18.085 Solutions to Quiz 1 October 5, 2001

1 Problem1

a). "As well as possible" means that we find a (the) vector $\vec{x}$ such that the vector

$$\vec{e} = \vec{b} - A\vec{x}$$

has the least possible length.

b). The fact the first (n-th) entry in $A^T e$ is zero means that e is orthogonal to the first (n-th) row of A^T or, in other words, e is orthogonal to the each column of A. Another way to put it is that e is orthogonal to the column space of A.

2 Problem 2

a) We have

$$A_0 = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}; \quad A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$K_0 = A_0^T A_0 = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$K = A^T A = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

b).

$$A^T b = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

And so

$$x = \left(A^T A\right)^{-1} A^T b = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$$

c). The equation

$$A_0^T A_0 \bar{x} = A_0^T b$$

cannot be solved because A_0 is not full rank. Since the third column of A_0 is a linear combination of the first two (it's minus their sum) it doesn't "contribute" to the column space and, therefore, the column space is the same.

What is the best $\bar{x}$? We can find it by "solving" $A_0^T A_0 \bar{x} = A_0^T b$, for in deriving this equation we never relied on A_0's being full rank. While it cannot be solved uniquely, it can be solved to "within" one degree of freedom Note that

$$\begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ 0 \end{bmatrix}$$

is a solution. All other solutions can be found by adding any vector from the null space of $A_0^T A_0$ to this one. The null space of $A_0^T A_0$ is clearly

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

and so

$$\bar{x} = \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ 0 \end{bmatrix} + C \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

d). Write down G. For the triangle, all nodes are interconnected so

$$G = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and

$$K_0 + G = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

For the 4-node structure, all nodes are connected except 1 and 4. So

$$G = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Also,

$$A_0 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

and

$$A_0^T A_0 + G = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

The resulting matrix simply tells us how many edges share a given node.

3 Problem 3

This is actually a very short problem. Label the edges 1, 2, and 3, starting with the left slanting one and going CW. Let the coordinate system be the standard right-handed Cartesian system. Order degrees of freedom as in (Left Horizontal, Left Vertical, Right Horizontal, Right Vertical).

a). It is clear that the structure can rotate with respect to the bottom pivots. The single word "rotation" would have commanded full credit on part a). Instantaneously, the motion is perpendicular to slanted bars which can be described by the following vector

$$u = \begin{bmatrix} \sin\alpha \\ \cos\alpha \\ \sin\alpha \\ \cos\alpha \end{bmatrix}$$

b). The incidence matrix is

$$A = \begin{bmatrix} -\cos\alpha & \sin\alpha & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & -\cos\alpha & \sin\alpha \end{bmatrix}$$

Now, confirming our intuition in a) we study Au:

$$Au = \begin{bmatrix} -\cos\alpha & \sin\alpha & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -\cos\alpha & \sin\alpha \end{bmatrix} \begin{bmatrix} \sin\alpha \\ \cos\alpha \\ \sin\alpha \\ \cos\alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$