18.086 Pset #1

Due before class starts on Feb. 27th. Late homework will not be given any credit. Indicate on your report whether you have collaborated with others and whom you have collaborated with.

1. (10 points) For the classical fourth order Runge-Kutta, prove the local error is of order four for the nonlinear ODE $u' = f(u)$.

2. (15-18 points) Consider the Kepler problem:

$$q' = H_p, \quad p' = -H_q,$$

where the Hamiltonian is given by

$$H(q,p) = \frac{p_1^2 + p_2^2}{2} - \frac{1}{r} - \frac{0.01}{2r^3}$$

and $r = \sqrt{q_1^2 + q_2^2}$. The initial conditions are

$$q_1(0) = 1 - \beta, q_2(0) = 0, p_1(0) = 0, p_2(0) = \sqrt{(1+\beta)/(1-\beta)}, \beta = 0.6.$$

Clearly, $H' = H_q q' + H_p p' = 0$, so $H(q(t),p(t)) = H(q(0),p(0))$. Thus $|H(q(t),p(t)) - H(q(0),p(0))|$ could be a simple error indicator, which *underestimates* the true error. The solution can be visualized by plotting $q_2(t)$ vs. $q_1(t)$ in the q_2-q_1 plane (phase plane portrait), which looks like curves on a torus. See Figure 0.1.

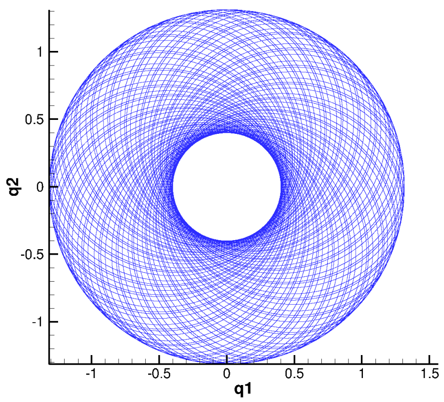

Figure 0.1: phase plane portrait

(a) Use the fourth-order Runge-Kutta to solve this ODE system with $\Delta t = 0.0005$ to the final time $T = 500$ as a reference solution.

(b) Consider the second-order accurate A-stable implicit midpoint method

$$\frac{u^{n+1} - u^n}{\Delta t} = f\left(\frac{u^{n+1} + u^n}{2}, t^{n+\frac{1}{2}}\right)$$

for the ODE $u' = f(u, t)$. Implement the midpoint method for the ODE system. Use Newton's iteration to solve the nonlinear equations.

(c) (**15 points**) Run the fourth-order Runge-Kutta and the midpoint method with $\Delta t = 0.1$ and $\Delta t = 0.01$ to the final time $T = 500$ (four runs in total). Plot the phase plane portraits (four figures with the same scale). Monitor the maximum of $|H(q(t), p(t)) - H(q(0), p(0))|$. Let $\bar{q}(t)$ denote the reference solution. Define the true error as $e(t) = |q(t) - \bar{q}(t)|$. Plot $\log e(t)$ vs. t (four curves in one figure).

(d) (**Bonus, up to 3 points**) What are your observations and conclusions for this particular problem?

3. (12 points) On page 484 of the textbook, Problem 10 and Problem 12.

4. (13-16 points) Consider $u_t = -u_{xxxx}$ on $-1 \leq x \leq 1$ with $u(x, 0) = (1-x^2)^2$ and boundary conditions $u(-1, t) = u(1, t) = u_x(-1, t) = u_x(1, t) = 0$. Design a consistent and stable finite difference method and use it to solve the problem.

(a) (**8 points**) Find a consistent spatial discretization. (Hint: use Taylor expansion).

(b) (**5 points**) Plot your numerical solution at $T = 0.04$ and $T = 0.08$.

(c) (**Bonus, 3 points**) Write down the semidiscrete scheme, which is an ODE system $\mathbf{y}' = A\mathbf{y}$. Choose a n-point grid fine enough to resolve the initial data. Give the explicit formulation of the $n \times n$ matrix A. Plot the stability region boundary curve of your ODE solver and all the eigenvalues of $\Delta t * A$ where Δt is a stable and accurate time step for this grid. What is the largest stable time step? Is it accurate?