

Your name is: _____

Grading

1
2
3
4Total

Thank you for making this course a pleasure to teach. The spring course 18.086 will be taught MW at 3–4:30pm by Dr. Mucha and I definitely recommend him.

1) (28 pts.)

- (a) Find the Fourier series for the delta function $\delta(x - \pi)$ with spike at $x = \pi$ (and then 2π -periodic, so there are spikes at $x = \pm\pi, \pm3\pi, \dots$).
- (b) Solve Laplace's equation inside the unit circle $r = 1$, with normal derivative boundary condition specified as $\delta(x) - \delta(x - \pi)$:

$$\frac{\partial u}{\partial r}(1, \theta) = F(\theta) = [\text{spike at } \theta = 0] - [\text{spike at } \theta = \pi].$$

Thus there is a source and a sink. First give a general form for the solution to Laplace's equation, then find the Fourier series for $F(\theta)$, then make the general form fit the boundary conditions.

- (c) This question is about the constant term a_0 in the solution $u(r, \theta)$. When you used the boundary condition $\partial u / \partial r = F$ in part (b), what information did that give about a_0 ? How is a_0 determined (if it is)?
- (d) What is the x -derivative $\frac{\partial u}{\partial x}$ at the center of the circle?

2) (24 pts.)

- (a) Start with the function $e^{\delta i x}$. Take its values (f_0, f_1, f_2, f_3) at the four points $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. Find the Discrete Fourier Transform $(\hat{f}_0, \hat{f}_1, \hat{f}_2, \hat{f}_3)$ of this vector.
- (b) Find a different function $e^{a i x}$ that would give exactly the same answer for part (a). In this case the number a is an “alias” for the number 6.
- (c) Sketch a graph of the function $f(x) = |x|$ on the interval $|x| \leq \pi$. Then roughly sketch what the sum of 8 terms of its Fourier series might look like. How fast do the Fourier coefficients decay?

3) (24 pts.)

- (a) Start with the 4th-order differential equation $u'''' + u = e^{3ix}$ and 2π -periodic boundary conditions

$$u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \quad u''(0) = u''(2\pi), \quad u'''(0) = u'''(2\pi).$$

If the solution is expanded in a Fourier series $u(x) = \sum_{-\infty}^{\infty} d_k e^{ikx}$, substitute into the differential equation to find the coefficients d_k .

- (b) If the right side of the equation changes to the unit delta function $\delta(x)$, what are now the coefficients d_k in the solution $u(x)$?
- (c) How fast do these coefficients in part (b) decay and how smooth is $u(x)$ (how many continuous derivatives)?

4) (24 pts.)

- (a) Write down the circulant matrix C that corresponds to cyclic convolution with the vector $f = (f_0, f_1, f_2, f_3)$. If you want to solve the 4 by 4 system $Cx = b$ using the Discrete Fourier Transforms \hat{f} and \hat{b} , how would you find the solution x ? Describe each step.
- (b) Suppose $f = (1, 1, -1, -1)$. Find its transform $\hat{f} = (\hat{f}_0, \hat{f}_1, \hat{f}_2, \hat{f}_3)$. Give two reasons why the circulant matrix C for this f is *not invertible*.
1. Give a reason based on your computed \hat{f} .
 2. Give a reason based directly on the matrix C , by finding a nonzero solution to $Cx = 0$.
- (c) What is the N -point Discrete Fourier Transform $(\hat{g}_0, \hat{g}_1, \dots, \hat{g}_{N-1})$ of the vector $g = (1, w, w^2, \dots, w^{N-1})$? Here $w = e^{2\pi i/N}$.