Problem Set # 04 (MIT 18.311, Spring 2008).

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March 13, 2008.

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Due (room 2-108): Friday March 21, 2008, before 3 PM.

Contents

NOTES.

n1. The first problem (*Rankine Hugoniot conditions for Isentropic Gas Dynamics*) involves carrying out (for each of the two conservation laws that make up the isentropic Euler equations of Gas Dynamics) exactly the same sort of calculation that was used in the lectures to derive the shock jump conditions for Traffic Flow or River Flows.

n2. The second problem (*Piston problem for Isentropic Gas Dynamics*) is a slight variation of the example done in the lectures (blowing/sucking air at a steady rate into/from a pipe).

n3. Part **A** (the **case** $V < 0$, see the hint) in the second problem (*Piston problem for Isentropic Gas Dynamics*) is .... **THE SPECIAL PART OF THIS PROBLEM SET.**
The rest of the second problem, as well as the
first problem, constitute .... **THE REGULAR PART OF THIS PROBLEM SET.**

1 Statements for the assigned problems.

1.1 Statement for GaDy03: Rankine Hugoniot conditions for Isentropic Gas Dynamics.

The one dimensional isentropic (constant entropy) Euler equations of Gas Dynamics are given by

$$\left.\begin{aligned} \rho_t + (\rho\, u)_x &= 0, \\ (\rho\, u)_t + (\rho\, u^2 + p)_x &= 0, \end{aligned}\right\} \tag{1.1}$$

where $\rho = \rho(x,\, t)$, $u = u(x,\, t)$, and $p = p(x,\, t)$ are the gas mass density, flow velocity, and pressure, respectively. The first equation implements the conservation of mass, and the second the conservation of momentum. These equations are complemented by an **equation of state**, relating the pressure to the density. This takes the form

$$p = P(\rho), \quad \text{where } P \text{ is a function satisfying } \frac{dP}{d\rho} > 0. \tag{1.2}$$

For example, for an ideal gas $P = \kappa\rho^\gamma$, where $\kappa > 0$ and $1 < \gamma < 2$ are constants.

Use conservation to derive the Rankine-Hugoniot jump conditions that shocks for the equations above must satisfy. In other words, consider a solution of the equations of the form

$$\left.\begin{array}{lllllll} \rho & = & \rho_L & \text{and} & u & = & u_L \quad \text{for} \quad x < s\,t, \\ \rho & = & \rho_R & \text{and} & u & = & u_R \quad \text{for} \quad x > s\,t, \end{array}\right\} \tag{1.3}$$

where $\rho_L > 0$, $\rho_R > 0$, u_L, u_R, and s are constants. Then find conditions that these constants have to satisfy so that mass and momentum are conserved.

Hint 1: In a frame of reference moving with the shock, the fluxes of mass and momentum on each side of the shock must be equal.

Hint 2: Alternatively: use the integral form of the conservation laws across an interval containing the shock; i.e.: $a < s\,t < b$.

Remark 1.1 *One may question whether it makes sense to consider shock wave solutions within the context of the constant entropy assumption.*[1] *However, it can be shown (though we will not do it here) that the amount of entropy produced within a shock is proportional to the cube of the shock strength for weak shocks. Hence,* as long as we use the equations in situations where only weak shocks arise, it does make sense to consider shocks within the context of a constant entropy assumption.

1.2 Statement for GaDy04: Piston problem for Isentropic Gas Dynamics.

Imagine a long pipe full of air at rest, with the piston in it. At time $t = 0$ you start moving the piston at a constant velocity V, which is neither too small, nor too large.[2] What happens?

[1]Constant entropy presumes adiabatic conditions, while changes at a shock are anything but slow. Hence shocks must produce entropy.

[2]The "neither too small" means that we cannot use linearized equations. The "nor too large" means that we can use simplifying assumptions, such as constant entropy flow.

A simple mathematical model for the situation above is as follows: we use the one dimensional isentropic Euler equations of Gas Dynamics (for a polytropic gas)

$$\left.\begin{array}{rcl}\rho_t + (\rho\, u)_x &=& 0,\\ (\rho\, u)_t + (\rho\, u^2 + p)_x &=& 0,\end{array}\right\} \tag{1.4}$$

to model the air,[3] where ρ is the gas density, u is the flow velocity, $p = \kappa\,\rho^\gamma$ is the pressure, $\kappa > 0$ is a constant, $1 < \gamma < 2$ is a constant, t is time, and x is the length along the pipe, measured from the point where the piston started. **The solution to these equations must then be found for $t > 0$ and $x > V\,t$, with the initial and boundary conditions given by**

$$\left.\begin{array}{rl}\rho(x,\,0) = \rho_0 \;\text{ and }\; u(x,\,0) = 0 & \text{for } \;x > 0.\\ u(x,\,t) = V & \text{for } \;x = V\,t > 0.\end{array}\right\} \tag{1.5}$$

Solve this problem.

Hint 1.1 *The equations have two sets of characteristics. On the* right moving ($\mathcal{C}_+$ characteristics) *given by*

$$\frac{dx}{dt} = u + c \quad \textit{the quantity} \quad r = u + \frac{2\,c}{\gamma - 1} \quad \textit{is constant.} \tag{1.6}$$

Similarly, on the left moving ($\mathcal{C}_-$ characteristics) *given by*

$$\frac{dx}{dt} = u - c \quad \textit{the quantity} \quad s = u - \frac{2\,c}{\gamma - 1} \quad \textit{is constant.} \tag{1.7}$$

Here $c = \sqrt{dp/d\rho} = \sqrt{\gamma\, p/\rho} > 0$ *is the sound speed,* r *is called the* right Riemann invariant*, and* s *is called the* left Riemann invariant.
You can now reformulate the problem completely in terms of r *and* s *as follows:*

$$\left.\begin{array}{lrcl}\textbf{1.}\;\; r \;\textit{ is constant along} & \dfrac{dx}{dt} &=& \dfrac{\gamma+1}{4}\,r + \dfrac{3-\gamma}{4}\,s,\\ \textbf{2.}\;\; s \;\textit{ is constant along} & \dfrac{dx}{dt} &=& \dfrac{3-\gamma}{4}\,r + \dfrac{\gamma+1}{4}\,s,\\ \textbf{3.}\;\; r = -s = \dfrac{2\,c_0}{\gamma-1} \quad \textit{on} & x &=& 0,\\ \textbf{4.}\;\; r + s = 2\,V \qquad\quad \textit{on} & x &=& V\,t,\end{array}\right\} \tag{1.8}$$

where c_0 *is the sound speed corresponding to* ρ_0*. Then* u *and* c *follow from* $u = (r+s)/2$ *and* $c = (\gamma-1)\,(r-s)/4$*. Since* c *is an increasing function of* ρ*,* ρ *follows from knowledge of* c*. Note that, since* $c > 0$*,* **a realistic physical solution requires $r > s$ everywhere.**

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[3]For dry air starting at one atmosphere and 15 degrees Celsius: $p = p_0\,(\rho/\rho_0)^\gamma$, where $p_0 = 1.013 \times 10^6$ dyn/cm^2, $\rho_0 = 1.226 \times 10^{-3}$ g/cm^3, and $\gamma = 1.401$.

Note also that the $\mathcal{C}_-$ characteristics that start on $x > 0$ move backwards through the gas particles (since their velocity $u - c$ is less than the gas velocity) and so must eventually approach the gas particles that start next to the piston at time $t = 0$. Hence it can be argued that

$$\textbf{Every parcel of gas, at any time, is connected with the initial data } (t = 0) \textbf{ on } x > 0 \textbf{ via a } \mathcal{C}_- \textbf{ characteristic.} \tag{1.9}$$

Since, along $x > 0$ for $t = 0$, $s = -2\,c_0/(\gamma - 1)$ is identically constant, and since s is constant along the $\mathcal{C}_-$ characteristics, we can use (1.9) to argue that

$$\textbf{The left Riemann invariant } s \equiv -2\,c_0/(\gamma - 1) \textbf{ is constant throughout the flow.} \tag{1.10}$$

HOWEVER, there is a proviso to this conclusion. *The argument that s is constant along the $\mathcal{C}_-$ characteristics depends on the solution being smooth. It will not apply across shocks. Thus* **(1.10) holds as long as the flow develops no shocks.**

At any rate, you can start solving the problem by assuming that (1.10) applies. Then the problem reduces to calculating the $\mathcal{C}_+$ characteristics, along which is r is constant. These are straight lines, since along each of them both r and s are constant, as follows from (1.10). But then the problem becomes very simple, and it is very similar to the light turns from red to green and from green to red type of problems in traffic flow.

There are two distinct cases that you must consider:

A. $\boxed{\textbf{Case } V < 0.}$ *In this case* **no shock waves arise.**

Start with the assumption that (1.10) holds, and then

- **a1.** *Solve for the $\mathcal{C}_+$ characteristics that start along $x > 0$ on $t = 0$. These carry the value $r = 2\,c_0/(\gamma - 1)$ —* **why?** *— and all have the same slope (calculate it).*
- **a2.** *Solve for the $\mathcal{C}_+$ characteristics that start along $x = V\,t$ for $t > 0$. These carry the value $r = 2\,V + 2\,c_0/(\gamma - 1)$ —* **why?** *— and all have the same slope (calculate it).*
- **a3.** *You will find that* **a1** *and* **a2** *leave an unfilled gap in the $\mathcal{C}_+$ characteristic field — a wedge shaped region in space-time, with its tip at the origin. An expansion fan is needed to fill this region.*

As long as $-2\,c_0/(\gamma - 1) = V_c < V < 0$, the process above will yield a complete solution for the problem. Give explicit formulas for u and c as functions of x and t.

CHALLENGE QUESTIONS: **CQ1.** *What is special about $V = V_c$?*
CQ2. *What is the solution for $V < V_c$?*

B. $\boxed{\textbf{Case } V > 0.}$ *In this case* **a shock wave arises.**

Start as in the prior case. When you reach step **a3**, *you will find that instead of a gap in the $\mathcal{C}_+$ characteristic field, there is region where the characteristics cross: a wedge in space time, with its tip at the origin. Thus a shock is needed to stop the $\mathcal{C}_+$ characteristics from crossing.*

Thus, look for a solution where there is a shock along a path $x = \sigma(t)$, into which the $\mathcal{C}_+$ characteristics converge (where they end, so the crossing is prevented). In order to proceed further, you will need the jump conditions at the shock. These arise from enforcing conservation of mass and momentum across the shock, which yields two equations. Do not worry about the details of these two equations. The important thing is that, at least in principle (see below for why "in principle"), you can think of them in the following way

The **shock jump conditions for a (right) moving shock with speed speed** U *have the form:*

$$U = f(r_L,\, r_R,\, s_R) \quad \textit{and} \quad s_L = g(r_L,\, r_R,\, s_R), \tag{1.11}$$

where

— f *and* g *are some (known) functions.*

— r_L *and* r_R *are the values of the right Riemann invariant along the* $\mathcal{C}_+$ *characteristics converging into the shock — one from the left, and the other from the right, respectively. The* condition that these characteristics converge into the shock is equivalent to

$$u_L + c_L > U > u_R + c_R, \tag{1.12}$$

since the $\mathcal{C}_+$ *characteristics have speed* $u + c$.

— **The $\mathcal{C}_-$ characteristics CROSS the shock path,** *entering from the right (fluid ahead of the shock) and leaving on the left. They carry the value* s_R *when they enter the shock, and come out on the other side with the value* s_L.

The best way to understand what all this means is to **draw a picture in space time, with the shock path, and some typical characteristics:** *the* $\mathcal{C}_+$ *converging into the shock and the* $\mathcal{C}_-$ *crossing it, right to left as time increases.*

Note: why do we say that the above is "in principle"? *The reason is that no explicit expressions for the functions* f *and* g *above exist. These functions are defined implicitly, and (to evaluate them) one must resort to numerical methods.*

IMPORTANT! *Once a shock is introduced,* **the validity of (1.10) must be questioned.** *When a* $\mathcal{C}_-$ *characteristic crosses a shock, the argument leading to the conclusion that along it* s *is constant breaks down — since it depends on derivatives existing. In fact,* s *will have a discontinuity across the shock, so that we* **can ONLY argue that**

The left Riemann invariant is constant $s \equiv \dfrac{-2\,c_0}{\gamma - 1}$ **ahead of the shock at** $x = \sigma(t)$**.**

Now you are ready to do the $V > 0$ case. Proceed as follows.

b1. *Argue that the solution to (1.4 – 1.5) should be a function of x/t only. Hence:*
—The shock speed $\dfrac{d\sigma}{dt} = U$; should be constant, with the shock path given by $x = Ut$.
—The the left Riemann invariant should be constant $s = s_1$ immediately behind the shock, that is: along $x = U\,t - 0$.

Hint: *Let $u = \tilde{u}(x,\, t)$ and $\rho = \tilde{\rho}(x,\, t)$ be the solution to the problem in (1.4 – 1.5). What problem does $u = \tilde{u}(\alpha\, x,\, \alpha\, t)$ and $\rho = \tilde{\rho}(\alpha\, x,\, \alpha\, t)$ (where $\alpha > 0$ is an arbitrary constant) solve? If the solution to (1.4 – 1.5) is unique, what do you conclude? Use this conclusion to show that $\tilde{u}$ and $\tilde{\rho}$ are, in fact, functions of x/t only.*

Now you should be able to argue **(do the argument!)** *that*

The left Riemann invariant is constant $s \equiv s_1$ behind the shock at $x = \sigma(t)$,

where s_1 is some **constant to be determined, different from the constant value ahead of the shock** $\boxed{s = s_0 = \dfrac{-2\,c_0}{\gamma - 1}.}$

b2. *Solve the problem in the region ahead of the shock $x > U\,t$. The solution here is determined by the characteristics (both $\mathcal{C}_+$ and $\mathcal{C}_-$) that start on $x > 0$ for $t = 0$. Write formulas for these characteristics.*

b3. *The solution in the region $V\,t < x < U\,t$ between the piston and the shock is determined as follows:*

-1- *The $\mathcal{C}_-$ characteristics start on $x > 0$ at $t = 0$, and move backwards, carrying the value $s = s_0 = -\dfrac{2\,c_0}{\gamma - 1}$ for the left Riemann invariant s. When they reach the shock, they go through it, and the value of s they carry jumps to a new value $s = s_1$, that should satisfy (1.11).*

-2- *When a $\mathcal{C}_-$ characteristic reaches the piston, the boundary condition there determines a value $r = r_1$ for the $\mathcal{C}_+$ characteristic that starts there.*

-3- *The $\mathcal{C}_+$ characteristic that start at the piston move to the right, and carry a value $r = r_1$ for the right Riemann invariant.*

-4- *When the $\mathcal{C}_+$ characteristics starting at the piston reach the shock, they end there. However, through (1.11), and the values carried by the other characteristics, they determine the shock speed U, and the value s_1 with which the $\mathcal{C}_-$ characteristics leave the shock (see item* **-1-***).*

Of course, you do not know the values of U, s_1, and r_1 mentioned above. However, you should be able to write equations (involving the functions f and g in (1.11)) that can be used to determine them. **WRITE THESE EQUATIONS.** *You will not be able to solve the equations because you do not have expressions for f and g. This is fine; however,* *note that it can be shown that the equations have a unique solution that satisfies (1.12).* **Describe what the solution to the problem looks like, in terms of these values.** **Hint/IMPORTANT:** *in order to fully understand what the stuff in items* **-1-** *through* **-4-** *above means, draw a picture in space-time showing typical characteristics and the shock path.*

C. *Finally:* **what happens in the case** $V = 0$**?**

THE END.