\left[e^{ik(j+\frac{1}{2})h} - e^{ik(j-\frac{1}{2})h}\right] = \frac{e^{ikh/2} - e^{-ikh/2}}{h}\, e^{ikjh} = \lambda e^{ikjh}.$$

So the exponentials e^{ikx} are eigenfunctions of the first difference Δ, and of the first derivative. The eigenvalues are different! The derivative gives ike^{ikx} with eigenvalue ik. The difference has λ close to ik when k is small:

$$\lambda = \frac{e^{ikh/2} - e^{ikh/2}}{h} = ik\left(\frac{\sin kh/2}{kh/2}\right) \approx ik. \tag{1}$$

Squaring d/dx will give the eigenvalue $(ik)^2 = -k^2$. The eigenfunction is still e^{ikx}. Squaring Δ will give the combination $2 - 2\cos kh$ that we see over and over in the book:

$$\lambda^2 = \left(\frac{e^{ikh/2} - e^{-ikh/2}}{h}\right)^2 = \frac{e^{ikh} - 2 + e^{-ikh}}{h^2} = -\frac{2 - 2\cos kh}{h^2}. \tag{2}$$

The middle expression shows the $1, -2, 1$ coefficients that come from a second difference.

Final step Compare these eigenvalues with the exact ik and $(ik)^2$. One way is to look at the differences $\lambda - ik$ and $\lambda^2 - (ik)^2$. This will show the second order accuracy of Δ and Δ^2:

$$ik - ik\left(\frac{\sin kh/2}{kh/2}\right) \approx \frac{ik}{6}(kh/2)^2 \tag{3}$$

In that step, $\sin\theta = \theta - \theta^3/6 + \cdots$ gives $(\sin\theta)/\theta = \operatorname{sinc}\theta \approx 1 - \theta^2/6$.

Squaring (for the second difference) gives $\operatorname{sinc}^2\theta \approx 1 - \theta^2/3$. Then the error term for second differences is $(ik)^2$ times $(kh/2)^2/3$.

A better comparison is to divide instead of subtract. The ratio λ/ik is approximate/exact:

$$\frac{\lambda}{ik} = \frac{\sin kh/2}{kh/2} = \operatorname{sinc}\left(\frac{kh}{2}\right). \tag{4}$$

For small k this ratio is near 1. Notice that the sinc function is normalized in signal processing (not here!) to be defined as $\sin(\pi x)/\pi x$. The crucial dimensionless quantity is clearly seen to be kh.

To double the range of frequency resolution, h must be cut in half. In other words we need a fixed number of meshpoints per wavelength, to maintain a specified accuracy. In practical wave problems, that fixed number of meshpoints in the shortest wavelength is about 10.