18.085 Quiz 2 April 12, 1996

1. An elastic bar made of two materials has a point load at $x = \frac{1}{2}$:

$$-\frac{d}{dx}\left(c\frac{du}{dx}\right) = \delta_{1/2}(x)\,.$$

It is fixed at **both ends:** $u(0) = u(1) = 0$. The materials meet at $x = a$ (and $a < \frac{1}{2}$). The material has $c = 1$ for $x < a$ and $c = 2$ for $x > a$. The **boundary conditions** are $u(0) = u(1) = 0$.

(a) Why can't we solve $-dw/dx = \delta_{1/2}$ first and then solve for $u(x)$?

(b) What is the general form of $u(x)$, **including** arbitrary constant(s), in the interval $[0, a]$ using the equation and the boundary condition $u(0) = 0$?

Answer: $u(x) =$ in the interval $[0, a]$.

(c) Continue $u(x)$ into the interval $[a, \frac{1}{2}]$ keeping the same arbitrary constant(s). How does the change in c (from 1 to 2) affect the slope of u?

____ .

$u(x) =$ in the interval $[a, \frac{1}{2}]$

(d) Continue $u(x)$ into the interval $[\frac{1}{2}, 1]$, keeping the same arbitrary constant(s). What happens to the slope of u at $x = \frac{1}{2}$?

____ .

$u(x) =$ in the interval $[\frac{1}{2}, 1]$

(e) Find the constant and **draw graphs** of $u(x)$ and $w(x) = c\dfrac{du}{dx}$ over the whole interval $[0, 1]$.

2. (a) For a function $f(x+iy)$, how is the y-derivative related to the x-derivative? Make this into an equation:

$$\frac{\partial f}{\partial y}(x+iy) \qquad \frac{\partial f}{\partial x}(x+iy)$$

If we split into $f(x+iy) = u(x,y) + is(x,y)$, what two equations involving u and s come from your one equation above?

(b) Suppose $f(x+iy) = \ln(x+iy)$. What are $u(x,y)$ and $s(x,y)$?? The *unit normal vectors* n_1 and n_2 to the equipotentials $u(x,y)$ and the streamlines $s(x,y)$? Verify that n_1 is perpendicular to n_2.

$u(x,y) =$

$s(x,y) =$

$n_1 =$

$n_2 =$

(c) To find the flux out of a region when the velocity is $v = \text{grad} u$ and the density is 1, what integral would you compute? Do that computation two ways—a **line integral** around the perimeter and a **double integral** using the divergence theorem for $u(x, y)$ above and the $\frac{1}{4}$-ring below.

Flux integral to compute is

For our specific example this integral is

Using the divergence theorem the flux integral is

(d) The flux out of a whole circle $r = R$ is **not zero**. How can this be true when $u_{xx} + u_{yy} = 0$??

3. (a) Suppose that our operators are $C = I$ and $v = w = Au = \left(2\frac{du}{dx}, 3\frac{du}{dy}\right)$. What is A^T? What is the differential equation $A^T C A u = F$?

A^T is

Diff. eqn. is

(b) Find functions to replace $f(x + iy)$ that will give pairs of solutions to the differential equation and **find** a 2nd degree pair. What zero boundary conditions might you impose at points on the boundary of a plane region?

Replace $f(x + iy)$ by ____________

Pair of 2nd degree solutions:

Boundary conditions:

(c) If I give you a vector $v(x, y) = (v_1(x, y), v_2(x, y))$, what are the conditions on v_1 and v_2 to decide whether this vector comes from a potential function u ($v = Au$ with A as above in part (a))?

Condition on v_1 and v_2

(d) Solve the beam equation with $c = 1$ and a point load at $x = \frac{1}{2}$ and fixed ends: $u(0) = u'(0) = u(1) = u'(1) = 0$ and $u'''' = \delta_{1/2}(x)$. What integral does your solution minimize?