On the Nature of Gödel's Second Incompleteness Theorem

Dan Willard
(University at Albany, SUNY)

Let $A(x, y, z)$ denote a 3-way predicate relation indicating that $x + y = z$, and let $M(x, y, z)$ indicate that $x * y = z$. Let us say an axiom system α *recognizes* addition and multiplication as **"Total"** functions iff it can prove:

$$\forall x \; \forall y \; \exists z \; A(x, y, z) \quad \text{AND} \quad \forall x \; \forall y \; \exists z \; M(x, y, z) \; .$$

In several recent articles, we have shown how such totality conditions are related to both generalizations and boundary-case exceptions for the Second Incompleteness Theorem.

For instance, our article [2] showed how essentially all axiom systems recognizing addition and multiplication as "total" functions are unable to prove a theorem affirming their semantic tableaux consistency. In contrast, [1] showed exceptions to the semantic tableaux version of the Second Incompleteness Theorem are feasible when an axiom system recognizes addition but not multiplications as a total function. In this talk, we will outline several more recent results we have published during the last 2 years:

1. Consider a language that has function symbols for the operations of addition and doubling, but which treats multiplication instead as a 3-way predicate. Let Π_n^* and Σ_n^* denote the analogs of classic arithmetic's Π_n and Σ_n sentences except that we now insist that the bounded quantifiers of these sentences are built out of addition and doubling (rather than from addition and multiplication). Let Level-n deduction refer to a hybrid deduction system that resembles semantic tableaux except it has a modus ponens cut-rule available for Π_n^* and Σ_n^* sentences. Then we have established the Second Incompleteness Theorem generalizes for most axiom systems using Level-2 deduction [3], but it admits significant boundary-case exceptions for Level-1 deduction under axiom systems that recognize addition but not multiplications as a total function [4].

2. An infinitized analog of a computer's floating point multiplication operation is very different from integer multiplication because the preceding boundary-case exception to the Second Incompleteness Theorem is compatible with floating point multiplication treated as a total function [5].

3. Also, [6] shows that *both the positive and negative results* of our research generalize from semantic tableaux to Hilbert deduction when one has "naming conventions" replace totality axioms.

References

[1] D. Willard, "Self-Verifying Systems, the Incompleteness Theorem and the Tangibility Reflection Principle", *Journal of Symbolic Logic* 66 #2 (2001) pp. 536-596

[2] D. Willard, "How to Extend The Semantic Tableaux And Cut-Free Versions of the Second Incompleteness Theorem Almost to Robinson's Arithmetic Q", *Journal of Symbolic Logic* 67 #1 (2002) pp. 465-496.

[3] D. Willard, "A Version of the Second Incompleteness Theorem For Axiom Systems that Recognize Addition But Not Multiplication as a Total Function", *First Order Logic Revisited,* Logos Verlag (Berlin) 2004, pp. 337–368.

[4] D. Willard, "An Exploration of the Partial Respects in which an Axiom System Recognizing Solely Addition as a Total Function Can Verify Its Own Consistency", *Journal of Symbolic Logic* 70 #4 (2005) pp. 1171-1209.

[5] D. Willard, "On the Partial Respects in which a Real Valued Arithmetic System Can Verify its Tableaux Consistency" SpringerVerlag LNCS#3702, (September 2005), pp. 292-306.

[6] D. Willard, "A New Variant of Hilbert Styled Generalization of the Second Incompleteness Theorem and Some Exceptions to It", to appear in 2006 in *Annals of Pure and Applied Logic* (Elsevier's galley proofs already posted).