HOMEWORK 4

II.3.7

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

The normal equation is $A^t A\hat{u} = A^t b$, so

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

The solution is

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

II.3.8

$$p = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

which lies on $3 - x$. The error is

$$e = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

and

$$A^t e = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = 0$$

II.3.12

In this case

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$$

The "unsolvable equations" are

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

The normal equation equation is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

The "unsolvable equations" for the cubic will be a 4×4 (nonsingular) matrix times four-vector equal to our b vector. In this case, since the matrix is invertible, there is an *actual* solution to the problem, so the error will be 0. In other words, through any 4 points in the plane, there exists a cubic.

Note: Completely by chance, the problem as given admits a *quadratic* exact solution, so in fact the error for the quadratic fit problem will also be 0. This is special, not general.

II.3.18

The answer is $\frac{9}{10}$.

II.3.22

$$\hat{u} = \left(\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 13 \\ 17 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$$

So the closest line is $9 + 4x$. The error is 0 because b is *in the column space of* A.

II.4.1

The incidence matrices are

$$A_{triangle} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$A_{square} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Note that the diagonal edge should be labeled 3.

$$(A_{triangle})^t A_{triangle} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$(A_{square})^t A_{square} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

II.4.3

One solution to *any* incidence matrix is a constant vector, so

$$u = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

is a solution to $A_{square}u = 0$.

For the transpose, we use cycles to get the two linearly independent solutions:

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

II.4.8

The process generates the following matrices step by step:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 & -c_1 & 0 & 0 \\ -c_1 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1+c_2 & -c_1 & -c_2 & 0 \\ -c_1 & c_1 & 0 & 0 \\ -c_2 & 0 & c_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1+c_2 & -c_1 & -c_2 & 0 \\ -c_1 & c_1+c_3 & -c_3 & 0 \\ -c_2 & -c_3 & c_2+c_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1+c_2+c_4 & -c_1 & -c_2 & -c_4 \\ -c_1 & c_1+c_3 & -c_3 & 0 \\ -c_2 & -c_3 & c_2+c_3 & 0 \\ -c_4 & 0 & 0 & c_4 \end{bmatrix}$$

$$\begin{bmatrix} c_1+c_2+c_4 & -c_1 & -c_2 & -c_4 \\ -c_1 & c_1+c_3+c_5 & -c_3 & -c_5 \\ -c_2 & -c_3 & c_2+c_3 & 0 \\ -c_4 & -c_5 & 0 & c_4+c_5 \end{bmatrix}$$

$$\begin{bmatrix} c_1+c_2+c_4 & -c_1 & -c_2 & -c_4 \\ -c_1 & c_1+c_3+c_5 & -c_3 & -c_5 \\ -c_2 & -c_3 & c_2+c_3+c_6 & -c_6 \\ -c_4 & -c_5 & -c_6 & c_4+c_5+c_6 \end{bmatrix}$$

II.4.12

A diagonal entry of $K * K^{-1}$ will be

$$\frac{1}{n}\left[2*(n-1)+\sum_{i=2}^{n-1}(-1)\right] = \frac{1}{n}\left[2*n-2-(n-2)\right] = 1$$

and an off diagonal entry will be

$$\frac{1}{n}\left[(n-1)-2+(-1*(n-3))\right] = 0$$

so the product is indeed the identity matrix.

As we've determined the eigenvalues are 1 and n (repeated $n-2$ times), we know all the eigenvalues are positive, so K is positive definite.

I would also add, since we know the eigenvalues, that the *Matrix Tree Theorem* states that if we take the determinant of the reduced graph laplacian for a connected graph we get the total number of spanning trees (connected subgraphs with no loops that touch every vertex). The determinant is the product of the eigenvalues, so the complete graph on n vertices has n^{n-2} spanning trees! This is an old theorem of Cayley (check this... but I'm pretty sure), proved very neatly.

II.2.17

(1) There are a total of $\begin{pmatrix} n \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ 2 \end{pmatrix} = 36$ possible edges in the graph, of which 12 are included. So there are 24 edges missing, so twice as many zeroes in A^tA (one for each end of the missing edge).

(2) It's main diagonal is the list of degrees, so

$$[2\,3\,2\,3\,4\,3\,2\,3\,2]$$

(3) The short answer is because there are four edges incident on the middle vertex.