18.085 FALL 2002 QUIZ 3 SOLUTIONS

PROBLEM 1

a) The graph looks like a symmetric butterfly. It is periodic with no discontinuities (but has kinks at $x = 0$ and $x = k\pi$, where the slope jumps).

$$\begin{aligned} c_k &= \frac{1}{2\pi}\left(\int_{-\pi}^{0} e^{x}e^{-ikx}dx + \int_{0}^{\pi} e^{-x}e^{-ikx}dx\right) \\ &= \frac{1}{2\pi}\left(\int_{-\pi}^{0} e^{x-ikx}dx + \int_{0}^{\pi} e^{-x-ikx}dx\right) \\ &= \frac{1}{2\pi}\left(\frac{1}{1-ik}\left(1-e^{-\pi}(-1)^k\right) + \frac{1}{-1-ik}\left(e^{-\pi}(-1)^k - 1\right)\right) \\ &= \frac{1-e^{-\pi}(-1)^k}{\pi\left(k^2+1\right)}, \text{ using } e^{ik\pi} = e^{-ik\pi} = (-1)^k \end{aligned}$$

: :

b) df/dx: The right half of the graph becomes $-e^{-x}$: now it's an odd butterfly. d^2f/dx^2: Back to the even butterfly with δ-functions $-2\delta(x)$ and $2e^{-\pi}\delta(x-\pi)$. The δ-functions come from jumps in df/dx.

c) From b)

$$-\frac{d^2f}{dx^2} + f = 2\delta(x) - 2e^{-\pi}\delta(x-\pi)$$

If

$$f(x) = \sum c_k e^{ikx}$$

then (recall $\delta(x) = \frac{1}{2\pi}\sum e^{ik\pi}$, $\frac{d}{dx} \to ik$):

$$\begin{aligned} \sum c_k k^2 e^{ikx} + \sum c_k e^{ikx} &= \frac{1}{\pi}\sum e^{ikx} - \frac{1}{\pi}e^{-\pi}\sum e^{ik(x-\pi)} \\ &= \frac{1}{\pi}\sum e^{ikx} - \frac{1}{\pi}(-1)^k e^{-\pi}\sum e^{ikx} \end{aligned}$$

Equate coefficients of like terms:

$$\left(k^2+1\right)c_k = \frac{1}{\pi}\left(1-(-1)^k e^{-\pi}\right)$$

The solution of this algebraic equation is the same answer as before,

$$c_k = \frac{1-(-1)^k e^{-\pi}}{\pi\left(k^2+1\right)}$$

PROBLEM 2

a) The easiest way to perform this convolution is the matrix way:

$$u * v = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}$$

b)

$$\begin{aligned} c &= F^{-1}u = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} \\ d &= F^{-1}v = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \\ 0 \end{bmatrix} \\ h &= cd = \begin{bmatrix} \frac{1}{4} \\ 0 \\ -\frac{1}{4} \\ 0 \end{bmatrix} \end{aligned}$$

Then

$$u * v = 4Fh = 4\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ 0 \\ -\frac{1}{4} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}$$

The answer in part a) is confirmed!

c) We have

$$DFT\,(u * z) = \left(\frac{1}{2}C_1, 0, \frac{1}{2}C_3, 0\right)$$

Therefore, we need $C_1 = 0$, $C_3 = 0$ in order to have $u * z = (0,0,0,0)$. So the Fourier coefficients of z must be $(0, a, 0, b)$ for any a and b. Then

$$z = F\begin{bmatrix} 0 \\ a \\ 0 \\ b \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} 0 \\ a \\ 0 \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ i\,(a-b) \\ -\,(a+b) \\ -i\,(a-b) \end{bmatrix}$$

This can be expressed more simply as

$$z = \begin{bmatrix} x \\ y \\ -x \\ -y \end{bmatrix}$$

We can check this result by convolution:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ -x \\ -y \end{bmatrix} = 0$$

PROBLEM 3

a) Since $f(x)$ vanishes outside of $[-1,1]$ the limits of integration are -1 and 1:

$$\hat{f}(k) = -\int_{-1}^{0} e^{-ikx} dx + \int_{0}^{1} e^{-ikx} dx = 2i \frac{\cos k - 1}{k}$$

b) Multiply by ik for the derivative:

$$\hat{D}(k) = 2 - e^{ik} - e^{-ik} = -2(\cos k - 1)$$

Convolution becomes multiplication in the Fourier domain. Therefore the transform is

$$\hat{C}(k) = \hat{D}(k) \hat{f}(k) = -4i \frac{(\cos k - 1)^2}{k}.$$

This confirms that the derivative $D = df/dx$ consists of δ-functions at $x = 0, -1, 1$.