

## Gradient and Divergence / Parallel Table

<b>Gradient</b>	<b>Divergence</b>
$v = \text{grad } u = \nabla u$	$\text{div } w = \nabla \cdot w = 0$
Potential $u(x, y)$ : $v_1 = \frac{\partial u}{\partial x}, v_2 = \frac{\partial u}{\partial y}$	Stream function $s(x, y)$ : $w_1 = \frac{\partial s}{\partial y}, w_2 = -\frac{\partial s}{\partial x}$
Test on $v$ : $\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 0$	Test on $w$ : $\frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} = 0$
Irrotational: zero vorticity	Solenoidal: zero source
Zero circulation around loops:	Zero flux through loops:
$\int v \cdot t \, ds = \int v_1 \, dx + v_2 \, dy = 0$	$\int w \cdot n \, ds = \int w_1 \, dy - w_2 \, dx = 0$
Kirchhoff's Voltage Law	Kirchhoff's Current Law
Equipotentials $u(x, y) = \text{constant}$	Streamlines $s(x, y) = \text{constant}$
$v$ is perpendicular to equipotentials	$w$ is tangent to streamlines

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Green-Gauss Formula  $\iint w \cdot \text{grad } u \, dx \, dy = \iint u(-\text{div } w) \, dx \, dy + \int u w \cdot n \, ds$

$(\text{grad})^T = -\text{div}$  from integration by parts:  $(Au)^T w = u^T (A^T w)$

### Connections when $(v_1, v_2) = (w_1, w_2)$

1. Equipotentials are perpendicular to streamlines
2. Laplace's equation  $\text{div}(\text{grad } u) = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right) = \nabla \cdot \nabla u = 0$
3. Cauchy-Riemann equations  $\frac{\partial u}{\partial x} = \frac{\partial s}{\partial y}$  and  $\frac{\partial u}{\partial y} = -\frac{\partial s}{\partial x}$  connecting  $u$  to  $s$
4. Laplace's equation for  $s$   $\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} = -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x} = 0$
5. Zero vorticity and zero source: Ideal potential flow
6. In two dimensions:  $u(x, y) + is(x, y)$  is a function  $f(x + iy)$